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Local limit theorems in relatively hyperbolic groups II: the non-spectrally degenerate case

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ABSTRACT

This is the second of a series of two papers dealing with local limit theorems in relatively hyperbolic groups. In this second paper, we restrict our attention to non-spectrally degenerate random walks and we prove precise asymptotics of the probability $p_n(e, e)$ of going back to the origin at time n. We combine techniques adapted from thermodynamic formalism with the rough estimates of the Green function given by part I to show that $p_n(e, e) \sim CR^{-n}n^{-3/2}$, where R is the inverse of the spectral radius of the random walk. This both generalizes results of Woess for free products and results of Gouëzel for hyperbolic groups.

1. Introduction

Consider a finitely generated group Γ and a probability measure μ on Γ. We define the μ-random walk on Γ, starting at $\gamma \in \Gamma$, as $X_n^\gamma = \gamma g_1 \ldots g_n$, where (g_k) are independent random variables of law μ in Γ. The law of X_n^γ is denoted by $p_n(\gamma, \gamma')$. For $\gamma = e$, it is given by the convolution powers μ^{*n} of the measure μ.

We say that μ is admissible if its support generates Γ as a semigroup. We say that μ is symmetric if $\mu(\gamma) = \mu(\gamma^{-1})$. Finally, if μ is admissible, we say that the random walk is aperiodic if $p_n(e, e) > 0$ for large enough n. The local limit problem consists in finding asymptotics of $p_n(e, e)$ when n goes to infinity. In many situations, if the μ-random walk is aperiodic, one can prove a local limit theorem of the form

$$p_n(e, e) \sim CR^{-n}n^{-\alpha}, \tag{1}$$

where $C > 0$ is a constant, $R \geq 1$, and $\alpha \in \mathbb{R}$.

If $\Gamma = \mathbb{Z}^d$ and μ is finitely supported and aperiodic, then classical Fourier computations show that $p_n(e, e) \sim Cn^{-d/2}$ if the random walk is centered and $p_n(e, e) \sim CR^{-n}n^{-d/2}$ with $R > 1$ if the random walk is non-centered. If Γ is a non-elementary Gromov-hyperbolic group and μ is finitely supported, symmetric and aperiodic, then one has $p_n(e, e) \sim CR^{-n}n^{-3/2}$, with $R > 1$, see [Gou14] and references therein.

Free products are a great source of examples for various local limit theorems, see, for example, [Car88, Car89, CG12]. Woess proved in [Woe86] that for a special class of nearest-neighbor random walks on free products, called the 'typical case' in [Woe00], one has a local limit of

the form (1), with $\alpha = 3/2$. This 'typical case' should be considered informally as a situation where the random walk only sees the underlying tree structure of the free product, and not what happens inside the free factors. Thus, in some sense, this coefficient $3/2$ is consistent with the hyperbolic case [Gou14]. Our main goal in this series of two papers is to extend Woess' results to any relatively hyperbolic group.

We give more details on relatively hyperbolic groups in § 2.1. Recall for now that a finitely generated group Γ is relatively hyperbolic with respect to a collection of subgroups Ω if it acts via a geometrically finite action on a proper geodesic Gromov hyperbolic space X, such that Ω is exactly the set of stabilizers of parabolic limit points for this action. Let Ω_0 be a set of representatives of conjugacy classes of elements of Ω. Such a set Ω_0 is finite.

Let μ be a probability measure on a relatively hyperbolic group Γ. Denote by R_μ the inverse of its spectral radius, that is, the radius of convergence of the Green function $G(x, y|r)$, defined as

$$G(x, y|r) = \sum_{n \geq 0} p_n(x, y) r^n.$$

This radius of convergence is independent of x, y. Let $\mathcal{H} \in \Omega_0$ be a parabolic subgroup. Denote by $p_{\mathcal{H}}$ the first return kernel to $\mathcal{H}$ associated to the measure $R_\mu \mu$. Say that a probability measure μ is spectrally degenerate along $\mathcal{H} \in \Omega_0$ if the spectral radius of $p_{\mathcal{H}}$ is one and that μ is non-spectrally degenerate if, for every $\mathcal{H} \in \Omega_0$, it is not spectrally degenerate along $\mathcal{H}$. This definition is independent of the choice of Ω_0. It was introduced in [DG21] and appeared to be crucial in the study of the stability of the Martin boundary of relatively hyperbolic groups.

In part I, we introduced the notion of spectral positive recurrence and proved a weaker form of (1) under this assumption, namely that there exists C such that

$$C^{-1} R^{-n} n^{-3/2} \leq p_n(e, e) \leq C R^{-n} n^{-3/2}.$$

In this part II, we prove a precise local limit theorem like (1), with $\alpha = 3/2$, for non-spectrally degenerate measures on relatively hyperbolic groups. As it was proved in part I, non-spectrally degenerate random walks are spectrally positive recurrent, so our assumptions here are stronger, but we prove a more precise result. We insist on the fact that our methods in both papers are very different and that this paper is not an enhanced version of part I, as it uses the results of part I. We make further comments in § 2.3.

Our main goal is to prove the following.

THEOREM 1.1. *Let Γ be a non-elementary relatively hyperbolic group. Let μ be a finitely supported, admissible, symmetric, and non-spectrally degenerate probability measure on Γ. Assume that the corresponding random walk is aperiodic. Then, for every $\gamma, \gamma' \in \Gamma$, there exists $C_{\gamma, \gamma'} > 0$ such that*

$$p_n(\gamma, \gamma') \sim C_{\gamma, \gamma'} R_\mu^{-n} n^{-3/2}.$$

If the μ-random walk is not aperiodic, similar asymptotics hold for $p_{2n}(\gamma, \gamma')$ if the distance between γ and γ' is even and for $p_{2n+1}(\gamma, \gamma')$ if this distance is odd.

This generalizes Woess's results [Woe86] on free products and known results on hyperbolic groups (see [GW86, Lal93, GL13, Gou14]). As a corollary, we also obtain the following.

COROLLARY 1.2. *Let Γ be a non-elementary relatively hyperbolic group. Let μ be a finitely supported, admissible, symmetric, and non-spectrally degenerate probability measure on Γ.*

Assume that the corresponding random walk is aperiodic. Denote by $q_n(x, y)$ the probability that the first visit in positive time at y starting at x is at time n. Then,

$$q_n(\gamma, \gamma') \sim C_{\gamma,\gamma'} R_\mu^{-n} n^{-3/2}.$$

In [Ger81], Gerl conjectured that if a local limit of the form $p_n(e, e) \sim C R^{-n} n^{-\alpha}$ holds for a finitely supported random walk, then α is a group invariant. This conjecture was disproved by Cartwright in [Car89]. He gave examples of local limit theorems on $\mathbb{Z}^d * \mathbb{Z}^d$, with $\alpha = d/2$ and examples on the same groups with $\alpha = 3/2$. Actually, if $d \neq 3$, then one can only get $\alpha = d/2$ if $d \geq 5$. There are some computations to explain why in [Car88] (see also [Woe00]). In [DG21, Proposition 6.1], we gave a geometric explanation of this fact and proved that if a parabolic subgroup $\mathcal{H}$ is virtually abelian of rank $d \leq 4$, the random walk cannot be spectrally degenerate along $\mathcal{H}$. As a particular case, we thus obtain the following corollary, for low-dimensional Kleinian groups.

THEOREM 1.3. *Let Γ be the fundamental group of a geometrically finite hyperbolic manifold of dimension $n \leq 5$. Let μ be a finitely supported, admissible, and symmetric probability measure on Γ. Assume that the μ-random walk is aperiodic. Then, for every $\gamma, \gamma' \in \Gamma$, there exists $C_{\gamma,\gamma'} > 0$ such that*

$$p_n(\gamma, \gamma') \sim C_{\gamma,\gamma'} R_\mu^{-n} n^{-3/2}.$$

If the μ-random walk is not aperiodic, similar asymptotics hold for $p_{2n}(\gamma, \gamma')$ if the distance between γ and γ' is even and for $p_{2n+1}(\gamma, \gamma')$ if it is odd.

Let us now give some details on the proofs. We have the same approach as Gouëzel and Lalley in [GL13, Gou14] and we begin by explaining their work.

The first step in both papers is to obtain an asymptotic differential equation satisfied by the Green function. Throughout this paper, we use the following notation: if two functions f and g satisfy that there exists some constant $C \geq 0$ such that $f \leq Cg$, then we write $f \lesssim g$. In addition, if $f \lesssim g$ and $g \lesssim f$, then we write $f \asymp g$. Whenever we need to be specific about the constant, or about its dependence on some parameters, we write the full inequalities to avoid being unclear. In [GL13, Gou14], the authors proved that

$$\frac{d^2}{dr^2} G(e, e|r) \asymp \left(\frac{d}{dr} G(e, e|r) \right)^3, \tag{2}$$

the implicit constant not depending on r. Integrating these inequalities yields

$$\left(\frac{d}{dr} G(e, e|r) \right)^{-2} - \left(\frac{d}{dr} G(e, e|R_\mu) \right)^{-2} \asymp R_\mu - r,$$

so that, assuming $(d/dr)G(e, e|R_\mu) = +\infty$ (which is proved in [GL13, Gou14]), one obtains

$$\frac{d}{dr} G(e, e|r) \asymp \frac{1}{\sqrt{R_\mu - r}}.$$

The rigorous way to proceed is to transform these *a priori* estimates (2) into an equivalent when r tends to R_μ, that is,

$$\frac{d^2}{dr^2} G(e, e|r) \underset{r \to R_\mu}{\sim} C \left(\frac{d}{dr} G(e, e|r) \right)^3. \tag{3}$$

Once this is established, one can prove that

$$\frac{d}{dr} G(e, e|r) \underset{r \to R_\mu}{\sim} \frac{C'}{\sqrt{R_\mu - r}}.$$

Finally, one can obtain asymptotics of $p_n(e, e)$ from asymptotics of $(d/dr)G(e, e|r)$ using Tauberian theorems and spectral theory. To go from (2) to (3), Lalley and Gouëzel used thermodynamic formalism. Precisely, they use Cannon's result and choose a finite automaton that encodes shortlex geodesics in the hyperbolic group Γ. They then define some potential function depending on r on the path space of this automaton, using the Green function. To prove that this potential function is Hölder continuous, they use the strong Ancona inequalities (see §2.4 for more details). They then apply the Ruelle–Perron–Frobenius theorem to the associated transfer operator to derive asymptotic properties of the first and second derivatives of the Green function, when r tends to R_μ, which, in turn, leads to (3).

Brief outline of the paper

We first compile in §2 the tools and results that are needed in the following. We also review in §3 the results of Sarig on thermodynamic formalism for countable shifts. The remaining sections are devoted to adapting the proofs of Lalley and Gouëzel to the relatively hyperbolic case.

Note that the first step, that is, obtaining an asymptotic differential equation satisfied by the Green function, is given by the results proved in part I. Precisely, [Dus22, Theorem 1.5] shows that (2) holds again in our situation. In the present paper, we mainly focus on the second step, that is deriving a precise equivalent such as (3) from the *a priori* estimates (2). We first give in §§4 and 5 an estimate for the first derivative of the Green function, in terms of the spectral data of a suited transfer operator. We then give in §6 an estimate of the second derivative. Combining these two estimates leads to (3). There are several difficulties here.

First, we do not have a finite automaton encoding geodesics. Anyway, geodesics are not so much interesting for our purpose. Indeed, Ancona inequalities that are used in [GL13, Gou14] to prove Hölder continuity do not hold along geodesics, but along relative geodesics in relatively hyperbolic groups. On the other hand, we proved in part I that there exists an automaton with finite set of vertices and countable set of edges that encodes relative geodesics, see precisely [Dus22, Theorem 4.2]. We use this automaton instead.

However, the associated path space will not be finite but countable. We thus have to use thermodynamic formalism for countable Markov shifts, which is more delicate to handle than thermodynamic formalism for Markov shifts of finite type. For example, there are situations where Ruelle–Perron–Frobenius theorem, which is a crucial tool in [Gou14], does not hold for countable shifts. We thus prove that the Hölder continuous function analogous to that introduced in [GL13, Gou14] is positive recurrent (using the terminology of Sarig in [Sar99]), which is sufficient to mimic some of the arguments of Lalley and Gouëzel.

Another difficulty is that the family of transfer operators $(\mathcal{L}_r)_{r \leq R_\mu}$ we introduce does not vary continuously in r for the operator norm. However, looking carefully at the proofs of [GL13, Gou14], one only needs continuity of the spectral data associated to this family of operators. We use an enhanced version of perturbations results due to Keller and Liverani [KL99] to prove this sort of continuity.

Finally, the last step, that is, getting the local limit theorem from the asymptotics of the Green function, is a combination of Tauberian theorems and spectral theory. We are able to use directly the results of [GL13] and so we have nothing to prove there to conclude. We also deduce Corollary 1.2 from Theorem 1.1 using directly results of [GL13]. We give more details in §7.

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2. Some background

2.1 Relatively hyperbolic groups

We first recall definitions and basic properties of relatively hyperbolic groups. More details are given in part I [Dus22]. Consider a finitely generated group Γ acting discretely and by isometries on a proper and geodesic hyperbolic space (X, d). We denote the limit set of Γ by $\Lambda\Gamma$, that is, the set of accumulation points in the Gromov boundary ∂X of an orbit $\Gamma \cdot o$, $o \in X$. A point $\xi \in \Lambda\Gamma$ is called conical if there is a sequence (γ_n) of Γ and distinct points ξ_1, ξ_2 in $\Lambda\Gamma$ such that $\gamma_n\xi$ converges to ξ_1 and $\gamma_n\zeta$ converges to ξ_2 for all $\zeta \neq \xi$ in $\Lambda\Gamma$. A point $\xi \in \Lambda\Gamma$ is called parabolic if its stabilizer in Γ is infinite, fixes exactly ξ in $\Lambda\Gamma$ and contains no loxodromic element. The stabilizer of a parabolic limit point is called a (maximal) parabolic subgroup. A parabolic limit point ξ in $\Lambda\Gamma$ is called bounded parabolic if is stabilizer in Γ is infinite and acts cocompactly on $\Lambda\Gamma \setminus \{\xi\}$. Finally, the action is called geometrically finite if the limit set only consists of conical limit points and bounded parabolic limit points.

DEFINITION 2.1. The group Γ is relatively hyperbolic with respect to Ω if it acts geometrically finitely on such a hyperbolic space (X, d) such that the stabilizers of the parabolic limit points are exactly the elements of Ω. In this situation, Γ is said to be non-elementary if its limit set is infinite.

One might choose different spaces X on which Γ can act geometrically finitely. However, different choices of X give rise to equivariantly homeomorphic limit sets $\Lambda\Gamma$. We call this limit set the Bowditch boundary of Γ and we denote it by $\partial_B\Gamma$.

Let Γ be a relatively hyperbolic group with respect to a collection of subgroups Ω. Let Ω_0 be a set of representatives of conjugacy classes of elements of Ω. Such a set Ω_0 is necessarily finite, according to [Bow12, Proposition 6.15]. Fix a finite generating set S for Γ. Denote by $\hat\Gamma$ the Cayley graph associated with the infinite generating set consisting of the union of S and of all parabolic subgroups $\mathcal{H} \in \Omega_0$. Endowed with the graph distance, that we write $\hat{d}$, the graph $\hat\Gamma$ is hyperbolic.

A relative geodesic is a geodesic in the graph $\hat\Gamma$. A relative quasi-geodesic is a path of *adjacent* vertices in $\hat\Gamma$, which is a quasi-geodesic for the distance $\hat{d}$. Following Osin [Osi06], a path is called without backtracking if once it has left a coset $\gamma\mathcal{H}$, for $\mathcal{H} \in \Omega_0$, it never goes back to it. Relative geodesic and relative quasi-geodesic satisfy the following property, called the BCP property.

PROPOSITION 2.1 (BCP property). *For all $\lambda \geq 1$ and $c \geq 0$, there exists a constant $C_{\lambda,c}$ such that for every pair (α_1, α_2) of relative (λ, c)-quasi geodesic paths without backtracking, starting and ending at the same point in Γ, the following hold:*

(i) *if α_1 travels more than $C_{\lambda,c}$ in a coset, then α_2 enters this coset;*
(ii) *if α_1 and α_2 enter the same coset, the two entering points and the two exit points are $C_{\lambda,c}$-close to each other in $\mathrm{Cay}(\Gamma, S)$.*

We need to study both geodesics in $\mathrm{Cay}(\Gamma, S)$ and relative geodesics in the following. We use the following terminology. Let α be a geodesic in $\mathrm{Cay}(\Gamma, S)$ and let $\eta_1, \eta_2 \geq 0$. A point γ on α is called an (η_1, η_2)-transition point if for any coset $\gamma_0\mathcal{H}$ of a parabolic subgroup, the part of α consisting of points at distance at most η_2 from γ is not contained in the η_1-neighborhood of $\gamma_0\mathcal{H}$.

Transition points are of great importance in relatively hyperbolic groups. They stay close to points on relative geodesics in the following sense.

LEMMA 2.2 [Hru10, Proposition 8.13]. *Fix a generating set S. For every large enough $\eta_1, \eta_2 > 0$, there exists $r \geq 0$ such that the following holds. Let α be a geodesic in $\mathrm{Cay}(\Gamma, S)$ and let $\hat{\alpha}$ be a relative geodesic path with the same endpoints as α. Then, for the distance in $\mathrm{Cay}(\Gamma, S)$, any (η_1, η_2)-transition point on α is within r of a point on $\hat{\alpha}$ and, conversely, any point on $\hat{\alpha}$ is within r of an (η_1, η_2)-transition point on α.*

2.2 Relatively automatic groups

Hyperbolic groups are known to be strongly automatic, meaning that for every generating set S, there exists a finite directed graph $\mathcal{G} = (V, E, v_*)$ encoding geodesics. It is quite implicit in Farb's work [Far94, Far98], although not formally stated, that relatively hyperbolic groups are relatively automatic in the following sense.

Let Γ be a finitely generated group and let Ω be a collection of subgroups invariant by conjugacy and such that there is a finite set Ω_0 of conjugacy classes representatives of subgroups in Ω.

DEFINITION 2.2. A relative automatic structure for Γ with respect to the collection of subgroups Ω_0 and with respect to some finite generating set S is a directed graph $\mathcal{G} = (V, E, v_*)$ with distinguished vertex v_* called the starting vertex, where the set of vertices V is finite and with a labelling map $\phi: E \to S \cup \bigcup_{\mathcal{H} \in \Omega_0} \mathcal{H}$ such that the following holds. If $\omega = e_1, \ldots, e_n$ is a path of adjacent edges in $\mathcal{G}$, define $\phi(e_1, \ldots, e_n) = \phi(e_1) \ldots \phi(e_n) \in \Gamma$. Then:

(i) no edge ends at v_*, except the trivial edge starting and ending at v_*;
(ii) every vertex $v \in V$ can be reached from v_* in $\mathcal{G}$;
(iii) for every path $\omega = e_1, \ldots, e_n$, the path $e, \phi(e_1), \phi(e_1 e_2), \ldots, \phi(\gamma)$ in Γ is a relative geodesic from e to $\phi(\gamma)$, that is, the image of $e, \phi(e_1), \phi(e_1 e_2), \ldots, \phi(\gamma)$ in $\hat{\Gamma}$ is a geodesic for the metric $\hat{d}$;
(iv) the extended map ϕ is a bijection between paths in $\mathcal{G}$ starting at v_* and elements of Γ.

Note that the union $S \cup \bigcup_{\mathcal{H} \in \Omega_0} \mathcal{H}$ is not required to be a disjoint union. Actually, the intersection of two distinct subgroups $\mathcal{H}, \mathcal{H}' \in \Omega_0$ can be non-empty. Also note that we require the vertex set V to be finite. However, the set of edges is infinite, except if all subgroups $\mathcal{H}$ in Ω_0 are finite.

If there exists a relative automatic structure for Γ with respect to Ω_0 and S, we say that Γ is automatic relative to Ω_0 and S. The following was proved in part I.

THEOREM 2.3 [Dus22, Theorem 4.2]. *Let Γ be a relatively hyperbolic group and let Ω_0 be a finite set of representatives of conjugacy classes of the maximal parabolic subgroups. For every symmetric finite generating set S of Γ, Γ is automatic relative to Ω_0 and S.*

Along the proof of this theorem, a lot of technical lemmas about relative geodesics were proved in [Dus22]. We use some of them repeatedly in this paper, so we restate them for convenience. We use the same notation as previously and so we fix a relatively hyperbolic group Γ and a finite set Ω_0 of conjugacy classes of parabolic subgroups.

LEMMA 2.4 [Dus22, Lemma 4.5]. *For every $K \geq 0$, there exists $C \geq 0$ such that the following holds. Let $(x_1, \ldots, x_n, \ldots)$ and $(x'_1, \ldots, x'_m, \ldots)$ be two infinite relative geodesics such that $d(x_1, x'_1) \leq K$ and x_n and x'_m converge to the same conical limit point ξ. Then, for every $j \geq 1$, there exists i_j such that $d(x_j, x'_{i_j}) \leq C$.*

LEMMA 2.5 [Dus22, Lemma 4.16]. *Let $(e, \gamma_1, \ldots, \gamma_n)$ and $(e, \gamma'_1, \ldots, \gamma'_m)$ be two relative geodesics. Assume that the nearest point projection of γ'_m on $(e, \gamma_1, \ldots, \gamma_n)$ is at γ_l. If there*

are several such nearest point projection, choose the closest to γ_n. Then, any relative geodesic from γ'_m to γ_n passes within a bounded distance of γ_l for the distance d. Moreover, if $\gamma_l \neq e$, then any relative geodesic from e to γ'_m passes within a bounded distance of γ_{l-1}.

2.3 Spectrally degenerate measures

We now consider a group Γ, hyperbolic relative to a collection of subgroups Ω. We fix a finite collection $\Omega_0 = \{\mathcal{H}_1, \ldots, \mathcal{H}_N\}$ of representatives of conjugacy classes of Ω. We assume that Γ is non-elementary. Let μ be a probability measure on Γ, R_μ the inverse of the spectral radius of the μ-random walk and $G(\gamma, \gamma'|r)$ the associated Green function, evaluated at r, for $r \in [0, R_\mu]$. If $\gamma = \gamma'$, we simply use the notation $G(r) = G(\gamma, \gamma|r) = G(e, e|r)$.

We denote by p_k the first return transition kernel to $\mathcal{H}_k$. Namely, if $h, h' \in \mathcal{H}_k$, then $p_k(h, h')$ is the probability that the μ-random walk, starting at h, eventually comes back to $\mathcal{H}_k$ and that its first return to $\mathcal{H}_k$ is at h'. In other words,

$$p_k(h, h') = \mathbb{P}_h(\exists n \geq 1, X_n = h', X_1, \ldots, X_{n-1} \notin \mathcal{H}_k).$$

More generally, for $r \in [0, R_\mu]$, we denote by $p_{k,r}$ the first return transition kernel to $\mathcal{H}_k$ for $r\mu$. Precisely, if $h, h' \in \mathcal{H}_k$, then

$$p_{k,r}(h, h') = \sum_{n \geq 1} \sum_{\substack{\gamma_1, \ldots, \gamma_{n-1} \\ \notin \mathcal{H}_k}} r^n \mu(h^{-1}\gamma_1)\mu(\gamma_1^{-1}\gamma_2) \ldots \mu(\gamma_{n-2}^{-1}\gamma_{n-1})\mu(\gamma_{n-1}^{-1}h').$$

We then denote by $p_{k,r}^{(n)}$ the convolution powers of this transition kernel, by $G_{k,r}(h, h'|t)$ the associated Green function, evaluated at t and by $R_k(r)$ the inverse of the associated spectral radius, that is, the radius of convergence of the power series $t \mapsto G_{k,r}(h, h'|t)$. For simplicity, write $R_k = R_k(R_\mu)$. If $h = h'$, we simply write $G_{k,r}(t) = G_{k,r}(h, h|t) = G_{k,r}(e, e|t)$.

According to [Dus22, Lemma 3.4], for any $r \in [0, R_\mu]$, for any $k \in \{1, \ldots, N\}$,

$$G_{k,r}(h, h'|1) = G(h, h'|r).$$

In addition, because Γ is non-elementary, it contains a free group and, hence, is non-amenable. It follows from a result of Guivarc'h (see [Gui80, p. 85, Remark b)]) that $G(R_\mu) < +\infty$. Thus, $G_{k,R_\mu}(1) < +\infty$. In particular, $R_k \geq 1$.

DEFINITION 2.3. We say that μ (or, equivalently, the random walk) is spectrally degenerate along $\mathcal{H}_k$ if $R_k = 1$. We say it is non-spectrally degenerate if, for every k, $R_k > 1$.

This definition was introduced in [DG21] to study the homeomorphism type of the Martin boundary at the spectral radius. As explained in [Dus22, § 3.3], it should be thought of as a notion of a spectral gap between the spectral radius of the random walk on the whole group and the spectral radii of the induced walks on the parabolic subgroups. Very roughly speaking, the random walk is not spectrally degenerate if the driving probability measure μ does not give too much weight to the parabolic subgroups. Let us give more details on this intuition now. For simplicity, we only consider the case of free products which was studied by Woess in [Woe86] and by Candellero and Gilch in [CG12]. We first need to introduce some notation.

Let $\Gamma = \Gamma_0 * \Gamma_1$ be a free product of two groups. Then, Γ is hyperbolic relative to the conjugates of the free factors Γ_0 and Γ_1. Consider a symmetric probability measure μ_0, respectively μ_1, whose finite support generates Γ_0, respectively Γ_1. For any $0 < \beta < 1$, set

$$\mu = \beta \mu_1 + (1 - \beta)\mu_0.$$

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Then, μ is a symmetric probability measure whose finite support generates Γ. Such a probability measure is called adapted to the free product structure: the random walk driven by μ can only move inside one of the free factors at each step.

In this situation, the Green function G_μ of μ and the Green function G_{μ_0} of μ_0 can be related by the following formula. For every $x, y \in \Gamma_0$, for every $r \leq R_\mu$,

$$\frac{G_\mu(x,y|r)}{G_\mu(e,e|r)} = \frac{G_{\mu_0}(x,y|\zeta_0(r))}{G_{\mu_0}(e,e|\zeta_0(r))}, \tag{4}$$

where ζ_0 is a continuous function of r, see [Woe00, Proposition 9.18] for an explicit formula. We always have $\zeta_0(R_\mu) \leq R_{\mu_0}$. A similar formula holds for $x, y \in \Gamma_1$.

Actually, because μ is adapted to the free product structure, the first return kernel $p_{\Gamma_0, r}$ can be written as

$$p_{\Gamma_0, r}(e, x) = (1 - \beta) r \mu_0 + w_0 \delta_{e,x},$$

where w_0 is the weight of the first return to e associated to $r\mu$, starting with a step in Γ_1. Thus, [Woe00, Lemma 9.2] shows that for any $x, y \in \Gamma_0$,

$$G_{\Gamma_0, r}(x, y|t) = \frac{1}{1 - w_0 t} G_{\mu_0}\left(x, y \;\middle|\; \frac{(1 - \beta) r t}{1 - w_0 t}\right). \tag{5}$$

In particular, for $t = 1$,

$$G_{\Gamma_0, r}(x, y|1) = \frac{1}{1 - w_0} G_{\mu_0}\left(x, y \;\middle|\; \frac{(1 - \beta) r}{1 - w_0}\right)$$

and so we recover (4) with $\zeta(r) = ((1 - \beta) r)/(1 - w_0)$.

Following Woess [Woe00], we call the situation where $\zeta_i(R_\mu) < R_i$ (the inverse of the spectral radius of μ_i) the 'typical case'. We prove that the 'typical case' is exactly the case where the measure is not spectrally degenerate.

PROPOSITION 2.6. *Consider an adapted random walk on a free product $\Gamma = \Gamma_0 * \Gamma_1$. The random walk is spectrally degenerate along Γ_0 if and only if $\zeta_0(R_\mu) = R_{\mu_0}$.*

Proof. Applying (5) to $t = 1 + \epsilon$ and $r = R_\mu$, we obtain

$$G_{\Gamma_0, R_\mu}(x, y|1 + \epsilon) = \frac{1}{1 - w_0(1 + \epsilon)} G_{\mu_0}\left(x, y \;\middle|\; \frac{(1 - \beta) R_\mu (1 + \epsilon)}{1 - w_0(1 + \epsilon)}\right).$$

If $\epsilon > 0$, then

$$\frac{(1 - \beta) R_\mu (1 + \epsilon)}{1 - w_0(1 + \epsilon)} > \frac{(1 - \beta) R_\mu}{1 - w_0} = \zeta_0(R_\mu).$$

Thus, there exists $t > 1$ such that $G_{\Gamma_0, R_\mu}(x, y|t)$ is finite if and only if there exists $z > \zeta_0(R_\mu)$ such that $G_{\mu_0}(x, y|z)$ is finite, which concludes the proof. $\square$

Hence, in the context of adapted random walks on free products, to check whether the random walk is non-spectrally degenerate, one has to check whether $\zeta_i(R_\mu) < R_i$ or not. This problem was studied by Candellero and Gilch. More specifically, in [CG12, § 7], they construct both spectrally degenerate and non-spectrally degenerate measures. In particular, in their example A, $\Gamma_i = \mathbb{Z}^{d_i}$, where $d_i \geq 5$. Then, if β is close enough to zero, the random walk is spectrally degenerate along Γ_0. Similarly, if β is close enough to one, the random walk is spectrally degenerate along Γ_1. In the middle case, the random walk is not spectrally degenerate.

Let us also mention their example F, where $\Gamma_0 = \mathbb{Z}^5$ and $\Gamma_1 = \mathbb{Z}^6$. They construct measures μ_0 and μ_1 such that the adapted measure μ is spectrally degenerate for every β. More precisely,

they show there exists a critical parameter β_c such that the following classification holds. For $\beta < \beta_c$, μ is spectrally degenerate along Γ_0 and is not spectrally positive recurrent. For $\beta > \beta_c$, μ is spectrally degenerate along Γ_1 and is not spectrally positive recurrent. On the other hand, for $\beta = \beta_c$, μ is spectrally positive recurrent. This yields a random walk which is spectrally degenerate, but which is spectrally positive recurrent. In this situation, one has a local limit theorem of the form (1) with $\alpha = 3/2$. This leads to the conjecture that spectral positive recurrence is a sufficient condition to obtain $\alpha = 3/2$.

Unfortunately, it seems difficult to construct (non-)spectrally degenerate measures on general relatively hyperbolic groups, so we do not know yet any example beyond free products, except low-dimensional Kleinian groups, according to Theorem 1.3. A reasonable approach would be to try to adapt the method of Candellero and Gilch in the following way. Consider a relatively hyperbolic group Γ and choose a parabolic subgroup $\mathcal{H}$. Start with an admissible, finitely supported and symmetric measure μ_0 on Γ and an admissible, finitely supported and symmetric measure $\mu_{\mathcal{H}}$ on $\mathcal{H}$. Define then $\mu = \beta\mu_0 + (1 - \beta)\mu_{\mathcal{H}}$. One would expect μ to be spectrally degenerate along $\mathcal{H}$ for small enough β. However, to actually prove that it is spectrally degenerate, one would need to prove a sort of continuity of the Green function and its derivatives with respect to the driving measure, which would, in turn, require some new material. Similar difficulties occur when trying to construct non-spectrally degenerate measures.

Finally, let us now recall some consequences of spectral degeneracy proved in [Dus22]. Let Γ be a relatively hyperbolic group. We introduce the following notation. We write

$$I^{(k)}(r) = \sum_{\gamma^{(1)},\ldots,\gamma^{(k)} \in \Gamma} G(\gamma, \gamma^{(1)}|r)G(\gamma^{(1)}, \gamma^{(2)}|r)\ldots G(\gamma^{(k-1)}, \gamma^{(k)}|r)G(\gamma^{(k)}, \gamma'|r).$$

Then, $I^{(k)}(r)$ is related to the kth derivative of the Green function. For a precise statement, we refer to [Dus22, Lemma 3.2]. For instance, we have the following.

LEMMA 2.7 [Dus22, Lemma 3.1]. *For every $\gamma, \gamma' \in \Gamma$, for every $r \in [0, R_\mu]$, we have*

$$\frac{d}{dr}(rG(\gamma_1, \gamma_2|r)) = \sum_{\gamma \in \Gamma} G(\gamma_1, \gamma|r)G(\gamma, \gamma_2|r).$$

If $\mathcal{H}$ is a parabolic subgroup, we also write

$$I_{\mathcal{H}}^{(k)}(r) = \sum_{\gamma^{(1)},\ldots,\gamma^{(k)} \in \mathcal{H}} G(\gamma, \gamma^{(1)}|r)G(\gamma^{(1)}, \gamma^{(2)}|r)\ldots G(\gamma^{(k-1)}, \gamma^{(k)}|r)G(\gamma^{(k)}, \gamma'|r).$$

Once $\Omega_0 = \{\mathcal{H}_1, \ldots, \mathcal{H}_N\}$ is fixed, we also write $I_j^{(k)}(r) = I_{\mathcal{H}_j}^{(k)}(r)$ for simplicity.

One of the main results of part I is that whenever $I_{\mathcal{H}}^{(2)}(r) < +\infty$ for every parabolic subgroup $\mathcal{H} \in \Omega_0$, then

$$I^{(2)}(r) \asymp (I^{(1)}(r))^3.$$

This is a consequence of the following result.

PROPOSITION 2.8 [Dus22, Proposition 5.6]. *For every $r \in [0, R_\mu)$, we have*

$$\frac{I^{(2)}(r)}{I^{(1)}(r))^3} \asymp 1 + \sum_j I_j^{(2)}(r).$$

In particular, if μ is non-spectrally degenerate, then

$$I^{(2)}(r) \asymp (I^{(1)}(r))^3.$$

The following results were also proved in part I. Note that we do not need to assume that μ is non-spectrally degenerate.

LEMMA 2.9 [Dus22, Lemma 5.4]. *There exists some uniform $C \geq 0$ such that for every $r \in [0, R_\mu]$, for every m,*

$$\sum_{\gamma \in \hat{S}_m} H(e, \gamma | r) \leq C.$$

COROLLARY 2.10 [Dus22, Corollary 5.5]. *For every parabolic subgroup $\mathcal{H} \in \Omega_0$ and every $r \in [0, R_\mu]$, we have $I_{\mathcal{H}}^{(1)}(r) < +\infty$.*

Finally, we also use the following.

PROPOSITION 2.11 [Dus22, Proposition 5.8]. *If μ is non-spectrally degenerate, then*

$$\frac{d}{dr}_{|r=R_\mu} G(e, e | R_\mu) = +\infty.$$

2.4 Relative Ancona inequalities

We consider a finitely generated group Γ, hyperbolic relative to a collection of subgroups Ω. Let Ω_0 be a finite set of representatives of conjugacy classes. We also consider a probability measure μ on Γ, whose finite support generates Γ as a semigroup and denote by R_μ the inverse of its spectral radius. As soon as Γ is non-elementary, it is non-amenable, so that $R_\mu > 1$ according to Kesten's results [Kes59].

In the case where Γ is hyperbolic, Ancona proved that the Green function G is roughly multiplicative along geodesics. Precisely, there exists $C \geq 1$ such that if x, y, z are elements along a geodesic in this order, then

$$\frac{1}{C} G(x, y) G(y, z) \leq G(x, z) \leq C G(x, y) G(y, z). \tag{6}$$

See [Anc88] for more details. The proof also works for the Green function evaluated at r, when $r < R_\mu$. Actually, the lower bound is always true, so that the content of Ancona inequalities really is

$$G(x, z | r) \leq C G(x, y | r) G(y, z | r).$$

In [Gou14], Gouëzel proved that these inequalities still hold at $r = R_\mu$ and that the constant C is uniform in r, when the measure is symmetric. He also gave a strengthened version of them. Namely, if x, x', y, y' are four points such that geodesics $[x, y]$ from x to y and $[x', y']$ from x' to y' fellow travel for a time at least n, then for $r \in [1, R_\mu]$,

$$\left| \frac{G(x, y | r) G(x', y' | r)}{G(x', y | r) G(x, y' | r)} - 1 \right| \leq C \rho^n, \tag{7}$$

where $C \geq 0$ and $0 < \rho < 1$ are uniform (see [Gou14, Theorem 2.9]). This strengthened version of Ancona inequalities was proved at $r < R_\mu$ in [INO08]. It was also already proved at the spectral radius by Gouëzel and Lalley in the case of co-compact Fuchsian groups (see [GL13, Theorem 4.6]). It allowed the authors to obtain Hölder regularity for Martin kernels on the Martin boundary and then to use thermodynamic formalism to deduce local limit theorems in hyperbolic groups in [Gou14, GL13].

Back to relatively hyperbolic groups, inequalities similar to (6) were obtained by Gekhtman, Gerasimov, Potyagailo, and Yang in [GGPY21]. Recall that if α is a geodesic in the Cayley graph $\mathrm{Cay}(\Gamma, S)$, a point on α is called a transition point if it is not deep in a parabolic subgroup. It is

proved in [GGPY21] that if x, y, z are elements on a geodesic in this order and if y is a transition point on this geodesic, then (6) holds. The proof actually works for $G(\cdot, \cdot | r)$ whenever $r < R_\mu$.

Because of our automatic structure, it is more convenient for us to work with relative geodesics rather than transition points on actual geodesics. We fix a generating set S and consider the Cayley graph and the graph $\hat{\Gamma}$ associated with S.

DEFINITION 2.4. Let Γ be a relatively hyperbolic group and let μ be a probability measure on Γ with Green function G. Let R_μ be the inverse of the spectral radius of μ. If $r \in [1, R_\mu]$, say that μ satisfies the weak r-relative Ancona inequalities if there exists $C \geq 0$ (which depends on r) such that for every $x, y, z \in \Gamma$ such that their images in $\hat{\Gamma}$ lie in this order on a relative geodesic,

$$\frac{1}{C}G(x, y|r)G(y, z|r) \leq G(x, z|r) \leq CG(x, y|r)G(y, z|r).$$

Say that μ satisfies the weak relative Ancona inequalities up to the spectral radius if it satisfies the r-relative Ancona inequalities for every $r \in [1, R_\mu]$ with a constant C not depending on r.

We also need the following enhanced version of relative Ancona inequalities.

DEFINITION 2.5. We say that two relative geodesics $[x, y]$ and $[x', y']$ c-fellow travel for a time n, for some $c \geq 0$, if there exist distinct points $\gamma_1, \ldots, \gamma_n$ which are at distance in $\mathrm{Cay}(\Gamma, S)$ at most c from points on $[x, y]$ and points on $[x', y']$.

DEFINITION 2.6. Let Γ be a relatively hyperbolic group and let μ be a probability measure on Γ with Green function G. Let R_μ be the inverse of the spectral radius of μ. If $r \in [1, R_\mu]$, say that μ satisfies the strong r-relative Ancona inequalities if for every $c \geq 0$, there exist $C \geq 0$ and $0 < \rho < 1$ such that if x, x', y, y' are four points such that relative geodesics $[x, y]$ from x to y and $[x', y']$ from x' to y' c-fellow travel for a time at least n, then

$$\left| \frac{G(x, y|r)G(x', y'|r)}{G(x', y|r)G(x, y'|r)} - 1 \right| \leq C\rho^n.$$

Say that μ satisfies the strong relative Ancona inequalities up to the spectral radius if it satisfies the strong r-relative Ancona inequalities for every $r \in [1, R_\mu]$ with constants C and ρ not depending on r.

Remark 2.1. If μ satisfies the strong or weak relative Ancona inequalities, then the reflected measure $\check{\mu}$, defined by $\check{\mu}(\gamma) = \mu(\gamma^{-1})$ also satisfies them. Indeed, if $\check{G}$ is the Green function of the reflected measure, then $\check{G}(x, y) = G(y, x)$.

The following is proved in [DG21].

THEOREM 2.12. *Let Γ be a non-elementary relatively hyperbolic group and let μ be a symmetric probability measure on Γ whose finite support generates Γ. Then μ satisfies both the weak and strong relative Ancona inequalities up to the spectral radius.*

Actually, these inequalities are stated in [DG21] using the Floyd distance, which is a suitable rescaling of the distance in $\mathrm{Cay}(\Gamma, S)$. However, [GP16, Corollary 5.10] relates the Floyd distance with transition points and Lemma 2.2 relates transition points with points on a relative geodesic. We deduce the above theorem combining these two results with [DG21, Theorem 1.6].

3. Thermodynamic formalism

3.1 Transfer operators with countably many symbols

We follow here the terminology of Sarig and recall some facts proved in [Sar99, Sar03]. We consider a countable set Σ (the set of symbols), and a matrix $A = (a_{s,s'})_{s,s' \in \Sigma}$, with entries zeros and ones (the transition matrix). We then define

$$\Sigma_A^* = \{x = (x_1, \ldots, x_n), x_i \in \Sigma, n \geq 0, \forall i, a_{x_i, x_{i+1}} = 1\}$$

and

$$\partial \Sigma_A^* = \{x = (x_1, \ldots, x_n, \ldots), x_i \in \Sigma, \forall i, a_{x_i, x_{i+1}} = 1\}.$$

Note that in the definition of Σ_A^*, n can be zero, so that the empty sequence, that we denote by $\emptyset$, is in Σ_A^*. We also define

$$\overline{\Sigma}_A = \Sigma_A^* \cup \partial \Sigma_A^*.$$

If $s_1, \ldots, s_k \in \Sigma$, we define the cylinder $[s_1, \ldots, s_k]$ as $\{x \in \overline{\Sigma}_A, x_1 = s_1, \ldots, x_k = s_k\}$.

Let $T : \overline{\Sigma}_A \to \overline{\Sigma}_A$ be given by

$$T((x_1, \ldots, x_n)) = (x_2, \ldots, x_n)$$

if $(x_1, \ldots, x_n) \in \Sigma_A^*$ and

$$T((x_1, \ldots, x_n, \ldots)) = (x_2, \ldots, x_n, \ldots)$$

if $(x_1, \ldots, x_n, \ldots) \in \partial \Sigma_A^*$. We call T the shift map and we call the pair $(\overline{\Sigma}_A, T)$ a Markov shift.

We say that the Markov shift is irreducible if for every $s, s' \in \Sigma$, there exists $N_{a,b}$ such that there exists $x \in \overline{\Sigma}_A$ with $x_1 = s$ and $x' \in \overline{\Sigma}_A$ with $x_1' = s'$ such that $T^{N_{a,b}} x = x'$. In other words, one can reach any cylinder $[s']$ from any cylinder $[s]$ with a finite number of iterations of the shift. We say it is topologically mixing if for every $s, s' \in \Sigma$, there exists $N_{a,b}$ such that for every $n \geq N_{a,b}$, there exists $x \in \overline{\Sigma}_A$ with $x_1 = s$ and $x' \in \overline{\Sigma}_A$ with $x_1' = s'$ such that $T^n x = x'$.

In [Sar99], everything is stated only using $\partial \Sigma_A^*$. However, up to considering a cemetery symbol $x_\dagger$, we can see finite sequences $(x_1, \ldots, x_n)$ in Σ_A^* as infinite ones, of the form $(x_1, \ldots, x_n, x_\dagger, \ldots, x_\dagger, \ldots)$. Thus, we can apply the terminology and results of [Sar99] to $\overline{\Sigma}_A$. In addition, for technical reasons, it will be convenient to assume that the empty sequence is not a preimage of itself by the shift. This can be done, for example, using a second cemetery symbol.

We also define a metric on $\overline{\Sigma}_A$, setting $d(x, y) = 2^{-n}$, where n is the first time that the two sequences x and y differ.

If $\varphi : \overline{\Sigma}_A \to \mathbb{R}$ is a function, define

$$V_n(\varphi) = \sup\{|\varphi(x) - \varphi(y)|, x_1 = y_1, \ldots, x_n = y_n\}.$$

For $\rho \in (0, 1)$, such a function φ is called ρ-locally Hölder continuous if it satisfies

$$\exists C, \forall n \geq 1, V_n(\varphi) \leq C \rho^n.$$

It is called locally Hölder continuous if it is ρ-locally Hölder continuous for some ρ. Note that nothing is required for $V_0(\varphi)$ and, in particular, φ can be unbounded. We can always change the metric d on $\overline{\Sigma}_A$, defining $d_\rho(x, y) = \rho^n$, where n is the first time that the two sequences x and y differ (and $0 < \rho < 1$). A ρ-locally Hölder continuous function is then a locally Lipschitz function for this new metric.

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If φ is locally Hölder continuous, we denote by $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ T^k$ its nth Birkhoff sum. We also define its transfer operator $\mathcal{L}_\varphi$ as

$$\mathcal{L}_\varphi f(x) = \sum_{Ty=x} e^{\varphi(y)} f(y).$$

It acts on several spaces of functions. We are interested in one in particular, described in the following, on which the transfer operator has a spectral gap. By definition, we have

$$(\mathcal{L}_\varphi^n f)(x) = \sum_{T^n y = x} e^{\varphi_n(y)} f(y).$$

We also define, for $s \in \Sigma$,

$$Z_n(\varphi, s) = \sum_{\substack{T^n x = x \\ x_1 = s}} e^{\varphi_n(x)}.$$

For every s, $(1/n) \log Z_n(\varphi, s)$ has a limit $P(\varphi, s)$. If the Markov shift is irreducible, then it is independent of s and we denote it by $P(\varphi)$. Moreover, $P(\varphi, s) > -\infty$ and if $\|\mathcal{L}_\varphi 1\|_\infty < +\infty$, then $P(\varphi, s) < +\infty$. We refer to [Sar99, Theorem 1] for a proof. Independence of s is proved under the assumption that the Markov shift is topologically mixing, although the proof only requires that it is irreducible. We call $P(\varphi, s)$ the Gurevic pressure of φ at s, or simply its pressure.

We say that φ is positive recurrent if for every $s \in \Sigma$, there exist $M_s \geq 1$ and $\lambda_s > 0$ such that for every large enough n, $Z_n(\varphi, s)/\lambda_s^n \in [M_s^{-1}, M_s]$. If it is the case, then one necessarily has $\log \lambda_s = P(\varphi, s)$. The main result of [Sar99] is that positive recurrence is a necessary and sufficient condition for convergence of the iterates of the transfer operator $\mathcal{L}_\varphi^n$ (see [Sar99, Theorem 4] for a precise statement).

If the set of symbols Σ is finite, then every Hölder continuous function is positive recurrent. Actually, we can say a little more in this case. The convergence of $\mathcal{L}_\varphi^n$ is exponentially fast. Precisely, if the Markov shift is topologically mixing, there exist $\lambda > 0$, a positive function h and a measure ν and constants $C \geq 0$ and $0 < \theta < 1$ satisfying, for all ρ-Hölder continuous function f and all $n \in \mathbb{N}$,

$$\left\| \lambda^{-n} \mathcal{L}_\varphi^n f - h \int f \, d\nu \right\| \leq C\theta^n \|f\|.$$

This is the so-called Ruelle–Perron–Frobenius theorem. Equivalently, λ is a positive eigenvalue of the operator $\mathcal{L}_\varphi$ acting on the space of Hölder continuous functions and the remainder of the spectrum is contained in a disk of radius strictly smaller than λ. In other words, $\mathcal{L}_\varphi$ acts on this space with a spectral gap

When the set of symbols is countable, it can happen that the convergence is not exponentially fast (see [Sar99, Example 1]). However, there are sufficient conditions for this to hold, studied by Aaronson, Denker, and Urbański among others (see [ADU93, AD01]; see also [Gou04]).

DEFINITION 3.1. Say that the Markov shift $(\overline{\Sigma}_A, T)$ has finitely many images if the set

$$\{T[s], s \in \Sigma\}$$

is finite. Equivalently, there is only a finite number of different rows (and, thus, a finite number of different columns) in the matrix A.

Fix $0 < \rho < 1$. Let β be the partition generated by the image sets, that is, β is the σ-algebra generated by $\{T[s], s \in \Sigma\}$. Then, define for a ρ-locally Hölder continuous function f,

$$D_{\rho,\beta}f = \sup_{b \in \beta} \sup_{x,y \in b} \frac{|f(x) - f(y)|}{d_\rho(x,y)},$$

where we recall that $d_\rho(x, y) = \rho^n$, where n is the first time that the two sequences x and y differ. Denote then $\|f\|_{\rho,\beta} = \|f\|_\infty + D_{\rho,\beta}f$ and define

$$\mathcal{B}_{\rho,\beta} = \{f, \|f\|_{\rho,\beta} < +\infty\}.$$

Then, $(\mathcal{B}_{\rho,\beta}, \|\cdot\|_{\rho,\beta})$ is a Banach space.

Having finitely many images is a sufficient condition to have a spectral gap on $(\mathcal{B}_{\rho,\beta}, \|\cdot\|_{\rho,\beta})$. Indeed, Mauldin and Urbański introduced in [MU01] the BIP property, which is automatically satisfied if the shift has finitely many images. Moreover, they proved that the BIP property is a sufficient condition for locally Hölder functions to have a Gibbs measure, whereas Sarig proved in [Sar99] that having a Gibbs measure is a sufficient condition to be positive recurrent and to have a spectral gap. In particular, we have the following two results.

PROPOSITION 3.1. *Let $(\overline{\Sigma}_A, T)$ be a topologically mixing countable Markov shift having finitely many images. Let φ be a locally Hölder continuous function with finite pressure $P(\varphi)$. Then, φ is positive recurrent.*

Proof. As φ is locally Hölder, the sum

$$\sum_{n \geq 1} V_n(\varphi) = \sum_{n \geq 1} \sup\{|\varphi(x) - \varphi(y)|, x_1 = y_1, \ldots, x_n = y_n\} \leq C \sum_{n \geq 1} \rho^n$$

is finite. Thus, [Sar03, Theorem 1] shows that φ has a Gibbs measure. Consequently, [Sar99, Theorem 8] shows that φ is positive recurrent. □

THEOREM 3.2 Sarig [Sar03, Corollary 3] and [Sar99, Theorem 4]. *Let $(\overline{\Sigma}_A, T)$ be a topologically mixing countable Markov shift having finitely many images. Let φ be a locally Hölder continuous function with finite pressure $P(\varphi)$. Then there exist a σ-finite measure ν and a function h bounded away from zero and infinity such that $\mathcal{L}_\varphi^* \nu = e^{P(\varphi)}\nu$ and $\mathcal{L}_\varphi h = e^{P(\varphi)}h$. There also exist $C \geq 0$ and $0 < \theta < 1$ such that for every $f \in \mathcal{B}_{\rho,\beta}$,*

$$\left\| e^{-nP(\varphi)}\mathcal{L}_\varphi^n f - h \int f \, d\nu \right\|_{\rho,\beta} \leq C\theta^n \|f\|_{\rho,\beta}.$$

Moreover, ν is supported on $\partial\Sigma_A^*$ and both measures ν and m defined by $dm = h \, d\nu$ are ergodic.

The fact that ν is ergodic is not stated in Corollary 3 but in Corollary 2 of [Sar03]. Ergodicity of m follows (see the remarks after [Sar99, Theorem 4]). Finally, the fact that h is bounded away from zero and infinity is deduced from the fact that the shift has finitely many images, see [Sar99, Proposition 2].

Actually, we never really use the $\|\cdot\|_{\rho,\beta}$ norm and all our bounds in the following are on the ρ-Hölder norm, that is, we both bound $\|\cdot\|_\infty$ and D_ρ, which is defined by

$$D_\rho f = \sup_{x,y \in \overline{\Sigma}_A} \frac{|f(x) - f(y)|}{d_\rho(x,y)}.$$

Obviously, a bound on D_ρ is stronger than a bound $D_{\rho,\beta}$. Moreover, when bounding $D_\rho f$, we have to bound $|f(x) - f(y)|/d_\rho(x, y)$. We always assume that x and y start with the same element, otherwise $d(x, y) = \rho$ and one can, thus, bound $|f(x) - f(y)|/d_\rho(x, y)$ by $2\rho^{-1}\|f\|_\infty$.

One issue we have to deal with, when applying these results to random walks on relatively hyperbolic groups in the next subsection, is that our Markov shift will not be topologically mixing. It will not even be irreducible (but will have only finitely many recurrent classes). This issue was already addressed in [Gou14] for a Markov shift with finitely many symbols, as we now explain.

In the following, we consider a countable Markov shift $(\overline{\Sigma}_A, T)$ with set of symbols Σ and transition matrix A and we assume that it has finitely many images. If the Markov shift is irreducible, but not topologically mixing, then there is a minimal period $p > 1$ such that for any symbol $s \in \Sigma$, if $T^{-n}[s] \cap [s] \neq \emptyset$, then $n = pk$ for some $k \geq 0$. Then, one can decompose the set of symbols as a finite union $\Sigma = \Sigma_A^{(1)} \sqcup \Sigma_A^{(2)} \sqcup \cdots \sqcup \Sigma_A^{(p)}$, such that for $i \in \mathbb{Z}/p\mathbb{Z}$, if $a_{s,s'} = 1$ and $s \in \Sigma^{(i)}$, then $s' \in \Sigma^{(i+1)}$. We call such a decomposition a cyclic decomposition. We denote by $\overline{\Sigma}_A^{(i)}$ the subset of $\overline{\Sigma}_A$ of sequences that begin with an element of $\Sigma_A^{(i)}$, so that the shift map T maps $\overline{\Sigma}_A^{(i)}$ to $\overline{\Sigma}_A^{(i+1)}$. Moreover, in this case, T^p acts on $\overline{\Sigma}_A^{(i)}$ and the induced Markov shift is topologically mixing. Using this decomposition together with Theorem 3.2, we get that if φ is locally Hölder continuous function with finite pressure $P(\varphi)$ and if the Markov shift has finitely many images, then there are positive functions $h^{(i)}$ on $\overline{\Sigma}_A^{(i)}$ and probability measures $\nu^{(i)}$ on $\overline{\Sigma}_A^{(i)}$ with $\int h^{(i)} d\nu^{(i)} = 1$ such that for $f \in \mathcal{B}_{\rho,\beta}$,

$$\left\| e^{-nP(\varphi)} \mathcal{L}_\varphi^n f - \sum_{i=1}^{p} h^{(i)} \int f \, d\nu^{((i-n) \bmod p)} \right\|_{\rho,\beta} \leq C\theta^n \|f\|_{\rho,\beta}.$$

Assume that the Markov shift is not irreducible. Then, because it has finitely many images, one can first decompose Σ as $\Sigma = \Sigma_{A,0} \sqcup \Sigma_{A,1} \sqcup \cdots \Sigma_{A,q}$, such that if a path starts at $s \in \Sigma_{A,0}$, then it never reaches s again and for $s, s' \in \Sigma$, one can reach s' starting at s and conversely if and only if s and s' are in the same subset $\Sigma_{A,j}$, $j \geq 1$. More formally, the decomposition of Σ satisfies the following properties.

(i) If $s \in \Sigma_{A,0}$, then for all $n \geq 1$, $T^{-n}[s] \cap [s] = \emptyset$.
(ii) If there exist n, n' such that $T^{-n}[s] \cap [s'] \neq \emptyset$ and $T^{-n'}[s'] \cap [s] \neq \emptyset$, then there exists $j \geq 1$ such that $s, s' \in \Sigma_{A,j}$.
(iii) Conversely, if s, s' lie in the same $\Sigma_{A,j}$, $j \geq 1$, then there exist n, n' such that $T^{-n}[s] \cap [s'] \neq \emptyset$ and $T^{-n'}[s'] \cap [s] \neq \emptyset$.

We call $\Sigma_{A,0}$ the transient component of Σ and the sets $\Sigma_{A,j}$, $j \geq 1$, the biconnected components of Σ. All the non-trivial dynamical behavior of the Markov shift happens in the biconnected components. We denote by $\overline{\Sigma}_{A,j}$ the subset of $\overline{\Sigma}_A$ of sequences x that stay in $\Sigma_{A,j}$, that is, for every n, $x_n \in \Sigma_{A,j}$. We similarly call the sets $\overline{\Sigma}_{A,j}$, $j \geq 1$ the biconnected components of $\overline{\Sigma}_A$.

Then, $\overline{\Sigma}_{A,j}$ is stable under the shift map T and we can apply the above discussion to $\overline{\Sigma}_{A,j}$. If φ is a locally Hölder continuous function on $\overline{\Sigma}_A$, denote by φ_j its restriction to the component $\overline{\Sigma}_{A,j}$, with associated transfer operator $\mathcal{L}_{\varphi_j}$. Denote the pressure of φ_j by $P_j(\varphi)$. Then, $\mathcal{L}_{\varphi_j}$ has a spectral gap and $e^{P_j(\varphi)}$ is its dominant eigenvalue. Let $P(\varphi)$ be the maximum of all the $P_j(\varphi)$ and call a component $\overline{\Sigma}_{A,j}$ maximal if $P_j(\varphi) = P(\varphi)$.

DEFINITION 3.2. We say that φ is semisimple if one cannot reach a maximal component from another. That is, for every two maximal components $\overline{\Sigma}_{A,j}$, $\overline{\Sigma}_{A,j'}$, $j \neq j'$, for any two symbols $s \in \Sigma_{A,j}$, $s' \in \Sigma_{A,j'}$, for any $n \geq 1$, $T^{-n}[s] \cap [s'] = \emptyset$.

Elaborating on ideas of Calegari and Fujiwara from [CF10], Gouëzel proved in [Gou14] a spectral gap theorem for the transfer operator of a semisimple Hölder continuous function, when

the set of symbols is finite. His proof works for a countable set of symbols, if the Markov shift has finitely many images and the Hölder continuous function is positive recurrent, because it is based on a spectral decomposition over the sets $\overline{\Sigma}_{A,j}$, on which one applies the Ruelle–Perron–Frobenius theorem (that we replace here with Theorem 3.2). Thus, combining Theorem 3.2 and the proof of [Gou14, Theorem 3.8], we obtain the following.

THEOREM 3.3. *Let $(\overline{\Sigma}_A, T)$ be a countable Markov shift with finitely many images. Let φ be a locally Hölder continuous function with finite maximal pressure $P(\varphi)$. Assume that φ is semisimple. Denote by $\overline{\Sigma}_{A,1}, \ldots \overline{\Sigma}_{A,k}$ the maximal components, with corresponding period $p_1, \ldots, p_k$ and consider a cyclic decomposition*

$$\Sigma_{A,j} = \Sigma_{A,j}^{(1)} \sqcup \cdots \Sigma_{A,j}^{(p_j)}.$$

Then, there exist functions $h_j^{(i)}$ and probability measures $\nu_j^{(i)}$ with $\int h_j^{(i)} d\nu_j^{(i)} = 1$ and such that $\mathcal{L}_\varphi^* \nu_j^{(i)} = \nu_j^{(i-1) \bmod p_j}$ and $\mathcal{L}_\varphi h_j^{(i)} = h_j^{(i-1) \bmod p_j}$. Moreover, for $f \in \mathcal{B}_{\rho,\beta}$,

$$\left\| e^{-nP(\varphi)} \mathcal{L}_\varphi^n f - \sum_{j=1}^k \sum_{i=1}^{p_j} h_j^{(i)} \int f \, d\nu_j^{((i-n) \bmod p_j)} \right\|_{\rho,\beta} \leq C\theta^n \|f\|_{\rho,\beta},$$

for some $C \geq 0$ and $0 < \theta < 1$. Finally, the functions $h_j^{(i)}$ are bounded away from zero and infinity on the support of $\nu_j^{(i)}$.

We also obtain the following result, again proved in [Gou14] for finite sets of symbols, which applies in our situation (see [Gou14, Lemma 3.7]).

LEMMA 3.4. *Let $(\overline{\Sigma}_A, T)$ be a countable Markov shift with finitely many images. Let φ be a locally Hölder continuous function with finite maximal pressure $P(\varphi)$. Let $s \in \Sigma$ and assume that there is a path starting with s that visits k maximal components. Then, for any non-negative function f with $f \geq 1$ on the set of paths starting with s, one has*

$$\mathcal{L}_\varphi^n f(\emptyset) \geq C n^{k-1} e^{nP(\varphi)},$$

where we recall that $\emptyset$ is the empty sequence in Σ_A^*. In particular, for $k = 2$, if φ is not semisimple, then

$$\mathcal{L}_\varphi^n 1(\emptyset) \geq C n e^{nP(\varphi)}.$$

3.2 Perturbation of the pressure

In [Gou14], Gouëzel proves a perturbation theorem for finite sets of symbols (see precisely [Gou14, Proposition 3.10]). Its proof remains valid for countable shifts with finitely many images. Denote by $\|\cdot\|_{\rho,\beta}$ the operator norm for operators acting on $(\mathcal{B}_{\rho,\beta}, \|\cdot\|_{\rho,\beta})$. To apply Gouëzel's perturbation theorem, one needs to control $\|\mathcal{L}_\varphi - \mathcal{L}_\psi\|_{\rho,\beta}$. However, estimating this norm will be very difficult in this paper, so we need finer results, that are based on the following theorem, proved by Keller and Liverani [KL99].

Consider a Banach $(V, \|\cdot\|)$ endowed with a norm $|\cdot|_w$, satisfying $|\cdot|_w \leq C\|\cdot\|$ for some uniform C. Letting $\mathcal{L} : V \to V$ be a linear operator, let

$$\|\mathcal{L}\| = \sup\{\|\mathcal{L}v\|, \|v\| \leq 1\}$$

denote the operator norm of $\mathcal{L}$ associated with $\|\cdot\|$ and let

$$\|\mathcal{L}\|_{s \to w} = \sup\{|\mathcal{L}v|_w, \|v\| \leq 1\}$$

be the operator norm of $\mathcal{L} : (V, \|\cdot\|) \to (V, |\cdot|_w)$.

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THEOREM 3.5. *Consider a family of bounded operators $\mathcal{L}_r : (V, \|\cdot\|) \to (V, \|\cdot\|)$, with r varying in $(0, R]$. Assume there exist $0 < \sigma < M$ and $C \geq 0$ and there exists a function $\tau(r)$ converging to zero as r tends to R such that the following hold.*

(i) *For every n, for every $v \in V$, $|\mathcal{L}_R^n v|_w \leq C M^n |v|_w$.*
(ii) *For every $r \leq R$, for every n, for every $v \in V$, $\|\mathcal{L}_r^n v\| \leq C \sigma^n \|v\| + C M^n |v|_w$.*
(iii) *For every $r \leq R$, $\||\mathcal{L}_r - \mathcal{L}_R\||_{s \to w} \leq \tau(r)$.*

For fixed $\rho > 0$ and $\rho' > 0$ let

$$A_{\rho,\rho'} = \{z \in \mathbb{C}, |z| \geq \sigma + \rho, d(z, \mathrm{spec}(\mathcal{L}_R)) \geq \rho'\}.$$

Then, for any $\rho, \rho' > 0$ there exist $\beta_0 < 1$ and $K_0 \geq 0$ and there exists r_0 such that for every $\beta \leq \beta_0$, for every $r \in [r_0, R]$ for every z $z \in A_{\rho,\rho'}$:

(a) *the operator $zI - \mathcal{L}_r : (V, \|\cdot\|) \to (V, \|\cdot\|)$ is invertible;*
(b) *the operator norm $\||(zI - \mathcal{L}_r)^{-1}\||$ is bounded independently of r;*
(c) *the norm $\||\cdot\||_{s \to w}$ satisfies $\||(zI - \mathcal{L}_r)^{-1} - (zI - \mathcal{L}_R)^{-1}\||_{s \to w} \leq K_0 \tau(r)^{\beta}$.*

Moreover, β_0 only depends on ρ and can be explicitly computed whenever $\sigma + \rho \leq M$. Indeed, one can then choose

$$\beta_0 = \frac{\log((\sigma + \rho)/\sigma)}{\log(M/\sigma)}.$$

In particular, β_0 converges to zero as ρ tends to zero and converges to one as ρ tends to $M - \sigma$.

For a proof, we refer to [Bal18, A.3]. Note that it is asked there that for every r, $|\mathcal{L}_r^n v|_w \leq C M^n |v|_w$, whereas our condition (i) only requires that this holds for $r = R$. However, the proof in [Bal18, A.3] only uses this inequality for $r = R$.

Let us apply this to transfer operators. Consider a countable shift with finitely many images (X_A, T) and a family of locally Hölder functions f_r, for $r \in (0, R]$. Let $\mathcal{L}_r = \mathcal{L}_{f_r}$ be the associated transfer operator. Assume that for every r, the maximal pressure P_r of f_r is finite and that f_r is semisimple. Let $h_j^{(i)}$ and $\nu_j^{(i)}$ be the functions and measures given by Theorem 3.3, associated with $\mathcal{L}_R$. Define the measure m_j as $dm_j = (1/p_j) \sum_{i=1}^{p_j} h_j^{(i)} \, d\nu_j^{(i)}$.

Let $m = \sum m_j$. Consider the Banach space $(V = H_{\rho,\beta}, \|\cdot\| = \|\cdot\|_{\rho,\beta})$ on $V = H_{\rho,\beta}$, endowed with the norm $|\cdot|_w = \|\cdot\|_{L^1(m)}$. As m is finite, we have $|\cdot|_w \leq C\|\cdot\|$. We deduce from Theorem 3.5 the following.

THEOREM 3.6. *With the same notation as previously, assume there exists σ such that $0 < \sigma < e^{P_R}$ and that there exist $C \geq 0$ and a function $\tau(r)$ converging to zero as r tends to R such that the following hold.*

(α) *For every $r \leq R$, for every n, for every $v \in V$,*

$$\|\mathcal{L}_r^n v\| \leq C \sigma^n \|v\| + C e^{n P_R} |v|_w.$$

(β) *For every $r \leq R$, $\||\mathcal{L}_r - \mathcal{L}_R\||_{s \to w} \leq \tau(r)$.*

Then, for every r which is close enough to R, there exist numbers $\tilde{P}_j(r)$ and eigenfunctions $\tilde{h}_{j,r}^{(i)}$ and eigenmeasures $\tilde{\nu}_{j,r}^{(i)}$ of $\mathcal{L}_r$ associated with the eigenvalue $e^{\tilde{P}_j(r)}$ such that

$$\left\| e^{-nP_R} \mathcal{L}_r^n g - \sum_{j=1}^{k} e^{n(\tilde{P}_j(r) - P_R)} \sum_{i=1}^{p_j} \tilde{h}_{j,r}^{(i)} \int g \, d\tilde{\nu}_{j,r}^{((i-n) \bmod p_j)} \right\|_{\rho,\beta} \leq C \theta^n \|g\|_{\rho,\beta}.$$

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The functions $\tilde{h}_j^{(i)}$ and the measures $\tilde{\nu}_j^{(i)}$ have the same support as $h_j^{(i)}$ and $\nu_j^{(i)}$, respectively. Moreover, $\|\tilde{h}_{j,r}^{(i)}\|$ is uniformly bounded. Finally, $\left|\tilde{h}_{j,r}^{(i)} - h_j^{(i)}\right|_w$ converges to zero as r tends to R and $\tilde{\nu}_{j,r}^{(i)}$ weakly converges to $\nu_j^{(i)}$ as r tends to R.

Proof. As $\mathcal{L}_R$ has a spectral gap according to Theorem 3.3, there exists $\sigma_0 < e^{P_R}$ such that the spectrum of $\mathcal{L}_R$ outside of the disk of radius σ_0 exactly consists of the eigenvalue e^{P_R}, with eigenfunctions $h_j^{(i)}$ and eigenmeasures $\nu_j^{(i)}$. The result is then a consequence of Theorem 3.5, choosing ρ such that $\sigma_0 < \sigma + \rho < e^{P_R}$. Indeed, condition (i) there is satisfied with $M = e^{P_R}$ because $\nu_j^{(i)}$ is an eigenmeasure of $\mathcal{L}_R$ associated with e^{P_R}. In addition, conditions (ii) and (iii) are direct consequences of assumptions (α) and (β). $\qquad\square$

Note that $\int \tilde{h}_{j,r}^{(i)} \, d\nu_j^{(i)} \neq 0$ for r close enough to R, because $\int h_j^{(i)} d\nu_j^{(i)} = 1$ and $\left|\tilde{h}_{j,r}^{(i)} - h_j^{(i)}\right|_w$ converges to zero. One can, thus, normalize $\tilde{h}_{j,r}$ declaring $\int \tilde{h}_{j,r}^{(i)} \, d\nu_j^{(i)} = 1$. We make this assumption in the following. We still have that $\|\tilde{h}_{j,r}^{(i)}\|$ is uniformly bounded and that $\left|\tilde{h}_{j,r}^{(i)} - h_j^{(i)}\right|_w$ converges to zero. The following result allows us to obtain a precise asymptotic of $\tilde{P}_{j,r} - P_R$ in the following sections.

PROPOSITION 3.7. *Under the assumptions of Theorem 3.6,*

$$e^{\tilde{P}_j(r)-P_R} - 1 = \int (e^{f_r-f_R} - 1) \, dm_j + \int (e^{f_r-f_R} - 1)\frac{1}{p_j}\left(\sum_{i=0}^{p_j-1} h_j^{(i)} - \tilde{h}_{j,r}^{(i)}\right) d\nu_j^{(i)},$$

where $dm_j = (1/p_j) \sum_{i=1}^{p_j} h_j^{(i)} \, d\nu_j^{(i)}$.

Proof. As $\tilde{h}_{j,r}^{(i)}$ are eigenfunctions of $\mathcal{L}_r$ and $\tilde{h}_{j,r}$ is normalized, we have

$$e^{\tilde{P}_j(r)} = \int \mathcal{L}_r \tilde{h}_{j,r}^{(i)} \, d\nu_j^{(i)}.$$

Consequently,

$$e^{\tilde{P}_j(r)} - e^{P_R} = \int (\mathcal{L}_r \tilde{h}_{j,r}^{(i)} - \mathcal{L}_R h_j^{(i)}) \, d\nu_j^{(i)}.$$

Note that for any function g, $\mathcal{L}_r g = \mathcal{L}_R(e^{f_r-f_R}g)$. In particular,

$$e^{\tilde{P}_j(r)} - e^{P_R} = \int \mathcal{L}_R(e^{f_r-f_R}\tilde{h}_{j,r}^{(i)} - h_j^{(i)}) \, d\nu_j^{(i)}.$$

Using that $d\nu_j^{(i)}$ is an eigenmeasure of $\mathcal{L}_R$ associated with the eigenvalue e^{P_R}, we obtain

$$e^{\tilde{P}_j(r)-P_R} - 1 = \int (e^{f_r-f_R}\tilde{h}_{j,r}^{(i)} - h_j^{(i)}) \, d\nu_j^{(i)}$$

$$= \int (e^{f_r-f_R} - 1)(\tilde{h}_{j,r}^{(i)} - h_j^{(i)}) \, d\nu_j^{(i)} + \int \tilde{h}_{j,r}^{(i)} \, d\nu_j^{(i)} - \int e^{f_r-f_R}h_j^{(i)} \, d\nu_j^{(i)}.$$

As $\int h_j^{(i)} d\nu_j^{(i)} = \int \tilde{h}_{j,r}^{(i)} d\nu_j^{(i)} = 1$, we thus obtain

$$e^{\tilde{P}_j(r)-P_R} - 1 = \int (e^{f_r-f_R} - 1)(\tilde{h}_{j,r}^{(i)} - h_j^{(i)}) \, d\nu_j^{(i)} + \int (e^{f_r-f_R} - 1)h_j^{(i)} \, d\nu_j^{(i)}.$$

This holds for every i, which concludes the proof summing over i and then dividing by p_j. $\quad\square$

4. Asymptotic of the first derivative of the Green function

In this section, we assume that Γ is hyperbolic relative to Ω and choose a system of representatives of conjugacy classes $\Omega_0 = \{\mathcal{H}_1, \ldots, \mathcal{H}_N\}$ of elements of Ω. We consider a probability measure μ on Γ that satisfies weak and strong relative Ancona inequalities up to the spectral radius and we denote by R_μ the inverse of this spectral radius. Note that we do not need to assume that the measure μ is symmetric but only that relative Ancona inequalities are satisfied.

We assume that μ is not spectrally degenerate. According to Proposition 2.11, we have $(d/dr)_{|r=R_\mu} G(e, e|r) = +\infty$, or equivalently $I^{(1)}(R_\mu) = +\infty$ by Lemma 2.7. In addition, Proposition 2.8 shows that

$$I^{(2)}(r) \asymp (I^{(1)}(r))^3.$$

Our goal in the following sections is to obtain a more precise statement, transforming $\asymp$ into $\sim$, when $r \to R_\mu$. Precisely, we prove the following.

THEOREM 4.1. *Under these assumptions, there exists $\xi > 0$ such that*

$$I^{(2)}(r) \underset{r \to R_\mu}{\sim} \xi (I^{(1)}(r))^3.$$

To do so, we use thermodynamic formalism, adapting [Gou14, GL13].

4.1 Transfer operator for the Green function

We choose a generating set S of Γ as in Theorem 2.3, so that Γ is automatic relative to Ω_0 and S, where Ω_0 is a finite set of representatives of conjugacy classes of the parabolic subgroups Let $\mathcal{G} = (V, E, v_*)$ be a graph and $\phi : E \to S \cup \bigcup_{\mathcal{H} \in \Omega_0} \mathcal{H}$ be a labelling map as in the definition of a relative automatic structure.

The set of vertices V is finite. Moreover, if $\sigma \in \Sigma_0 = S \cup \bigcup_{\mathcal{H} \in \Omega_0} \mathcal{H}$ and if $v \in V$, there is at most one edge that leaves v and that is labelled with σ. Thus, the set of edges E is countable. Set $\Sigma = E$ and consider the transition matrix $A = (a_{s,s'})_{s,s' \in \Sigma}$, defined by $a_{s,s'} = 1$ if the edges s and s' are adjacent in $\mathcal{G}$ and $a_{s,s'} = 0$ otherwise. We then define Σ_A^*, $\partial \Sigma_A$ and $\overline{\Sigma}_A$ as previously. According to the definition of a relative automatic structure, elements of $\overline{\Sigma}_A$ represent relative geodesics and relative geodesic rays.

We decompose $\Sigma_0 = S \cup \bigcup_{\mathcal{H} \in \Omega_0} \mathcal{H}$ as follows. The sets $\mathcal{H}_j \cap \mathcal{H}_k$ are finite if $j \neq k$ (see, e.g., [DS05, Lemma 4.7]). We can, thus, consider $\mathcal{H}'_k = \mathcal{H}_k \setminus \cup_{j \neq k} \mathcal{H}_j$ and $\mathcal{H}'_0 = \Sigma_0 \setminus \cup_k \mathcal{H}'_k$. Then, $\mathcal{H}'_0$ remains finite and the sets $\mathcal{H}'_k$ are disjoint. By analogy with free factors in a free product, we introduce the following terminology.

DEFINITION 4.1. We call the sets $\mathcal{H}'_k$ the factors of the relatively automatic structure

Paths of length n in $\mathcal{G}$ beginning at v_* are in bijection with the relative sphere $\hat{S}_n$. Moreover, infinite paths in $\mathcal{G}$ starting at v_* give relative geodesic rays starting at e. Denote by $E_* \subset E$ the set of edges that starts at v_*. The labelling map ϕ can be extended to infinite paths. When restricted to infinite words starting in E_*, it gives a surjective map from paths beginning at v_* to the Gromov boundary of the graph $\hat{\Gamma}$, which is by definition the set of conical limit points of Γ, included in the Bowditch boundary. Restricting the distance $d_\rho(x, y) = \rho^{-n}$ to E_*, this induced map is continuous, endowing the Bowditch boundary with the usual topology. A formal way of restricting our attention to elements of the group and to conical limit points is to consider the function 1_{E_*} on $\overline{\Sigma}_A$ which takes value one on sequences in $\overline{\Sigma}_A$ beginning with an edge in E_* and that takes value zero elsewhere. This function 1_{E_*} is locally Hölder continuous.

We have the following, which proves that every locally Hölder continuous function with finite pressure is positive recurrent, according to Proposition 3.1.

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LEMMA 4.2. *The Markov shift $(\overline{\Sigma}_A, T)$ has finitely many images.*

Proof. If an edge s in Σ ends at some vertex v in $\mathcal{G}$, then the only edges s' such that $a_{s,s'} = 1$ are those that start at v. Thus, if two edges end at the same vertex v, they have the same row in the matrix A. The lemma follows, because there is only a finite number of vertices. $\square$

Recall that R_μ is the inverse of the spectral radius of the μ-random walk on Γ. Also recall that for $r \in [0, R_\mu]$, we write $H(e, \gamma|r) = G(e, \gamma|r)G(\gamma, e|r)$. For $r \in [1, R_\mu]$, we define the function φ_r on Σ_A^* by $\varphi_r(\emptyset) = 1$ and

$$\varphi_r(x = x_1, \ldots, x_n) = \log\left(\frac{H(e, \phi(x)|r)}{H(e, \phi(Tx)|r)}\right) = \log\left(\frac{H(e, \phi(x_1 \ldots x_n)|r)}{H(e, \phi(x_2 \ldots x_n)|r)}\right).$$

Using equivariance of the Green function, we also have

$$\varphi_r(x = x_1, \ldots, x_n) = \log\left(\frac{H(e, \phi(x_1 \ldots x_n)|r)}{H(\phi(x_1), \phi(x_1 \ldots x_n)|r)}\right).$$

LEMMA 4.3. *For every $r \in [1, R_\mu]$, the function φ_r can be extended to $\overline{\Sigma}_A$. It is then locally Hölder continuous on $\overline{\Sigma}_A$.*

Proof. Let $n \geq 1$ and let $x, y \in \Sigma_A^*$ be such that $x_1 = y_1, \ldots, x_n = y_n$. Hence, in Γ, we have $\phi(x_1) = \phi(y_1), \ldots, \phi(x_n) = \phi(y_n)$. This means that relative geodesics from e to $\phi(x)$ and from $\phi(x_1) = \phi(y_1)$ to $\phi(y)$ fellow-travel for a time at least $n - 1$. According to strong relative Ancona inequalities, we thus have, for some $C \geq 0$ and $0 < \rho < 1$,

$$\left| \frac{G(e, \phi(x)|r)G(\phi(y_1), \phi(y)|r)}{G(\phi(x_1), \phi(x)|r)G(e, \phi(x)|r)} - 1 \right| \leq C\rho^n.$$

Weak relative Ancona inequalities also show that

$$G(e, \phi(x)|r)G(\phi(y_1), \phi(y)|r) \geq \frac{1}{C}G(e, \phi(x_1)|r)G(\phi(x_1), \phi(x)|r)G(\phi(y_1), \phi(y)|r)$$

and because $x_1 = y_1$, we obtain

$$G(e, \phi(x)|r)G(\phi(y_1), \phi(y)|r) \geq \frac{1}{C^2}G(e, \phi(y)|r)G(\phi(x_1), \phi(x)|r).$$

Thus, $G(e, \phi(x)|r)G(\phi(y_1), \phi(y)|r)/G(\phi(x_1), \phi(x)|r)G(e, \phi(x)|r)$ is bounded away from zero, so that

$$\left| \log\left(\frac{H(e, \phi(x)|r)}{H(\phi(x_1), \phi(x)|r)}\right) - \log\left(\frac{H(e, \phi(y)|r)}{H(\phi(y_1), \phi(y)|r)}\right) \right|$$

$$\leq C_1 \left| \frac{G(e, \phi(x)|r)G(\phi(y_1), \phi(y)|r)}{G(\phi(x_1), \phi(x)|r)G(e, \phi(y)|r)} - 1 \right| \leq C\rho^n. \tag{8}$$

This proves that if $x = (x_1, \ldots, x_n, \ldots) \in \partial\Sigma_A^*$, then the sequence $\varphi_r(x_1, \ldots, x_k)$ is Cauchy, so that it converges to some well-defined limit $\varphi_r(x)$. This extended function φ_r on $\overline{\Sigma}_A$ still satisfies (8), so that it is locally Hölder continuous. $\square$

We denote by $\mathcal{L}_r$ the transfer operator associated with the function φ_r.

LEMMA 4.4. *For every $r \in [1, R_\mu]$, the function φ_r has finite pressure and is positive recurrent.*

Proof. As noted previously, because the Markov shift has finitely many images, Proposition 3.1 shows that any locally Hölder function with finite pressure is positive recurrent. Thus, we only need to prove that φ has finite pressure, which is equivalent to proving that $\|\mathcal{L}_r 1\|_\infty < +\infty$

by [Sar99, Theorem 1]. For $x \in \overline{\Sigma}_A$, let X^1_x be the set of symbols that can precede x in the automaton $\mathcal{G}$. Then, by weak relative Ancona inequalities,

$$\mathcal{L}_r 1(x) = \sum_{\sigma \in X^1_x} \frac{H(e, \sigma x | r)}{H(\sigma, \sigma x | r)} \lesssim \sum_{k=0}^{N} \sum_{\sigma \in \mathcal{H}'_k} H(e, \sigma | r).$$

This last sum is bounded by Corollary 2.10, which concludes the proof. $\qquad\square$

Let $P_j(r)$ be the pressure of the restriction of φ_r to a component $\overline{\Sigma}_{A,j}$ of the Markov shift and let $P(r)$ be the maximal pressure, that is, the maximum of the $P_j(r)$. Recall that we declared that the empty sequence is not a preimage of the empty sequence. This will simplify the following.

The main reason for introducing this function φ_r is that

$$\mathcal{L}_r^n 1_{E_*}(\emptyset) = \frac{1}{H(e, e | r)} \sum_{\gamma \in \hat{S}_n} H(e, \gamma | r),$$

where we recall that 1_{E_*} is the function on $\overline{\Sigma}_A$ that takes value one on paths that start at v_* in the automaton $\mathcal{G}$ and zero elsewhere. Indeed, to prove Theorem 4.1, we want to understand $I^{(1)}(r) = \sum_{\gamma \in \Gamma} H(e, \gamma | r)$. Thus, we want to understand the behavior of $\sum_{\gamma \in \hat{S}_n} H(e, \gamma | r)$, which is thus the same as understanding the behavior of $\mathcal{L}_r^n 1_{E_*}(\emptyset)$.

4.2 Continuity properties of the transfer operator

Our goal in this subsection is to prove that the map $r \mapsto \mathcal{L}_r$ is continuous in a weak sense. We begin by the following result.

LEMMA 4.5. *There exists $C > 0$ such that for all $r \in [1, R_\mu)$,*

$$\frac{1}{C} \frac{1}{\sqrt{R_\mu - r}} \le \sum_{\gamma \in \Gamma} H(e, \gamma | r) \le C \frac{1}{\sqrt{R_\mu - r}}.$$

Proof. Let $I_1(r) = \sum_\gamma H(e, \gamma | r)$ and $F(r) = r^2 I_1(r)$. According to [Dus22, Lemma 3.2],

$$F'(r) = 2r \sum_{\gamma, \gamma'} G(e, \gamma' | r) G(\gamma', \gamma | r) G(\gamma, e | r).$$

Proposition 2.8 gives

$$\frac{1}{C'} \le \frac{F'(r)}{F(r)^3} \le C'$$

and Proposition 2.11 gives $F(R_\mu) = +\infty$. Thus, integrating the inequality above between r and R_μ, we obtain

$$\frac{1}{C'}(R_\mu - r) \le \frac{1}{F(r)^2} \le C'(R_\mu - r),$$

which leads to the desired inequality. $\qquad\square$

We also prove the following.

LEMMA 4.6. *For any $r \in [1, R_\mu]$, $P(r) \le 0$. Moreover, $P(R_\mu) = 0$ and φ_{R_μ} is semisimple.*

Proof. If $P(r)$ were positive, then Theorem 3.3 would show that

$$\mathcal{L}_r 1_{E_*}(\emptyset) = \frac{1}{H(e, e | r)} \sum_{\gamma \in \hat{S}_n} H(e, \gamma | r)$$

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tends to infinity. However, Lemma 2.9 shows that this quantity is bounded, so we obtain a contradiction.

If $P(R_\mu)$ were negative, then $\sum_{\gamma \in \hat{S}_n} H(e, \gamma | r)$ would converge to zero exponentially fast, according to Theorem 3.3. In particular, $\sum_{\gamma \in \Gamma} H(e, \gamma | r)$ would be finite, that is, using Lemma 2.7, $(d/dr)_{|r=R_\mu} G(e, e | r)$ would be finite. This would be a contradiction with Proposition 2.11.

Finally, if φ_{R_μ} were not semisimple, then Lemma 3.4 would again show that $\sum_{\gamma \in \hat{S}_n} H(e, \gamma | r)$ would tend to infinity, because we already know that $P(R_\mu) = 0$. Again, Lemma 2.9 shows that this quantity is bounded. $\qquad \square$

Let $h_j^{(i)}$ and $\nu_j^{(i)}$ be the functions and measures given by Theorem 3.3, associated with $\mathcal{L}_{R_\mu}$. Let m_j be the measure defined as $dm_j = (1/p_j) \sum_{i=1}^{p_j} h_j^{(i)} d\nu_j^{(i)}$. According to [Sar99, Proposition 4], m_j is a Gibbs measure. However, we have to apply this proposition to each component $\overline{\Sigma}_{A,j}$ of the shift, so that we do not have that $m_j([x_1, \ldots, x_n]) \asymp H(e, x_1 \ldots x_n | R_\mu)$ for any cylinder $[x_1, \ldots, x_n]$. We still deduce that there exists $C \geq 0$ such that for any n, for any cylinder $[x_1, \ldots, x_n]$,

$$m_j([x_1 \ldots x_n]) \leq CH(e, x_1 \ldots x_n | R_\mu). \tag{9}$$

Furthermore, letting $m = \sum_j m_j$, there exists $C \geq 0$ such that for any n, for any cylinder $[x_1 \ldots x_n]$ in the support of one of the measures m_j,

$$\frac{1}{C} H(e, x_1 \ldots x_n | R_\mu) \leq m([x_1, \ldots, x_n]) \leq CH(e, x_1 \ldots x_n | R_\mu). \tag{10}$$

PROPOSITION 4.7. *There exists a non-negative function φ on $\overline{X}_A$, possibly taking the value $+\infty$ on $\partial \Sigma_A$, which is integrable with respect to the measure m and such that for every $x \in \Sigma_A$ and for every $1 \leq r, r' \leq R_\mu$,*

$$|\varphi_r(x) - \varphi_{r'}(x)| \leq 2\varphi(x)\sqrt{|r - r'|}.$$

Integrating over r, this proposition is a direct consequence of the following lemma.

LEMMA 4.8. *There exists a non-negative function φ on $\overline{X}_A$, possibly taking the value $+\infty$ on $\partial \Sigma_A$, which is integrable with respect to the measure m and such that for every $x \in \Sigma_A$ and for every $1 \leq r < R_\mu$,*

$$\left| \frac{d}{dr} \varphi_r(x) \right| \leq \varphi(x) \frac{1}{\sqrt{R_\mu - r}}.$$

Proof. Fix $x \in \Sigma_A$. We compute the derivative of $r \mapsto \varphi_r(x)$. To simplify the notation, we identify x with $\phi(x) \in \Gamma$. We obtain

$$\frac{d}{dr} \varphi_r(x) = \frac{d}{dr} \log \frac{H(e, x | r)}{H(x_1, x | r)} = \frac{d}{dr} \log r^2 H(e, x | r) - \frac{d}{dr} \log r^2 H(x_1, x | r).$$

In [Dus22, Lemma 3.2] we showed that

$$\frac{d}{dr} \varphi_r(x) = \sum_{y \in \Gamma} \frac{G(e, y | r)G(y, x | r)G(x, e | r) + G(e, x | r)G(x, y | r)G(y, e | r)}{rH(e, x | r)}$$

$$- \sum_{y \in \Gamma} \frac{G(x_1, y | r)G(y, x | r)G(x, x_1 | r) + G(x_1, x | r)G(x, y | r)G(y, x_1 | r)}{rH(x_1, x | r)}. \tag{11}$$

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We first give an upper bound for

$$\left| \frac{\sum_{y\in\Gamma} G(e,y|r)G(y,x|r)G(x,e|r)}{rH(e,x|r)} - \frac{\sum_{y\in\Gamma} G(x_1,y|r)G(y,x|r)G(x,x_1|r)}{rH(x_1,x|r)} \right|.$$

The remaining term in (11) will be bounded in the same way. Putting together these two sums, we obtain

$$\frac{1}{rH(e,x|r)}\sum_{y\in\Gamma} G(e,y|r)G(y,x|r)G(x,e|r) - G(x_1,y|r)G(y,x|r)G(x,x_1|r)\frac{H(e,x|r)}{H(x_1,x|r)}.$$

We rewrite this as

$$\frac{1}{rH(e,x|r)}\sum_{y\in\Gamma} G(e,y|r)G(y,x|r)G(x,e|r)\left(1 - \frac{G(x_1,y|r)G(e,x|r)}{G(x_1,x|r)G(e,y|r)}\right).$$

We decompose the sum over Γ in the following way. Let $n = \hat{d}(e,x)$, so that the relative geodesic $[e,x]$ has length n. Denote by $e, x_1, \ldots, x_n$ successive points on this relative geodesic. In addition, for $0 \le k \le n$, let Γ_k be the set of $y \in \Gamma$ whose projection on $[e,x]$ which is closest to x is exactly at x_k.

We use Lemma 2.5 several times. Let us first focus on the sum over Γ_0. If $y \in \Gamma_0$, then any relative geodesic from y to x passes within a bounded distance of e. Weak relative Ancona inequalities show that

$$G(y,x|r) \lesssim G(y,e|r)G(e,x|r).$$

Similarly, any relative geodesic from x_1 to y passes within a bounded distance of e, hence

$$G(x_1,y|r) \lesssim G(x_1,e|r)G(e,y|r).$$

We also have

$$G(e,x|r) \lesssim G(e,x_1|r)G(x_1,x|r),$$

so that

$$\left| \frac{1}{rH(e,x|r)}\sum_{y\in\Gamma_0} G(e,y|r)G(y,x|r)G(x,e|r)\left(1 - \frac{G(x_1,y|r)G(e,x|r)}{G(x_1,x|r)G(e,y|r)}\right) \right|$$

$$\lesssim \sum_{y\in\Gamma} H(e,y|r)(1 + H(e,x_1|r)).$$

As $H(e,x_1|r)$ is uniformly bounded, we deduce from Lemma 4.5 that

$$\left| \frac{1}{rH(e,x|r)}\sum_{y\in\Gamma_0} G(e,y|r)G(y,x|r)G(x,e|r)\left(1 - \frac{G(x_1,y|r)G(e,x|r)}{G(x_1,x|r)G(e,y|r)}\right) \right|$$

$$\lesssim \frac{1}{\sqrt{R_\mu - r}}.$$

Let us focus on the sum over Γ_1 now. Let $\mathcal{H}_1$ be the union of parabolic subgroups containing x_1. Let $y \in \Gamma_1$ and denote by σ its projection on $\mathcal{H}_1$. Then, any relative geodesic from e to y passes within a bounded distance of σ and any relative geodesic from y to x passes first to a point within a bounded distance of σ, then to a point within bounded distance of x_1. We thus obtain

$$G(e,y|r) \lesssim G(e,\sigma|r)G(\sigma,y|r)$$

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and
$$G(y,x|r) \lesssim G(y,\sigma|r)G(\sigma,x_1|r)G(x_1,x|r).$$

Similarly,
$$\frac{G(x_1,y|r)G(e,x|r)}{G(x_1,x|r)G(e,y|r)} \lesssim \frac{G(x_1,\sigma|r)G(e,x_1|r)}{G(e,\sigma|r)} \lesssim 1.$$

Letting Γ_1^σ be the set of y whose projection on $\mathcal{H}_1$ is at σ, we obtain
$$\left| \frac{1}{rH(e,x|r)} \sum_{y\in\Gamma_1} G(e,y|r)G(y,x|r)G(x,e|r)\left(1 - \frac{G(x_1,y|r)G(e,x|r)}{G(x_1,x|r)G(e,y|r)}\right)\right|$$
$$\lesssim \sum_{\sigma\in\mathcal{H}_1} \sum_{y\in\Gamma_1^\sigma} \frac{G(e,\sigma|r)G(\sigma,x_1|r)}{G(e,x_1|r)} H(\sigma,y|r).$$

We bound the sum over $y \in \Gamma_1^\sigma$ by a sum over $y \in \Gamma$, so that
$$\left| \frac{1}{rH(e,x|r)} \sum_{y\in\Gamma_1} G(e,y|r)G(y,x|r)G(x,e|r)\left(1 - \frac{G(x_1,y|r)G(e,x|r)}{G(x_1,x|r)G(e,y|r)}\right)\right|$$
$$\lesssim \frac{1}{\sqrt{R_\mu - r}} \sum_{\sigma\in\mathcal{H}_1} \frac{G(e,\sigma|r)G(\sigma,x_1|r)}{G(e,x_1|r)}.$$

Suppose now that $k \geq 2$ and consider the sum over Γ_k. For any $y \in \Gamma_k$, relative geodesic from x_1 to y and from e to x travel together for a time at least $k-1$. We deduce from strong relative Ancona inequalities that
$$\left|1 - \frac{G(x_1,y|r)G(e,x|r)}{G(x_1,x|r)G(e,y|r)}\right| \lesssim \rho^k$$

for some $0 < \rho < 1$. Letting $\mathcal{H}_k$ be the union of parabolic subgroups containing $x_{k-1}^{-1}x_k$, we also obtain
$$\left| \frac{1}{rH(e,x|r)} \sum_{y\in\Gamma_k} G(e,y|r)G(y,x|r)G(x,e|r)\left(1 - \frac{G(x_1,y|r)G(e,x|r)}{G(x_1,x|r)G(e,y|r)}\right)\right|$$
$$\lesssim \rho^k \frac{1}{\sqrt{R_\mu - r}} \sum_{\sigma\in\mathcal{H}_k} \frac{G(x_{k-1},x_{k-1}\sigma|r)G(x_{k-1}\sigma,x_k|r)}{G(x_{k-1},x_k|r)}.$$

Putting everything together and letting $x_0 = e$, we obtain
$$\left| \frac{1}{rH(e,x|r)} \sum_{y\in\Gamma} G(e,y|r)G(y,x|r)G(x,e|r)\left(1 - \frac{G(x_1,y|r)G(e,x|r)}{G(x_1,x|r)G(e,y|r)}\right)\right|$$
$$\lesssim \frac{1}{\sqrt{R_\mu - r}}\left(1 + \sum_{k=0}^{n-1} \rho^k \sum_{\sigma\in\mathcal{H}_k} \frac{G(x_k,x_k\sigma|r)G(x_k\sigma,x_{k+1}|r)}{G(x_k,x_{k+1}|r)}\right). \tag{12}$$

We now bound
$$\sum_{k=0}^{n} \rho^k \sum_{\sigma\in\mathcal{H}_k} \frac{G(x_k,x_k\sigma|r)G(x_k\sigma,x_{k+1}|r)}{G(x_k,x_{k+1}|r)}$$

by an integrable function independently of r and n. We first prove the following.

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LEMMA 4.9. *There exists Λ such that the following holds. Let $\mathcal{H}$ be a parabolic subgroup. For every x in $\mathcal{H}$ and for any $1 \leq r \leq R_\mu$, we have*

$$\sum_{y \in \mathcal{H}} G(e, y|r) G(y, x|r) \leq (\Lambda d(e, x) + \Lambda) G(e, x|r).$$

Proof. We write $G_r^{(1)}(e, x) = (d/dt)_{|t=1}(G_r(e, x|t))$, where G_r is the Green function associated with the first return kernel p_r to $\mathcal{H}$. According to Lemma 2.7, it is enough to prove that

$$G_r^{(1)}(e, x) \leq (\Lambda d(e, x) + \Lambda) G(e, x|r).$$

As we are assuming that μ is non-spectrally degenerate along $\mathcal{H}$, there exists $\rho < 1$ such that for any x, for any n,

$$p_r^{(n)}(e, x) \leq p_{R_\mu}^{(n)}(e, x) \leq \rho^n,$$

where $p_r^{(n)}$ denotes the nth power of convolution of p_r. By definition,

$$G_r^{(1)}(e, x) = \sum_{n \geq 0} n p_r^{(n)}(e, x).$$

Note then that

$$\sum_{n \geq \Lambda d(e,x) + \Lambda} n p_r^{(n)}(e, x) \lesssim (\rho')^{\Lambda d(e,x) + \Lambda}.$$

As $r \geq 1$, for any x, $G(e, x|r) \geq p^{d(e,x)}$ for some $p < 1$ and so

$$G_r^{(1)}(e, x) \geq G(e, x|r) \geq p^{d(e,x)}.$$

If Λ is large enough, we thus have

$$\sum_{n \geq \Lambda d(e,x) + \Lambda} n p_r^{(n)}(e, x) \leq \frac{1}{2} G_r^{(1)}(e, x),$$

so that

$$G_r^{(1)}(e, x) \leq 2 \sum_{n \leq \Lambda d(e,x) + \Lambda} n p_r^{(n)}(e, x) \leq (2\Lambda d(e, x) + 2\Lambda) G(e, x|r).$$

This concludes the proof. □

Going back to the proof of Lemma 4.8, we obtain the upper bound

$$\sum_{k=0}^{n} \rho^k \sum_{\sigma \in \mathcal{H}_k} \frac{G(x_k, x_k \sigma|r) G(x_k \sigma, x_{k+1}|r)}{G(x_k, x_{k+1}|r)} \leq \sum_{k=0}^{n} \rho^k (\Lambda d(x_k, x_{k+1}) + \Lambda).$$

We fix N and define $\varphi_1^{(N)}$ by

$$\varphi_1^{(N)}(x) = 1 + \sum_{k=0}^{n-1} \rho^k (\Lambda d(x_k, x_{k+1}) + \Lambda) \tag{13}$$

for any word x of length $n \leq N$. We extend $\varphi_1^{(N)}$ to a function on $\overline{\Sigma}_A$ declaring $\varphi_1^{(N)}$ to be constant on cylinders of length N. For fixed x, the sequence $\varphi_1^{(N)}(x)$ is non-decreasing. Let $\varphi_1(x)$ be its limit, possibly infinite if $x \in \partial \Sigma_A$. We now prove that φ_1 is integrable with respect to m. This is based on the following.

Letting E be a set, a transition kernel p is a function $p : E \times E \to [0, +\infty)$. We fix a base point $x_0 \in E$. We say that p is finite if its total mass is finite, that is,

$$\sum_{x \in E} p(x_0, x) < +\infty.$$

If the total mass is one, then p defines a Markov chain Z_n on E. Otherwise, p still defines a chain Z_n with transition given by p. We let z_n be the increments of this chain. Whenever E is endowed with a distance d, we say that p is C-quasi-invariant if there exists C such that for any k,

$$d(Z_k, Z_{k+1}) \leq C d(e, z_{k+1}).$$

LEMMA 4.10. *Let p be a finite C-quasi-invariant transition kernel on a countable metric space (E, d). Let x_0 be a fixed point in E. Assume that p has exponential moments in the sense that*

$$\sum_{x \in E} p(x_0, x) e^{\alpha d(x_0, x)}$$

for some positive α. Then, for any $\beta > 0$, there exists $\lambda > 0$ and C_λ such that for any $x \in E$,

$$\sum_{n \leq d(e,x)/\lambda} p^{(n)}(x_0, x) \leq C_\lambda e^{-\beta d(e,x)},$$

where $p^{(n)}$ denotes the nth power of convolution of p.

Proof. The proof is contained in the proof of [BHM11, Lemma 3.6], although the statement and the assumptions there are different, so we rewrite it for convenience. To simplify the notation, we assume that the total mass of p is one, so that p defines a Markov chain Z_n. The general proof is the same. By assumption, we have

$$\mathbb{E}(e^{\alpha d(x_0, Z_1)}) = E < +\infty.$$

For any λ, Markov inequality shows that

$$\mathbb{P}\left(\sup_{1 \leq k \leq n} d(e, Z_k) \geq \lambda n\right) \leq e^{-(\alpha/C)\lambda n} \mathbb{E}(e^{(\alpha/C) \sup_{1 \leq k \leq n} d(e, Z_k)}).$$

As p is C-quasi-invariant, we have for any $k \leq n$, letting $Z_0 = x_0$,

$$d(x_0, Z_k) \leq \sum_{0 \leq j \leq n-1} d(Z_j, Z_{j+1}) \leq C \sum_{1 \leq j \leq n} d(x_0, z_j).$$

As the z_j are independent and follow the same law as Z_1, we obtain

$$\mathbb{P}\left(\sup_{1 \leq k \leq n} d(e, Z_k) \geq \lambda n\right) \leq e^{-(\alpha/C)\lambda n} E^n \leq e^{n(-(\alpha/C)\lambda + \log E)}.$$

We choose λ large enough so that $-(\alpha/C)\lambda + \log E \leq -2\beta$. Then,

$$\sum_{n \leq d(e,x)/\lambda} p^{(n)}(x_0, x) \leq \frac{d(e, x)}{\lambda} e^{-2\beta d(e,x)} \lesssim e^{-\beta d(e,x)}.$$

This concludes the proof. $\qquad\square$

We apply this in our situation. Let $\mathcal{H}$ be a parabolic subgroup. For any $\eta > 0$, we let p_{η, R_μ} be the first return kernel associated with $R_\mu \mu$ to the η-neighborhood $\mathcal{N}_\eta(\mathcal{H})$ of $\mathcal{H}$. Then, [DG21, Lemma 4.6] shows that if η is large enough, then p_{η, R_μ} has exponential moments. As it is defined as the first return associated with $R_\mu \mu$, it is C-quasi-invariant for the induced metric

on $\mathcal{N}_\eta(\mathcal{H})$ (it is actually invariant for this distance). Thus, for any $\beta > 0$, there exists λ and C_λ such that

$$\sum_{n \leq d(e,x)/\lambda} p_{\eta,R_\mu}^{(n)}(e,x) \leq C_\lambda e^{-\beta d(e,x)}.$$

The Green function associated with p_{η,R_μ} coincides with the restriction of the Green function associated with $R_\mu \mu$ on $\mathcal{N}_\eta(\mathcal{H})$, see [Dus22, Lemma 3.4] for a proof. As there exists $q < 1$ such that $G(e,x|R_\mu) \geq q^{d(e,x)}$, we can choose λ so that

$$\sum_{n \leq d(e,x)/\lambda} p_{\eta,R_\mu}^{(n)}(e,x) \leq \frac{1}{2} G(e,x|R_\mu)$$

and so

$$G(e,x|R_\mu) \leq 2 \sum_{n \geq d(e,x)/\lambda} p_{\eta,R_\mu}^{(n)}(e,x) \leq \frac{2\lambda}{d(e,x)} \sum_{n \geq d(e,x)} n p_{\eta,R_\mu}^{(n)}(e,x).$$

According to Lemma 2.7,

$$G(e,x|R_\mu) \leq \frac{2\lambda}{d(e,x)} \sum_{y \in \mathcal{N}_\eta(\mathcal{H})} G(e,y|R_\mu) G(y,x|R_\mu).$$

Finally, any point in $\mathcal{N}_\eta(\mathcal{H})$ is within η of a point in $\mathcal{H}$, hence for any $x \in \mathcal{H}$,

$$d(e,x) G(e,x|R_\mu) \lesssim \sum_{y \in \mathcal{H}} G(e,y|R_\mu) G(y,x|R_\mu), \tag{14}$$

because η is fixed.

Recall that we want to prove that φ_1 is integrable with respect to m. As $\varphi_1^{(n)}$ is non-decreasing, it is enough to show that there exists a uniform $C \geq 0$ such that for any n,

$$\int \varphi_1^{(n)} \, dm \leq C.$$

By definition, $\varphi_1^{(n)}$ is constant on cylinders of the form $[x_1, \ldots, x_n]$. According to (9), we just need to show that for every n,

$$\sum_{x \in \hat{S}^N} H(e,x|R_\mu) \sum_{k=0}^{n-1} \rho^k d(x_k, x_{k+1}) \tag{15}$$

is uniformly bounded.

We decompose $x \in \hat{S}^n$ as $x = x_1 \ldots x_n$. For any $y \in \hat{S}^k$, denote by X_1^y the set of symbols σ which can follow y in the automaton $\mathcal{G}$. More generally, denote by X_j^y the set of words of length j which can follow y. For fixed k writing $\sigma_{k+1} = x_k^{-1} x_{k+1}$ and $y = x_{k+1}^{-1} x$, we have, using weak relative Ancona inequalities,

$$\sum_{x \in \hat{S}^n} H(e,x|R_\mu) d(x_k, x_{k+1})$$

$$\lesssim \sum_{x_k \in \hat{S}^k} \sum_{\sigma_{k+1} \in X_1^{x_k}} \sum_{y \in X_{n-k-1}^{x_{k+1}}} H(e,x_k|R_\mu) H(e,y|R_\mu) d(e,\sigma_{k+1}) G(e,\sigma_{k+1}|R_\mu) G(\sigma_{k+1}, e|R_\mu)$$

$$\lesssim \sum_{x_k \in \hat{S}^k} \sum_{\sigma_{k+1} \in X_1^{x_k}} \sum_{y \in X_{n-k-1}^{x_{k+1}}} H(e,x_k|R_\mu) H(e,y|R_\mu) \sum_{\sigma \in \mathcal{H}_k} G(e,\sigma|R_\mu) G(\sigma,\sigma_{k+1}|R_\mu) G(\sigma_{k+1}, e|R_\mu).$$

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Lemma 2.9 shows that

$$\sum_{y \in X_{n-k-1}^{x_{k+1}}} H(e, y | R_\mu) \lesssim 1.$$

As μ is not spectrally degenerate,

$$\sum_{\sigma \in \mathcal{H}_k} G(e, \sigma | R_\mu) G(\sigma, \sigma_{k+1} | R_\mu) G(\sigma_{k+1}, e | R_\mu) \lesssim 1.$$

Using again Lemma 2.9,

$$\sum_{x_k \in \hat{S}^k} H(e, x_k | R_\mu) \lesssim 1.$$

Finally, we find that (15) is bounded by $C \sum_{k=0}^{n-1} \rho^k$ for some C. As $\rho < 1$, this last sum is uniformly bounded.

To conclude, we give a similar bound for the remaining term in (11), with an integrable function φ_2. We set $\varphi = \varphi_1 + \varphi_2$. This concludes the proof. $\square$

The function φ is constructed as the non-decreasing limit of functions $\varphi^{(n)}$ which are uniformly integrable and satisfy that for any word x of length n,

$$\left| \frac{d}{dr} \varphi_r(x) \right| \le \varphi^{(n)}(x) \frac{1}{\sqrt{R_\mu - r}}. \tag{16}$$

Let us note that we proved something a bit stronger than $\int \varphi^{(n)} \, dm \lesssim 1$. Indeed, we proved there exists $C \ge 0$ such that for every n,

$$\sum_{x \in \hat{S}^n} H(e, x | R_\mu) \varphi^{(n)}(x) \le C. \tag{17}$$

We both use (16) and (17) in the following. However, to simplify the notation, we only stated Lemma 4.8 using φ and m.

We also prove the following result. We do not use in full generality, but only for $x = \emptyset$.

PROPOSITION 4.11. *For every $x \in \Sigma_A$, there exists C_x such that for every $r, r' \le R_\mu$ and for every bounded function f,*

$$\left| (\mathcal{L}_r f)(x) - (\mathcal{L}_{r'} f)(x) \right| \le C_x \|f\|_\infty \sqrt{|r - r'|}.$$

Proof. Fix $x \in \Sigma_A$ and let $n = \hat{d}(e, x)$. Let X_x^1 be the set of symbols which can precede x in the automaton $\mathcal{G}$. Then,

$$(\mathcal{L}_r f)(x) - (\mathcal{L}_{r'} f)(x) = \sum_{\sigma \in X_x^1} (e^{\varphi_r(\sigma x)} - e^{\varphi_{r'}(\sigma x)}) f(\sigma x).$$

Differentiating in r the quantity

$$\sum_{\sigma \in X_x^1} e^{\varphi_r(\sigma x)} f(\sigma x),$$

we obtain

$$\sum_{\sigma \in X_x^1} \left(\frac{d}{dr} \varphi_r(\sigma x) \right) e^{\varphi_r(\sigma x)} f(\sigma x).$$

Using Lemma 4.8, this is bounded by

$$\|f\|_\infty \frac{1}{\sqrt{R_\mu - r}} \sum_{\sigma \in X_x^1} \varphi(\sigma x) e^{\varphi_r(\sigma x)}.$$

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We deduce from weak relative Ancona inequalities that this is bounded by

$$\|f\|_\infty \frac{1}{\sqrt{R_\mu - r}} \sum_{\sigma \in X_x^1} \varphi(\sigma x) H(e, \sigma | r).$$

As $\sum_{\sigma \in X_x^1} \varphi(\sigma x) H(e, \sigma | r)$ only depends on x, it is enough to show that this last sum is bounded independently of r. This is done exactly like showing that the sum (15) is bounded. $\square$

We want to apply Theorem 3.6, so we now prove that the assumptions of this theorem are satisfied, for $\tau(r) = \sqrt{R_\mu - r}$. Recall that $m = \sum_j m_j$.

PROPOSITION 4.12. *There exist constants $0 < \sigma < 1$ and $C \geq 0$ such that for every $1 \leq r \leq R_\mu$, for every n, for every function $f \in H_{\rho,\beta}$,*

$$\|\mathcal{L}_r^n f\|_{\rho,\beta} \leq C\sigma^n \|f\|_{\rho,\beta} + C \int |f| dm$$

and

$$\int \left| (\mathcal{L}_r - \mathcal{L}_{R_\mu}) f \right| dm \leq C \|f\|_{\rho,\beta} \sqrt{R_\mu - r}.$$

The proof of this proposition is postponed to the end of the section. We first state the following corollary which is deduced from Theorem 3.6.

COROLLARY 4.13. *For every r close enough of R_μ, there exist numbers $\tilde{P}_j(r)$, eigenfunctions $\tilde{h}_{j,r}^{(i)}$, and eigenmeasures $\tilde{\nu}_{j,r}^{(i)}$ of $\mathcal{L}_r$ associated with the eigenvalue $e^{\tilde{P}_j(r)}$ such that for every $g \in H_{\rho,\beta}$,*

$$\left\| \mathcal{L}_r^n g - \sum_{j=1}^{k} e^{n(\tilde{P}_j(r))} \sum_{i=1}^{p_j} \tilde{h}_{j,r}^{(i)} \int g \, d\tilde{\nu}_{j,r}^{((i-n) \bmod p_j)} \right\|_{\rho,\beta} \leq C\theta^n \|g\|_{\rho,\beta},$$

where $C \geq 0$ and $0 < \theta < 1$. The functions $\tilde{h}_j^{(i)}$ and the measures $\tilde{\nu}_j^{(i)}$ have the same support as the functions $h_j^{(i)}$ and the measures $\nu_j^{(i)}$, respectively. Moreover, $\|\tilde{h}_{j,r}^{(i)}\|_{\rho,\beta}$ is uniformly bounded. Finally,

$$\int \left| \tilde{h}_{j,r}^{(i)} - h_j^{(i)} \right| dm \xrightarrow[r \to R_\mu]{} 0$$

and $\tilde{\nu}_{j,r}^{(i)}$ weakly converges to $\nu_j^{(i)}$ as r tends to R_μ.

To conclude, note that Proposition 4.11 yields

$$|\mathcal{L}_r f(\emptyset) - \mathcal{L}_{r'} f(\emptyset)| \lesssim \|f\|_{\rho,\beta} \sqrt{|r' - r|}. \tag{18}$$

Consequently,

$$|\tilde{h}_{j,r}^{(i)}(\emptyset) - h_j^{(i)}(\emptyset)| \xrightarrow[r \to R_\mu]{} 0. \tag{19}$$

Indeed, this last estimate (18) shows that we can replace the norm $|\cdot|_w$ when applying Theorem 3.6 with the norm $|\cdot|'_w$ defined by

$$|f|'_w = |f(\emptyset)| + \int |f| \, dm.$$

Although we use (19) in the following, we preferred using the norm $|\cdot|_w$ in the statements and in the proofs for convenience.

We now prove Proposition 4.12. We use repeatedly strong relative Ancona inequalities to obtain stronger and stronger continuity statements. We first give an upper bound for $\|\mathcal{L}_r^n f\|_\infty$

and then one for $D_{\rho,\beta}(\mathcal{L}_r^n f)$, that actually use the first upper bound. This will conclude the proof of the first statement in the proposition. The proof of the second statement is similar but a bit more technically involved.

Proof. Let $f \in H_{\rho,\beta}$ and let $x \in \overline{\Sigma}_A$. Denote by $S_n\varphi_r$ the nth Birkhoff sum of φ_r and let X_x^n the set of words of length n which can precede x in the automaton $\mathcal{G}$. Then,

$$\mathcal{L}_r^n f(x) = \sum_{\gamma \in X_x^n} e^{S_n\varphi_r(\gamma x)} f(\gamma x).$$

Let $f^{(n)}$ be the function which is constant on cylinders of length n and which is equal to f elsewhere. In particular,

$$f^{(n)}(\gamma x) = f(\gamma)$$

for $x \in \overline{\Sigma}_A$ and $\gamma \in X_x^n$. As f is ρ-locally Hölder,

$$|f^{(n)}(\gamma x) - f(\gamma x)| \leq \rho^n D_{\rho,\beta}(f).$$

Hence,

$$\left| \mathcal{L}_r^n f(x) \right| \leq \rho^n D_{\rho,\beta}(f) \sum_{\gamma \in X_x^n} e^{S_n\varphi_r(\gamma x)} + \sum_{\gamma \in X_x^n} e^{S_n\varphi_r(\gamma x)} |f^{(n)}(\gamma x)|.$$

To simplify, we identify an element $\gamma \in X_x^1$ with the corresponding element in $\hat{S}^n \subset \Gamma$. Note that $e^{S_n\varphi_r(\gamma x)} = H(e, \gamma x|r)/H(\gamma, \gamma x|r)$. Using weak relative Ancona inequalities, we obtain

$$\left| \mathcal{L}_r^n f(x) \right| \lesssim \rho^n D_{\rho,\beta}(f) \sum_{\gamma \in \hat{S}^n} H(e, \gamma|R_\mu) + \sum_{\gamma \in \hat{S}^n} H(e, \gamma|R_\mu) |f^{(n)}(\gamma x)|.$$

For every $\gamma \in \Gamma$, we can use the automaton $\mathcal{G}$ and choose a relative geodesic from e to γ whose increments we denote by $x_1, \ldots, x_n$. Let $[\gamma]$ be the corresponding cylinder $[x_1, \ldots, x_n]$. Let $\hat{S}_{\max}^n$ be the set of $\gamma \in \hat{S}^n$ such that the cylinder $[\gamma]$ is in a maximal component. As $f^{(n)}$ is constant on cylinders of length n, (10) shows that

$$\sum_{\gamma \in \hat{S}_{\max}^n} H(e, \gamma|R_\mu) |f^{(n)}(\gamma x)| \lesssim \int |f^{(n)}| \, dm.$$

In addition, by the definition of maximal components, there exists $\rho' < e^{P(R_\mu)} = 1$ such that

$$\sum_{\gamma \in \hat{S}^n \setminus \hat{S}_{\max}^n} H(e, \gamma|R_\mu) |f^{(n)}(\gamma x)| \lesssim (\rho')^n \|f\|_\infty.$$

Using that f is ρ-locally Hölder, we obtain

$$\int |f^{(n)}| \, dm \lesssim \rho^n D_{\rho,\beta}(f) + \int |f| \, dm.$$

Lemma 2.9 shows that the sum $\sum_{\gamma \in \hat{S}^n} H(e, \gamma|R_\mu)$ is bounded independently of n. We thus obtain

$$\left| \mathcal{L}_r^n f(x) \right| \lesssim 2\rho^n D_{\rho,\beta}(f) + (\rho')^n \|f\|_\infty + \int |f| \, dm \lesssim \sigma^n \|f\|_{\rho,\beta} + \int |f| \, dm, \tag{20}$$

where $\sigma = \max(\rho, \rho')$. We can thus control $\|\mathcal{L}_r^n f\|_\infty$.

We now focus on $D_{\rho,\beta}(\mathcal{L}_r f)$. Let $x, x' \in \overline{\Sigma}_A$ and let $\rho^m = d_\rho(x, x')$, $m \geq 1$. We have

$$\mathcal{L}_r^n f(x) - \mathcal{L}_r^n f(x') = \sum_{\gamma \in X_x^n} (e^{S_n \varphi_r(\gamma x)} - e^{S_n \varphi_r(\gamma x')}) f(\gamma x)$$
$$+ \sum_{\gamma \in X_x^n} e^{S_n \varphi_r(\gamma x')} (f(\gamma x) - f(\gamma x')).$$

On the one hand, using that f is ρ-locally Hölder and using weak relative Ancona inequalities to bound $e^{S_n \varphi_r(\gamma x)}$ by $H(e, \gamma|r)$ as previously, we obtain

$$\sum_{\gamma \in X_x^n} e^{S_n \varphi_r(\gamma x')} |(f(\gamma x) - f(\gamma x'))| \lesssim \sum_{\gamma \in \hat{S}^n} H(e, \gamma|R_\mu) \rho^{n+m} D_{\rho,\beta}(f).$$

It follows from Lemma 2.9 that the sum $\sum_{\gamma \in \hat{S}^n} H(e, \gamma|R_\mu)$ is bounded and because $\rho^m = d_\rho(x, x')$, we have

$$\sum_{\gamma \in X_x^n} e^{S_n \varphi_r(\gamma x')} |(f(\gamma x) - f(\gamma x'))| \lesssim \rho^n \|f\|_{\rho,\beta} d_\rho(x, x').$$

On the other hand,

$$\sum_{\gamma \in X_x^n} (e^{S_n \varphi_r(\gamma x)} - e^{S_n \varphi_r(\gamma x')}) f(\gamma x)$$
$$= \sum_{\gamma \in X_x^n} e^{S_n \varphi_r(\gamma x)} (1 - e^{S_n \varphi_r(\gamma x') - S_n \varphi_r(\gamma x)}) f(\gamma x).$$

By definition,

$$(1 - e^{S_n \varphi_r(\gamma x') - S_n \varphi_r(\gamma x)}) = \left(1 - \frac{H(e, \gamma x|r) H(\gamma, \gamma x'|r)}{H(\gamma, \gamma x|r) H(e, \gamma x'|r)}\right).$$

As relative geodesics $[e, x]$ and $[e, x']$ fellow travel for a time at least m and because γ both precedes x and x' in the automaton $\mathcal{G}$, relative geodesics $[e, \gamma x]$ and $[\gamma, \gamma x']$ also fellow travel for a time at least m. Strong relative Ancona inequalities thus yield

$$\left|1 - e^{S_n \varphi_r(\gamma x') - S_n \varphi_r(\gamma x)}\right| \lesssim \rho^m$$

and so

$$\left|\sum_{\gamma \in X_x^n} (e^{S_n \varphi_r(\gamma x)} - e^{S_n \varphi_r(\gamma x')}) f(\gamma x)\right| \lesssim \rho^m \sum_{\gamma \in X_x^n} e^{S_n \varphi_r(\gamma x)} |f(\gamma x)|.$$

We bound $\sum_{\gamma \in X_x^n} e^{S_n \varphi_r(\gamma x)} |f(\gamma x)|$ by $\sigma^n \|f\|_{\rho,\beta} + \int |f| \, dm$ as previously to obtain

$$\left|\mathcal{L}_r f(x) - \mathcal{L}_r f(x')\right| \lesssim \left(\sigma^n \|f\|_{\rho,\beta} + \int |f| \, dm\right) d_\rho(x, x'). \tag{21}$$

We deduce from (20) and (21) that

$$\left\|\mathcal{L}_r^n f\right\|_{\rho,\beta} \lesssim \sigma^n \|f\|_{\rho,\beta} + \int |f| \, dm,$$

which concludes the first part of the proposition.

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We now prove that

$$\int \left|(\mathcal{L}_r - \mathcal{L}_{R_\mu})f\right| dm \le C\|f\|_{\rho,\beta}\sqrt{R_\mu - r}. \tag{22}$$

The function f is ρ-locally Hölder. As the operator $\mathcal{L}_r - \mathcal{L}_{R_\mu}$ is bounded on $(H_{\rho,\beta}, \|\cdot\|_{\rho,\beta})$, for every word $x = x_1 \ldots x_n$ of length n and for every $y \in [x_1, \ldots, x_n]$,

$$\left|(\mathcal{L}_r - \mathcal{L}_{R_\mu})f(y) - (\mathcal{L}_r - \mathcal{L}_{R_\mu})f(x)\right| \lesssim \rho^n \|f\|_{\rho,\beta}.$$

Hence,

$$\left|(\mathcal{L}_r - \mathcal{L}_{R_\mu})f(y)\right| \lesssim \rho^n \|f\|_{\rho,\beta} + \left|(\mathcal{L}_r - \mathcal{L}_{R_\mu})f(x)\right|.$$

Fixing r, we choose n large enough so that $\rho^n \le \sqrt{R_\mu - r}$. Let $\tilde{f}_r$ be the function which is constant on cylinders of length n and which is equal to $(\mathcal{L}_r - \mathcal{L}_{R_\mu})f(x)$ elsewhere. To prove (22), we just need to show that

$$\int |\tilde{f}_r| dm \lesssim \|f\|_{\rho,\beta}\sqrt{R_\mu - r}.$$

For every x of length n and for every y in $[x_1, \ldots, x_n]$, we have

$$\tilde{f}_r(y) = \sum_{\sigma \in X_x^1} (e^{\varphi_r(\sigma x)} - e^{\varphi_{R_\mu}(\sigma x)})f(\sigma x).$$

Differentiating this, we obtain

$$\sum_{\sigma \in X_x^1} \left(\frac{d}{dr}\varphi_r(\sigma x)\right) e^{\varphi_r(\sigma x)} f(\sigma x).$$

Using (16), we bound the absolute value of this term by

$$\frac{1}{\sqrt{R_\mu - r}}\|f\|_\infty \sum_{\sigma \in X_x^1} \varphi^{(n+1)}(\sigma x)e^{\varphi_r(\sigma x)}.$$

Inverting the sum and the derivative is legitimate because weak relative Ancona inequalities show that $e^{\varphi_r(\sigma x)} \lesssim H(e, \sigma)$. As in the proof of Lemma 4.8, we show that the sum

$$\sum_{\sigma \in X_x^1} \varphi^{(n+1)}(\sigma x)e^{\varphi_r(\sigma x)}$$

is finite. Therefore,

$$\left|\frac{d}{dr}\tilde{f}_r(y)\right| \lesssim \frac{1}{\sqrt{R_\mu - r}}\|f\|_\infty \sum_{\sigma \in X_x^1} H(e, \sigma|R_\mu)\varphi^{(n+1)}(\sigma x).$$

Integrating this, we obtain

$$\int |\tilde{f}_r| dm \lesssim \|f\|_{\rho,\beta}\sqrt{R_\mu - r} \int \sum_{\sigma \in X_x^1} H(e, \sigma|R_\mu)\varphi^{(n+1)}(\sigma x)\, dm(x).$$

As $x \mapsto \varphi^{(n+1)}(\sigma x)$ is constant on cylinders of length n, (10) shows that

$$\int |\tilde{f}_r| dm \lesssim \|f\|_{\rho,\beta}\sqrt{R_\mu - r} \sum_{x \in \hat{S}^n} \sum_{\sigma \in X_x^1} H(e, \sigma|R_\mu)H(e, x|R_\mu)\varphi^{(n+1)}(\sigma x)$$

$$\lesssim \|f\|_{\rho,\beta}\sqrt{R_\mu - r} \sum_{y \in \hat{S}^{n+1}} H(e, y|R_\mu)\varphi^{(n+1)}(y).$$

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According to (17), we thus have

$$\int |\tilde{f}_r|\, dm \lesssim \|f\|_{\rho,\beta} \sqrt{R_\mu - r},$$

which concludes the proof. □

5. Evaluating the pressure

Our goal in this section is to estimate the numbers $\tilde{P}_j(r)$ given by Corollary 4.13 and to compare them with the maximal pressure $P(r)$. This allows us to obtain a precise estimate of $I^{(1)}(r)$.

According to Corollary 4.13, if $R_\mu - r$ is small enough, then

$$\mathcal{L}_r^n 1_{E_*}(\emptyset) = \sum_{j=1}^k e^{n\tilde{P}_j(\varphi_r)} \sum_{i=1}^{p_j} \tilde{h}_j^{(i)}(\emptyset) \int 1_{E_*}\, d\tilde{\nu}_j^{((i-n) \bmod p_j)} + O(\theta^n).$$

Here, $0 < \theta < 1$ and k is the number of maximal components for the function φ_{R_μ}. Denote by p the least common multiple of the periods of these components, so that if $n \geq 0$ and $0 \leq q < p$, then $d\tilde{\nu}_j^{((i-np+q) \bmod p_j)}$ only depends on q. In particular, we can write

$$\mathcal{L}_r^{np+q} 1_{E_*}(\emptyset) = \sum_{j=1}^k e^{(np+q)\tilde{P}_j(\varphi_r)} \xi_{q,j}(r) + O(\theta^{np+q}),$$

where $\xi_{q,j}$ is a non-negative function of r defined on some fixed neighborhood of R_μ. Note that $\xi_{q,j}(r)$ only depends on $\tilde{h}_{j,r}(\emptyset)$ and on $\int 1_{E_*}\, d\tilde{\nu}_{j,r}^{(i)}$ and that it is continuous in r according to Corollary 4.13 and (19).

If $r < R_\mu$, then $\sum_\gamma H(e,\gamma|r)$ is finite, so the numbers $\tilde{P}_j(\varphi_r)$ are negative. Summing over n and $q \in \{0, \ldots, p-1\}$, we obtain

$$\sum_{\gamma \in \Gamma} H(e,\gamma|r) = H(e,e|r) \sum_{n,q} \mathcal{L}_r^{np+q} 1_{E_*}(\emptyset) = \sum_{j=1}^k \frac{\xi_j(r)}{|\tilde{P}_j(\varphi_r)|} + O(1), r \to R_\mu, \qquad (23)$$

for some non-negative functions ξ_j, which are continuous in r on some neighborhood of R_μ.

We have the following result, which shows that the pressure is asymptotically independent of the maximal components. Its proof is postponed to the following subsections.

PROPOSITION 5.1. *For every $j \in \{1, \ldots, k\}$, $\tilde{P}_j(\varphi_r)/P(r)$ tends to one when r tends to R_μ, where $P(r)$ is the maximal pressure of the function φ_r.*

Combining Proposition 5.1 and (23), we obtain that

$$I^{(1)}(r) = \sum_{\gamma \in \Gamma} H(e,\gamma|r) = \frac{\xi(r)}{|P(r)|} + O(1), r \to R_\mu, \qquad (24)$$

for some non-negative function ξ, which is continuous in r on some neighborhood of R_μ. Recall that $I^{(1)}(r) \asymp \sqrt{R_\mu - r}$ and $|P(r)| \asymp \sqrt{R_\mu - r}$. Therefore, $\xi(R_\mu) > 0$, so that $\xi(r)$ is bounded away from zero on a neighborhood of R_μ.

The remainder of this section is devoted to proving Proposition 5.1. An analogous result is proved in [Gou14] for hyperbolic groups. This is done by showing the following.

LEMMA 5.2. *For $r \in [1, R_\mu]$, $\int \varphi_r dm_j$ does not depend on j, where m_j is the measure in Proposition 3.7.*

This lemma, in turn, is proved in several steps.

Step 1. Fix c and define $U(c) \subset \partial\hat\Gamma$ as the set of points $\xi \in \partial\hat\Gamma$ such that if x is an infinite sequence in $\partial\Sigma_A^*$ defining ξ (that is, denoting $\gamma_n = x_1 \ldots x_n$, $e, \gamma_1, \ldots, \gamma_n, \ldots$ is a relative geodesic ray that converges to ξ in $\partial\hat\Gamma$), then $\log H(e, \gamma_n|r)/\hat d(e, \gamma_n)$ converges to c. Then, the definition of $U(c)$ does not depend on the choice of the sequence x_n. Moreover, $U(c)$ is Γ-invariant, that is, for any $\gamma \in \Gamma$, $\gamma \cdot U(c) = U(c)$.

Step 2. Define the sequence of measures $\lambda_n = \sum_{\gamma \in \hat S_n} H(e, \gamma|R_\mu)\delta_\gamma$ on Γ. Define $\tilde\lambda_N$ as

$$\tilde\lambda_N := \sum_{n=1}^N \lambda_n \Big/ \left(\sum_{n=1}^N \lambda_n(\Gamma)\right). \tag{25}$$

Then, up to a subsequence, $\tilde\lambda_N$ converges weakly to a probability measure on $\partial\hat\Gamma$, which we denote by λ_{R_μ}.

Step 3. The limit measure λ_{R_μ} is ergodic for the action of Γ on $\partial\hat\Gamma$.

Step 4. Let $c_j = \int \varphi_r \, dm_j$. Then, $\lambda_{R_\mu}(U(c_j)) > 0$. As λ_{R_μ} is ergodic and $U(c_i)$ is Γ-invariant, we thus have $\lambda_{R_\mu}(U(c_j)) = 1$ for all j. In particular, all the sets $U(c_j)$ intersect, which proves that c_j is independent of j.

5.1 Proof of step 1

We prove here the following lemma.

LEMMA 5.3. *The sets $U(c)$ as previously are well defined and are Γ-invariant.*

Step 1 is stated in [Gou14, §3.5] using the Gromov boundary $\partial\Gamma$ of Γ instead of $\partial\hat\Gamma$, because groups are hyperbolic in there and not relatively hyperbolic. It is a consequence of the fact that geodesics converging to $\xi \in \partial\Gamma$ in a hyperbolic group stay within a bounded distance of each other. This property still holds in our situation as we show in the proof of Lemma 5.3 that we now present.

Proof. First, let us show that the definition of $U(c)$ does not depend on the choice of the sequence x defining ξ. Assume that x and x' are two sequences such that, setting $\gamma_n = x_1 \ldots x_n$ and $\gamma_n' = x_1' \ldots x_n'$, both sequences $e, \gamma_1, \ldots, \gamma_n, \ldots$ and $e, \gamma_1', \ldots, \gamma_n', \ldots$ are relative geodesics converging to ξ. Then, according to Lemma 2.4, for every n, there exists k_n such that $d(\gamma_n, \gamma_{k_n}') \leq C$, so that $H(e, \gamma_n|r) \asymp H(e, \gamma_{k_n}'|r)$. We thus have $|\log H(e, \gamma_n|r) - \log H(e, \gamma_{k_n}'|r)| \leq C'$. Moreover, because $d(\gamma_n, \gamma_{k_n}') \leq C$, $\hat d(\gamma_n, \gamma_{k_n}') \leq C$, so that $|n - k_n| \leq C''$ and, thus, $|\hat d(e, \gamma_n') - \hat d(e, \gamma_{k_n}')| \leq C''$. This proves that $\log H(e, \gamma_n|r)/\hat d(e, \gamma_n)$ and $\log H(e, \gamma_n'|r)/\hat d(e, \gamma_n')$ have the same limit.

Let $\gamma \in \Gamma$ and let $\xi \in U(c)$. We want to prove that $\gamma \cdot \xi \in U(c)$. Consider a sequence x defining ξ and a sequence x' defining $\gamma \cdot \xi$, that is, setting $\gamma_n = x_1 \ldots x_n$ and $\gamma_n' = x_1' \ldots x_n'$, the sequence $e, \gamma_1, \ldots, \gamma_n, \ldots$ is a relative geodesic converging to ξ and the sequence $e, \gamma_1', \ldots, \gamma_n', \ldots$ is a relative geodesic converging to $\gamma \cdot \xi$. Then, $\gamma, \gamma\gamma_1, \ldots, \gamma\gamma_n, \ldots$ is a relative geodesic starting at γ and converging to $\gamma \cdot \xi$. According to Lemma 2.4, for every n, there exists k_n such that $d(\gamma\gamma_n, \gamma_{k_n}') \leq C$. This time, the bound C depends on γ, but not on the sequences γ_n and γ_n'. This shows that $H(\gamma^{-1}, \gamma_n|r) = H(e, \gamma\gamma_n|r) \asymp H(e, \gamma_{k_n}'|r)$, and because γ is fixed, $H(\gamma^{-1}, \gamma_n|r) \asymp H(e, \gamma_n|r)$, so that $H(e, \gamma_n|r) \asymp H(e, \gamma_{k_n}'|r)$. The same proof then shows that $\log H(e, \gamma_n|r)/\hat d(e, \gamma_n)$ and $\log H(e, \gamma_n'|r)/\hat d(e, \gamma_n')$ have the same limit. $\qquad\square$

5.2 Proof of step 2

We prove here the following lemma.

LEMMA 5.4. *Up to a subsequence, $\tilde{\lambda}_N$ as defined in (25) converges weakly to a probability measure on $\partial\hat{\Gamma}$, which we denote by λ_{R_μ}.*

An analogous result follows directly from the convergence properties of the transfer operator $\mathcal{L}_r$ in [Gou14] and one does not need to extract a subsequence. However, in our situation, we can only prove convergence of $\int f \, d\lambda_n$ for functions $f \in \mathcal{B}_{\rho,\beta}$. As our space is not compact and not even locally compact, this set of functions is not dense in the set of all continuous and bounded functions for the $\|\cdot\|_\infty$ norm. To fix this problem, we need to consider a compact space that contains $\partial\hat{\Gamma}$ so that $\tilde{\lambda}_N$ converges to a measure on this compact space (up to a subsequence). We then prove that this limit measure gives full measure to $\partial\hat{\Gamma}$. The compact space in question is a version of the Martin boundary that we define. Actually, we deal both with the Martin boundary and the Bowditch boundary at the same time.

We first define the Green distance at the inverse of the spectral radius as

$$d_G(\gamma, \gamma') = -\log F(\gamma, \gamma'|R_\mu)F(\gamma', \gamma|R_\mu),$$

where $F(\gamma, \gamma'|R_\mu)$ is the first visit Green function at R_μ. More precisely, we have

$$F(\gamma, \gamma'|r) = \sum_{n \geq 0} r^n \mathbb{P}(X_0 = \gamma, X_n = \gamma', X_k \neq \gamma', 1 \leq k \leq n-1), \tag{26}$$

where X_k is the position of the μ-random walk at time k. Note that for $r = 1$, $F(\gamma, \gamma'|1)$ is the probability of ever reaching γ' starting at γ.

Using the relation

$$G(\gamma, \gamma'|r) = F(\gamma, \gamma'|r)G(\gamma', \gamma'|r) = F(\gamma, \gamma'|r)G(e, e|r)$$

(see [Woe00, Lemma 1.13(b)]), we also have that

$$d_G(\gamma, \gamma') = -\log G(\gamma, \gamma'|R_\mu) - \log G(\gamma', \gamma|R_\mu) + 2G(e, e|R_\mu).$$

Actually, the Green distance was introduced by Blachère and Brofferio in [BB07] as $d_G(\gamma, \gamma') = -\log F(\gamma, \gamma'|1)$. What we call the Green distance here is, thus, a symmetrized version at the spectral radius of what they call the Green distance.

In general, in any metric space (X, d), one can consider a compactification given by the distance called the horofunction compactification. It was introduced by Kuratowski in [Kur35] and used a lot by Gromov (see, for example, [BGS85]). It is the smallest compact set H such that the function $\phi : (x, y) \mapsto d(x, y) - d(x_0, y)$ extends continuously to $X \times H$, where x_0 is a base point. Its homeomorphism type does not depend on x_0. The horofunction boundary is the complement of Γ in the horofunction compactification. We refer to [MT18, § 3] for a construction and many more details.

Define the Martin kernel as

$$\tilde{K}(\gamma, \gamma') = \frac{G(\gamma, \gamma'|R_\mu)G(\gamma', \gamma|R_\mu)}{G(e, \gamma'|R_\mu)G(\gamma', e|R_\mu)}.$$

The Martin compactification is defined as the horofunction compactification for the Green distance. In other words, a sequence γ_n in Γ converges to a point ξ in the Martin boundary if and only if the Martin kernel $\tilde{K}(\cdot, \gamma_n)$ converge pointwise to a limit function $\tilde{K}(\cdot, \xi)$. Usually, the

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Martin compactification is defined using the Martin kernel

$$K(\gamma, \gamma') = \frac{G(\gamma, \gamma')}{G(e, \gamma')}.$$

Again, our Martin compactification is a symmetrized version of the usual Martin compactification.

It is proved in [GGPY21] that as soon as weak relative Ancona inequalities are satisfied, there is a one-to-one continuous map from $\Gamma \cup \partial \hat{\Gamma}$ to the Martin compactification, which is a homeomorphism on its image. Actually, this is proved for the usual definition of the Martin boundary. Although the proof still works for our symmetrized version, the terminology is a bit different and we give a proof for completeness.

LEMMA 5.5. *There is a one-to-one continuous map from $\Gamma \cup \partial \hat{\Gamma}$ to the Martin compactification, which is a homeomorphism on its image.*

Proof. Let ξ be a conical limit point and let $[e, \xi)$ be a relative geodesic ray from e to ξ. Let γ_n be a sequence along $[e, \xi)$ converging to ξ. Let $\gamma \in \Gamma$ and let $\tilde{\gamma}$ be its projection on $[e, \xi)$ in $\hat{\Gamma}$. Lemma 2.5 shows that for large enough n, a relative geodesic from γ to γ_n passes within a bounded distance of $\tilde{\gamma}$. In addition, [Dus22, Lemma 4.17] shows that for large enough n, relative geodesics from e to γ_n and from γ to γ_n fellow travel for an arbitrarily long time, when n goes to infinity. Then, strong relative Ancona inequalities show that for every γ, $\tilde{K}(\gamma, \gamma_n)$ converges to some limit $\tilde{K}_\xi(\gamma)$, exactly as in the proof of Lemma 4.3. We thus proved that γ_n converges to a limit that we still denote by ξ in the Martin boundary.

More generally, let ξ be a conical limit point and let ξ_n be a sequence in $\Gamma \cup \partial \hat{\Gamma}$ converging to ξ. Let α be a relative geodesic ray from e to ξ and let α_n be a (finite or infinite) relative geodesic from e to ξ_n. Let d_μ be an arbitrary distance on the Martin compactification. Then, there exists $\gamma_n \in \Gamma$ on α_n such that $d_\mu(\gamma_n, \xi_n) \leq 1/n$. If $\xi_n \in \Gamma$, we can choose $\xi_n = \gamma_n$. Otherwise, we use what we just proved previously. Up to choosing $\hat{d}(e, \gamma_n)$ large enough, we can also assume that γ_n converges to ξ in $\Gamma \cup \partial \hat{\Gamma}$. Thus, there exists a sequence k_n going to infinity such that the projection $\tilde{\gamma}_n$ of γ_n on α in $\hat{\Gamma}$ satisfies $\hat{d}(e, \tilde{\gamma}_n) \geq k_n$. In particular, $\tilde{\gamma}_n$ converges to ξ in the Martin boundary, that is, for any γ, $\tilde{K}(\gamma, \tilde{\gamma}_n)$ converges to $\tilde{K}_\xi(\gamma)$. Let $\gamma \in \Gamma$. Then, according to [Dus22, Lemma 4.17] applied twice, relative geodesics from e to γ_n and from γ to $\tilde{\gamma}_n$ fellow-travel for an arbitrarily long time, when n goes to infinity. Strong relative Ancona inequalities show that $\tilde{K}(\gamma, \gamma_n)$ also converges to $\tilde{K}_\xi(\gamma)$. Thus, $d_\mu(\xi, \gamma_n)$ goes to zero. As $d_\mu(\gamma_n, \xi_n) \leq 1/n$, we also have that ξ_n converges to ξ in the Martin boundary.

We have, thus, constructed a map from $\Gamma \cup \partial \hat{\Gamma}$ to the Martin compactification. We also proved that this map is continuous. Let us prove that it is one-to-one. Let $\xi \neq \xi'$ be two conical limit points. We just need to prove that $\xi \neq \xi'$ in the Martin boundary. Consider two relative geodesics $[e, \xi)$ and $[e, \xi')$ from e to ξ and from e to ξ'. Let γ_n and γ'_n be a sequence on $[e, \xi)$ and $[e, \xi')$, respectively, converging to ξ and ξ', respectively. As $\xi \neq \xi'$, the projection of γ_n on $[e, \xi')$ in $\hat{\Gamma}$ stays within a bounded distance of e. Thus, for large enough n and m, a relative geodesic from γ_n to γ'_m passes within a bounded distance of e. Weak relative Ancona inequalities show that $\tilde{K}(\gamma_n, \gamma'_m) \asymp H(\gamma_n, e|R_\mu)$. Letting m tend to infinity, we thus have that $\tilde{K}_{\xi'}(\gamma_n) \asymp H(\gamma_n, e|R_\mu)$, so that $\tilde{K}_{\xi'}(\gamma_n)$ converges to zero. Weak relative Ancona inequalities also show that if $n < m$, we have $\tilde{K}(\gamma_n, \gamma_m) \asymp 1/H(e, \gamma_n|R_\mu)$. Letting m tend to infinity, we obtain $\tilde{K}_\xi(\gamma_n) \asymp 1/H(\gamma_n, e|R_\mu)$, so that $\tilde{K}_\xi(\gamma_n)$ goes to infinity. We can thus find n such that $\tilde{K}_\xi(\gamma_n) \neq \tilde{K}_{\xi'}(\gamma_n)$ and so $\xi \neq \xi'$ in the Martin boundary.

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Finally, we prove that this map is a homeomorphism on its image. Let ξ_n be a sequence in $\Gamma \cup \partial\hat{\Gamma}$ converging to ξ in the Martin compactification. Assume by contradiction that it does not converge to ξ in $\Gamma \cup \partial\hat{\Gamma}$. Fix a relative geodesic α from e to ξ and for every n, a relative geodesic α_n from e to ξ_n. Then, up to choosing a subsequence, we can assume that the projection of α_n on α in $\hat{\Gamma}$ stays within a uniform bounded distance of e. In particular, if γ_m is a sequence on α converging to ξ and if γ'_k is a sequence on α_n converging to ξ_n, then a relative geodesic from γ_m to γ'_k passes within a bounded distance of e, independently of k, m, n. Weak relative Ancona inequalities show that $\tilde{K}(\gamma_m, \gamma'_k) \asymp H(\gamma_m, e|R_\mu)$ so that letting k tend to infinity, $\tilde{K}_{\xi_n}(\gamma_m) \asymp H(\gamma_m, e|R_\mu)$. In particular, $\tilde{K}_{\xi_n}(\gamma_m) \leq C$ for some uniform C. However, as we saw above, $\tilde{K}_\xi(\gamma_m)$ tends to infinity, so there exists m such that $\tilde{K}_\xi(\gamma_m) \geq C + 1$. Fixing such an m, we obtain a contradiction, since $\tilde{K}_{\xi_n}(\gamma_m)$ converges to $\tilde{K}_\xi(\gamma_m)$ when n tends to infinity. $\qquad\square$

We first prove that $\tilde{\lambda}_N$ converges to a probability measure on the Bowditch compactification. We then prove that it also converges to a probability measure on the Martin compactification.

PROPOSITION 5.6. *Up to a subsequence, the measure $\tilde{\lambda}_N$ weakly converges to a measure λ_{R_μ} on the Bowditch compactification. This limit measure gives full measure to the set of conical limit points.*

Proof. Convergence up to a subsequence follows directly from compactness of the Bowditch compactification. We just need to prove that any limit measure of $\tilde{\lambda}_N$ gives full measure to the set of conical limit points. Recall that $\lambda_n = \sum_{\gamma \in \hat{S}_n} H(e, \gamma|R_\mu)\delta_\gamma$ and $\tilde{\lambda}_N = \sum_{n=1}^N \lambda_n / (\sum_{n=1}^N \lambda_n(\Gamma))$.

First, we prove that any limit measure λ_{R_μ} of $\tilde{\lambda}_N$ gives full mass to the Bowditch boundary. Let $K \subset \Gamma$ be a compact subset. Then, K is finite, so that for any N, $\sum_{n=1}^N \lambda_n(K)$ is bounded, independently of N. Moreover, according to Proposition 2.11, $\sum_{n=1}^N \lambda_n(\Gamma)$ tends to infinity. This proves that for any subsequence $\tilde{\lambda}_{N_j}$ of $\tilde{\lambda}_N$, $\tilde{\lambda}_{N_j}(K)$ converges to zero when j tends to infinity. As K is both open and closed in the Bowditch compactification, the Portmanteau theorem shows that $\lambda_{R_\mu}(K) = 0$.

Consider a parabolic limit point ξ in the Bowditch boundary. As the set of parabolic limit points is countable, we just need to prove that $\lambda_{R_\mu}(\{\xi\}) = 0$ to conclude. Let $\mathcal{H}$ be the corresponding parabolic subgroup, that is, $\mathcal{H}$ is the stabilizer of ξ. Choose $\mathcal{H}_0 \in \Omega_0$ so that $\mathcal{H}$ is conjugated to $\mathcal{H}_0$, say $\mathcal{H} = \gamma_0 \mathcal{H}_0 \gamma_0^{-1}$ Denote by $U_{\xi,n}$ the set of $\gamma \in \Gamma$ such that the projection of γ on $\gamma_0 \mathcal{H}_0$ in the Cayley graph $\mathrm{Cay}(\Gamma, S)$ is at d-distance at least n from e. Let $V(\xi, n)$ be the closure of $U(\xi, n)$ in the Bowditch compactification. Then, $V(\xi, n)$ contains $\{\xi\}$ so we only need to prove that $\lambda_{R_\mu}(V(\xi, n))$ converges to zero when n tends to infinity.

According to the BCP property, if $\zeta \in V(\xi, n)$, then there exists $\gamma \in \mathcal{H}_0$ such that $\gamma_0\gamma$ is within a bounded distance of a relative geodesic from e to ζ. In particular, if N is large enough, for every $\gamma' \in V(\xi, n) \cap \Gamma$ such that $\hat{d}(e, \gamma') = N$, weak relative Ancona inequalities show that

$$H(e, \gamma'|R_\mu) \lesssim H(e, \gamma_0|R_\mu)H(e, \gamma|R_\mu)H(\gamma_0\gamma, \gamma'|R_\mu).$$

Thus,

$$\lambda_N(V(\xi, n)) \lesssim \sum_{\gamma \in \mathcal{H}_0, d(e, \gamma_0\gamma) \geq n} H(e, \gamma|R_\mu) \sum_{k=1}^N \lambda_k(\Gamma)$$

and so

$$\tilde{\lambda}_N(V(\xi, n)) \lesssim \sum_{\gamma \in \mathcal{H}_0, d(e, \gamma_0\gamma) \geq n} H(e, \gamma|R_\mu).$$

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As γ_0 is fixed, this proves that

$$\lambda_{R_\mu}(V(\xi, n)) \lesssim \sum_{\gamma \in \mathcal{H}_0, d(e,\gamma) \geq n} H(e, \gamma | R_\mu).$$

According to Corollary 2.10, this last term converges to zero when n tends to infinity, which concludes the proof. $\square$

We can thus see the measure λ_{R_μ} as a measure on $\partial\hat{\Gamma}$. We fix a subsequence $\tilde{\lambda}_{N_k}$ such that $\tilde{\lambda}_{N_k}$ weakly converges to λ_{R_μ}. We can now prove Lemma 5.4.

Proof. There is a one-to-one and continuous map from $\Gamma \cup \partial\hat{\Gamma}$ to the Bowditch compactification, which is a homeomorphism on its image. As the limit measure λ_{R_μ} does not give any mass to the complement of $\Gamma \cup \partial\hat{\Gamma}$, the Portmanteau theorem shows that $\tilde{\lambda}_{N_k}$ also weakly converges to λ_{R_μ} on $\Gamma \cup \partial\hat{\Gamma}$. $\square$

We also prove the following corollary.

LEMMA 5.7. *The measure $\tilde{\lambda}_{N_k}$ also weakly converges to λ_{R_μ} on the Martin compactification.*

Proof. There is a one-to-one and continuous map from $\Gamma \cup \partial\hat{\Gamma}$ to the Martin compactification, which is a homeomorphism on its image. Let f be a bounded continuous function on the Martin compactification. Its restriction $\tilde{f}$ to $\Gamma \cup \partial\hat{\Gamma}$ also is bounded continuous, so $\tilde{\lambda}_{N_k}(f) = \tilde{\lambda}_{N_k}(\tilde{f})$ converges to $\lambda_{R_\mu}(\tilde{f})$. This proves that $\tilde{\lambda}_{N_k}$ also weakly converges to λ_{R_μ} on the Martin compactification. $\square$

5.3 Proof of step 3

We prove here the following.

LEMMA 5.8. *The limit measure λ_{R_μ} is ergodic for the action of Γ on $\partial\hat{\Gamma}$.*

This step is a bit more complicated. To show that λ_{R_μ} is ergodic, we follow the strategy of [MYJ20]. We first prove that λ_{R_μ} is conformal for the Green distance defined previously.

Let $\gamma \in \Gamma$ and let L_γ the operator of multiplication by γ on the left.

LEMMA 5.9. *For every γ, we have*

$$\frac{d(L_\gamma)_* \lambda_{R_\mu}}{d\lambda_{R_\mu}}(\xi) = \tilde{K}_\xi(\gamma).$$

Proof. Direct computation shows that for fixed γ_0, one has, for any γ and any N such that $N \geq \hat{d}(e, \gamma_0) + \hat{d}(e, \gamma)$,

$$(L_{\gamma_0})_* \tilde{\lambda}_N(\gamma) = H(\gamma_0, \gamma | R_\mu)/H(e, \gamma | R_\mu)\tilde{\lambda}_N(\gamma) = \tilde{K}(\gamma_0, \gamma)\tilde{\lambda}_N(\gamma). \tag{27}$$

Lemma 2.9 shows that $\sum_{n=N-\hat{d}(e,\gamma_0)}^{N} \sum_{\gamma \in \hat{S}_n} H(e, \gamma | R_\mu)$ is bounded. Thus, according to Proposition 2.11,

$$\frac{\sum_{n=N-\hat{d}(e,\gamma_0)}^{N} H(e, \gamma | R_\mu)}{\sum_{n \leq N} H(e, \gamma | R_\mu)} \xrightarrow[N \to \infty]{} 0.$$

Combined with (27), this shows that $((L_{\gamma_0})_* \tilde{\lambda}_N - \tilde{K}(\gamma_0, \cdot)\tilde{\lambda}_N)$ converges to zero in total variation norm.

By definition, for fixed γ_0, the function $\tilde{K}(\gamma_0, \cdot)$ is continuous and bounded on the Martin compactification. Thus, $\tilde{K}(\gamma_0, \cdot)\tilde{\lambda}_{N_k}$ weakly converges to $\tilde{K}(\gamma_0, \cdot)\lambda_{R_\mu}$. Moreover, left multiplication by γ_0 on Γ extends to a homeomorphism on the Martin compactification, so that $(L_{\gamma_0})_*\tilde{\lambda}_{N_k}$ weakly converges to $(L_{\gamma_0})_*\lambda_{R_\mu}$. We have, thus, proved that $(L_{\gamma_0})_*\lambda_{R_\mu} = \tilde{K}(\gamma_0, \cdot)\lambda_{R_\mu}$. $\qquad\square$

We use this property to prove the following.

LEMMA 5.10. *The measure λ_{R_μ} on $\partial\hat{\Gamma}$ has no atom.*

Proof. Assume in contrast that there exists $\xi \in \partial\hat{\Gamma}$ such that $\lambda_{R_\mu}(\xi) > 0$. Consider a sequence γ_n converging along a relative geodesic ray to ξ. Then, weak relative Ancona inequalities show that if $n \leq m$, then $\tilde{K}(\gamma_n, \gamma_m) \geq C/H(e, \gamma_n|R_\mu)$. Consequently, letting m tend to infinity, we see that $\tilde{K}_\xi(\gamma_n) \geq C/H(e, \gamma_n|R_\mu)$. As $d(\gamma_n, e)$ tends to infinity, $H(e, \gamma_n|R_\mu)$ converges to zero and so $\tilde{K}_\xi(\gamma_n)$ tends to infinity. Lemma 5.9 shows that $\lambda_{R_\mu}(\gamma_n^{-1}\xi) = \tilde{K}_\xi(\gamma_n)\lambda_{R_\mu}(\xi)$, which goes to infinity. This is a contradiction, because λ_{R_μ} is a probability measure. $\qquad\square$

In the following, it is simpler to see the measure λ_{R_μ} as a measure on the Bowditch boundary that gives full mass to the set of conical limit points. We used the symmetrized Martin boundary to prove Lemmas 5.9 and 5.10. In the hyperbolic setting, using results of Coornaert (see [Coo93]), conformal measures for hyperbolic distances are ergodic. Actually, [Coo93] only deals with geodesic distances and this was generalized by [BHM11] for distances that are hyperbolic and quasi-isometric to a word distance, such as the Green distance as long as weak Ancona inequalities hold (this is also proved in [BHM11]). Comparing a geodesic distance with the Green distance is more difficult here and we need another approach. We use instead the same strategy as in [MYJ20, Theorem 4.1] to prove Lemma 5.8, which generalizes Coornaert's result.

Before proving this proposition, let us introduce some notions of geometric measure theory from [MYJ20] and some constructions of [DG20] and [Yan22]. Let Λ be a metric space. A covering relation $\mathcal{C}$ is a subset of the set of all pairs (ξ, S) such that $\xi \in S \subset \Lambda$. A covering relation $\mathcal{C}$ is said to be fine at $\xi \in \Lambda$ if there exists a sequence S_n of subsets of Λ with $(\xi, S_n) \in \mathcal{C}$ and such that the diameter of S_n converges to zero. Let $\mathcal{C}$ be a covering relation. For any measurable subset $E \subset \Lambda$, define $\mathcal{C}(E)$ to be the collection of subsets $S \subset \Lambda$ such that $(\xi, S) \in \mathcal{C}$ for some $\xi \in E$. A covering relation $\mathcal{C}$ is said to be a Vitali relation for a finite measure κ on Λ if it is fine at every point of Λ and if the following holds: if $\mathcal{C}' \subset \mathcal{C}$ is fine at every point of Λ then for every measurable subset E, $\mathcal{C}'(E)$ has a countable disjoint subfamily $\{S_n\}$ such that $\kappa(E \setminus \bigcup_{n=1}^{\infty} S_n) = 0$. We use the letter $\mathcal{V}$ to denote a Vitali relation in the following.

Recall that an (η_1, η_2)-transition point on a geodesic α in the Cayley graph $\mathrm{Cay}(\Gamma, S)$ is a point γ such that for any coset $\gamma_0\mathcal{H}$ of a parabolic subgroup, the part of α consisting of points at distance at most η_2 from γ is not contained in the η_1-neighborhood of $\gamma_0\mathcal{H}$. Let ξ be a conical limit point. Following Yang [Yan22], the partial shadow $\Omega_{\eta_1, \eta_2}(\gamma)$ at $\gamma \in \Gamma$ is the set of points ξ in the Bowditch boundary such that there is a geodesic ray $[e, \xi)$ in $\mathrm{Cay}(\Gamma, S)$ containing an (η_1, η_2)-transition point in the ball $B(\gamma, 2\eta_2)$.

We define the following relation $\mathcal{V}_{\eta_1, \eta_2}$ on the Bowditch boundary. For ξ parabolic, we declare $(\xi, \{\xi\}) \in \mathcal{V}_{\eta_1, \eta_2}$. For ξ conical, we declare $(\xi, \Omega_{\eta_1, \eta_2}(\gamma)) \in \mathcal{V}_{\eta_1, \eta_2}$ whenever $\xi \in \Omega_{\eta_1, \eta_2}(\gamma)$. According to [DG20, Proposition 3.3], the relation $\mathcal{V}_{\eta_1, \eta_2}$ is fine at every limit point in the Bowditch boundary.

Let $\gamma \in \Gamma$ and let $\eta_1, \eta_2 > 0$. Consider a neighborhood U of $\Omega_{\eta_1, \eta_2}(\gamma)$ in the Bowditch compactification. One can choose U such that for any point ξ in U, γ is within a bounded distance of a transition point on a geodesic from e to ξ. According to Lemma 2.2, γ is within a bounded distance of a point on a relative geodesic from e to ξ. In particular, weak relative Ancona inequalities

imply that there exists a constant C (depending on η_2) such that for N large enough,

$$\tilde{\lambda}_N(\Omega_{\eta_1,2\eta_2}(\gamma)) \leq C\tilde{\lambda}_N(\Omega_{\eta_1,\eta_2}(\gamma)),$$

so that

$$\lambda_{R_\mu}(\Omega_{\eta_1,2\eta_2}(\gamma)) \leq C\lambda_{R_\mu}(\Omega_{\eta_1,\eta_2}(\gamma)).$$

The measure λ_{R_μ} on the Bowditch boundary gives full measure to the set of conical limit points. Thus, [DG20, Proposition 3.4] shows that the relation $\mathcal{V}_{\eta_1,\eta_2}$ is a Vitali relation for λ_{R_μ}.

To prove Lemma 5.8, we need the following two results.

LEMMA 5.11 [MYJ20, Theorem 4.2]. *Let E be a measurable subset of the Bowditch boundary. Then, for λ_{R_μ}-almost every point $\xi \in E$, one has*

$$\frac{\lambda_{R_\mu}(E \cap S_n)}{\lambda_{R_\mu}(S_n)} \xrightarrow[n\to\infty]{} 1$$

for every sequence $\{S_n\}$ such that $(\xi, S_n) \in \mathcal{V}_{\eta_1,\eta_2}$ for all n and such that the diameter of S_n converges to 0 when n tends to infinity.

For the second result, we need to choose a distance on the Bowditch boundary. To simplify the argument, we choose the shortcut distance, so that the following holds. We refer to [Yan22, § 2.4] for the definition of the shortcut distance.

LEMMA 5.12. *For every $\epsilon > 0$ and $\eta_1 > 0$, there exists $\eta_2 > 0$ such that for every $\gamma \in \Gamma$, the diameter of the complement of $\gamma^{-1}\Omega_{\eta_1,\eta_2}(\gamma)$ is smaller than ϵ.*

Proof. This is exactly the content of the remark inside the proof of [Yan22, Lemma 4.1]. $\square$

Actually, the choice of the distance is not relevant and with a bit of work, one could have proved the same result for a visual distance on the Bowditch boundary, adapting the arguments of [MYJ20, Proposition 2.10]. We only chose the shortcut distance to avoid reproving this technical claim.

We can finally prove Lemma 5.8. Everything is settled so that we can easily adapt the arguments of [MYJ20, Theorem 4.1]. We still rewrite the proof for convenience.

Proof. Denote by $\partial_B\Gamma$ the Bowditch boundary of Γ. Consider a Γ-invariant measurable subset E of the Bowditch boundary, such that $\lambda_{R_\mu}(E) > 0$. Assume, by contradiction, that $\lambda_{R_\mu}(E) < 1$. We fix $\epsilon > 0$ arbitrarily small. For technical reasons, we assume that $\lambda_{R_\mu}(\partial_B\Gamma) \geq 2\epsilon$, that is, $\epsilon \leq 1/2$.

According to Lemma 5.11, if η_2 is large enough, for λ_{R_μ}-almost every ξ in E^c,

$$\frac{\lambda_{R_\mu}(E \cap \Omega_{\eta_1,\eta_2}(\gamma_n))}{\lambda_{R_\mu}(\Omega_{\eta_1,\eta_2}(\gamma_n))} \xrightarrow[n\to\infty]{} 0,$$

whenever γ_n converges to ξ along a relative geodesic ray. Take such a ξ and such a sequence γ_n. Up to taking a subsequence, we can assume that γ_n^{-1} converges to a point ζ in the Bowditch boundary. According to Lemma 5.10, $\lambda_{R_\mu}(\zeta) = 0$.

Then, because the Bowditch boundary is compact, there exists $\delta > 0$ (not depending on ζ) such that the ball centered at ζ of radius δ has measure at most ϵ. Moreover, according to Lemma 5.12, there exists $\eta_2 > 0$ such that the diameter of the complement of $\gamma_n^{-1}\Omega_{\eta_1,\eta_2}(\gamma_n)$ is smaller than δ. Fixing such an $\eta_2 > 0$, for large enough n, we have that $\zeta \notin \gamma_n^{-1}\Omega_{\eta_1,\eta_2}(\gamma_n)$, so that the complement of $\gamma_n^{-1}\Omega_{\eta_1,\eta_2}(\gamma_n)$ is contained in the ball of center ζ and of radius δ.

In particular,

$$\lambda_{R_\mu}(\partial_B \Gamma \setminus \gamma_n^{-1} \Omega_{\eta_1, \eta_2}(\gamma_n)) \le \epsilon.$$

As $\lambda_{R_\mu}(\partial_B \Gamma) \ge 2\epsilon$, we thus also have

$$\lambda_{R_\mu}(\gamma_n^{-1} \Omega_{\eta_1, \eta_2}(\gamma_n)) \ge \epsilon.$$

As E is Γ-invariant, we have

$$\lambda_{R_\mu}(E \cap \gamma_n^{-1} \Omega_{\eta_1, \eta_2}(\gamma_n)) = (L_{\gamma_n})_* \lambda_{R_\mu}(E \cap \Omega_{\eta_1, \eta_2}(\gamma_n)).$$

Weak relative Ancona inequalities show that if $\xi' \in \Omega_{\eta_1, \eta_2}(\gamma_n)$, then

$$\frac{1}{C(\eta_2)} \frac{1}{H(e, \gamma_n | R_\mu)} \le \tilde{K}_{\xi'}(\gamma_n) \le C(\eta_2) \frac{1}{H(e, \gamma_n | R_\mu)}.$$

In particular, Lemma 5.9 shows that

$$\lambda_{R_\mu}(E \cap \gamma_n^{-1} \Omega_{\eta_1, \eta_2}(\gamma_n)) \le C(\eta_2) \frac{1}{H(e, \gamma_n | R_\mu)} \lambda_{R_\mu}(E \cap \Omega_{\eta_1, \eta_2}(\gamma_n)).$$

Similarly, we have

$$\lambda_{R_\mu}(\gamma_n^{-1} \Omega_{\eta_1, \eta_2}(\gamma_n)) \ge \frac{1}{C(\eta_2)} \frac{1}{H(e, \gamma_n | R_\mu)} \lambda_{R_\mu}(\Omega_{\eta_1, \eta_2}(\gamma_n)).$$

This proves that

$$\frac{\lambda_{R_\mu}(E \cap \gamma_n^{-1} \Omega_{\eta_1, \eta_2}(\gamma_n))}{\lambda_{R_\mu}(\gamma_n^{-1} \Omega_{\eta_1, \eta_2}(\gamma_n))} \le C'(\eta_2) \frac{\lambda_{R_\mu}(E \cap \Omega_{\eta_1, \eta_2}(\gamma_n))}{\lambda_{R_\mu}(\Omega_{\eta_1, \eta_2}(\gamma_n))}.$$

The right-hand side of this last equation converges to zero when n tends to infinity. This proves that $\lambda_{R_\mu}(E \cap \gamma_n^{-1} \Omega_{\eta_1, \eta_2}(\gamma_n))$ converges to zero when n tends to infinity, because $\lambda_{R_\mu}(\gamma_n^{-1} \Omega_{\eta_1, \eta_2}(\gamma_n)) \ge \epsilon$. Finally, recall that $\lambda_{R_\mu}(\partial_B \Gamma \setminus \gamma_n^{-1} \Omega_{\eta_1, \eta_2}(\gamma_n)) \le \epsilon$, so that $\lambda_{R_\mu}(E) \le 2\epsilon$. As ϵ is arbitrarily small, we get that $\lambda_{R_\mu}(E) = 0$, which is a contradiction. $\square$

5.4 Proof of step 4

We prove here the following.

LEMMA 5.13. *Let $c_j = \int \varphi_r \, dm_j$. With the notation as previously, we have*

$$\lambda_{R_\mu}(U(c_j)) > 0.$$

This lemma is a consequence of the two following results. We use the notation $\alpha_j = \sum_i \nu_j^{(i)}$, where the measures $\nu_j^{(i)}$ are given by Theorem 3.3.

LEMMA 5.14. *The measure $T_* \alpha_j$ is absolutely continuous with respect to the measure α_j.*

Proof. Since by Lemma 4.6 the maximal pressure at the spectral radius is zero, we have

$$\mathcal{L}_{R_\mu}^* \alpha_j = \alpha_j.$$

Denote by $[x_1, \ldots, x_n]$ the cylinder consisting of elements of $\overline{\Sigma}_A$ starting with the symbols $x_1, \ldots, x_n$. Then, we have

$$\alpha_j([x_1, \ldots, x_n]) = \alpha_j(\mathcal{L}_{R_\mu} 1_{[x_1, \ldots, x_n]}).$$

Moreover, weak Ancona inequalities show that

$$\alpha_j(\mathcal{L}_{R_\mu} 1_{[x_1, \ldots, x_n]}) \le C H(e, x_1 | R_\mu) \alpha_j([x_2, \ldots, x_n]).$$

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Now, $T^{-1}[x_2, \ldots, x_n]$ is contained in the union of cylinders of the form $[x_1, \ldots, x_n]$, where $x_1 \in S$ or $x_1 \in \mathcal{H}$ for some parabolic subgroup. According to Corollary 2.10, the sum $\sum_{\sigma \in \mathcal{H}} H(e, \sigma | r)$ is uniformly bounded, so that

$$\alpha_j(T^{-1}[x_2, \ldots, x_n]) \leq C\alpha_j([x_2, \ldots, x_n]).$$

This is true for any cylinder $[x_2, \ldots, x_n]$. It follows that $\alpha_j(T^{-1}E) \leq C\alpha_j(E)$ for any measurable set $E \in \overline{\Sigma}_A$. $\qquad\square$

Recall that ϕ maps paths of $\overline{\Sigma}_A$ that start with v_* to $\Gamma \cup \partial\Gamma$. Let $\alpha_j(\cdot \cap E_*)$ be the measure α_j restricted to paths that start at v_*. Then, $\phi_* \alpha_j(\cdot \cap E_*)$ is a measure on $\partial\hat{\Gamma}$.

LEMMA 5.15. *The measure $\phi_* \alpha_j(\cdot \cap E_*)$ is absolutely continuous with respect to the measure λ_{R_μ}.*

Proof. The sequence of measures $\tilde{\lambda}_{N_k}$ weakly converges to λ_{R_μ} in the Bowditch compactification. Recall that according to Lemma 5.7, it also weakly converges to λ_{R_μ} in $\Gamma \cup \partial\hat{\Gamma}$.

If f is a function defined on $\partial\hat{\Gamma}$, then $f \circ \phi$ is defined on $\overline{\Sigma}_A$ and it vanishes on the complement of E_*. We see $\partial\hat{\Gamma}$ as the Gromov boundary of the hyperbolic space $\hat{\Gamma}$ and we endow $\partial\hat{\Gamma}$ with a visual distance d_v, as in [GH90]. Then, there exists $\epsilon > 0$ such that $d_v(\xi, \xi') \leq e^{-\epsilon(\xi, \xi')_e}$, where $(\xi, \xi')_e$ is the Gromov product of ξ and ξ', based at e, see [GH90, Proposition 7.3.10]. In particular, if f is a bounded locally Hölder continuous function on $\partial\hat{\Gamma}$, then $f \circ \phi$ is in $\mathcal{B}_{\rho,\beta}$. Theorem 3.3 shows that $\mathcal{L}_{R_\mu}^{np+q}(f \circ \phi)(\emptyset)$ converges to $\sum_{j=1}^{k} \sum_{i=1}^{p_j} h_j^{(i)} \int (f \circ \phi) \, d\nu_j^{((i-n) \bmod p_j)}$. Also note that

$$\sum_{\gamma \in \hat{S}_n} H(e, \gamma | R_\mu) = H(e, e | R_\mu) \mathcal{L}_{R_\mu}^n (1_{E_*} f \circ \phi)(\emptyset).$$

Let $\beta_j = \phi_* \alpha_j(\cdot \cap E_*)$. As the functions $h_j^{(i)}$ are bounded away from zero and infinity on the support of $\nu_j^{(i)}$, this proves that for any bounded locally hölder continuous function f on $\partial\hat{\Gamma}$, we have

$$\beta_j(f) \leq C\lambda_{R_\mu}(f). \tag{28}$$

Bounded Hölder continuous functions are not dense in bounded continuous functions for the supremum norm, but they are dense in the space of integrable functions for the L^1-norm, see [AB06, Corollary 3.14]. Let $A \subset \partial\hat{\Gamma}$ be any measurable set. We want to prove that $\beta_j(A) \leq C\lambda_{R_\mu}(A)$. Let $\epsilon > 0$. There exists a bounded locally Hölder continuous function f such that

$$\|f - 1_A\|_{L^1(\partial\hat{\Gamma}, \beta_j + \lambda_{R_\mu})} \leq \epsilon.$$

Then, $\beta_j(A) \leq \beta_j(|f|) + \epsilon$. As $|f|$ still is locally Hölder continuous, (28) shows that

$$\beta_j(A) \leq C\lambda_{R_\mu}(|f|) + \epsilon \leq C\lambda_{R_\mu}(A) + (1 + C)\epsilon.$$

As ϵ is arbitrary, this concludes the proof. $\qquad\square$

Those two lemmas allow us to conclude the proof of Lemma 5.13 We only outline the proof and refer to the end of the proof of [Gou14, Proposition 3.16] for the details. The probability measure $dm_j = (1/p_j) \sum_{i=1}^{p_j} h_j^{(i)} \, d\nu_j^{(i)}$ is invariant and ergodic for the shift T (see, for example, [Sar99, Lemma 11]). Fix r close enough to R_μ so that the conclusions of Theorem 3.6 hold. Let O_j be the set of points where the Birkhoff sums $(1/n) \sum_{1 \leq k \leq n} \varphi_r \circ T^k$ converge to $c_j = \int \varphi_r dm_j$. By the Birkhoff ergodic theorem, $m_j(O_j) = 1$. We first deduce that $\alpha_j(O_j \cap \overline{\Sigma}_{A,j}) > 0$. Using that $T_* \alpha_j$ is absolutely continuous with respect to α_j, we then deduce that

$\alpha_j(O_j) > 0$. Then, using that $\phi_* \alpha_j(\cdot \cap E_*)$ is absolutely continuous with respect to λ_{R_μ}, we deduce that $\lambda_{R_\mu}(\phi(O_j \cap E_*)) > 0$. Finally, direct computation shows that $\phi(O_j \cap E_*) \subset U(c_j)$. This proves that $\lambda_{R_\mu}(U(c_j)) > 0$.

We now conclude the proof of Lemma 5.2.

Proof. As $U(c_j)$ is invariant and λ_{R_μ} is ergodic, we thus have $\lambda_{R_\mu}(U(c_j)) = 1$. This holds for all j, so we finally obtain that every $U(c_j)$ intersect, so that c_j does not depend on j. $\qquad \square$

5.5 End of the proof of Proposition 5.1

We want to prove that $\tilde{P}_j(\varphi_r)/P(r)$ tends to one when r tends to R_μ. According to Proposition 4.12, the assumptions of Proposition 3.7 are satisfied, so that

$$e^{\tilde{P}_j(r)} - 1 = \int (e^{\varphi_r - \varphi_{R_\mu}} - 1) \, dm_j$$

$$+ \int (e^{\varphi_r - \varphi_{R_\mu}} - 1) \frac{1}{p_j} \left(\sum_{i=0}^{p_j-1} h_j^{(i)} - \tilde{h}_{j,r}^{(i)} \right) d\nu_j^{(i)}. \tag{29}$$

Lemma 4.5 and (23) show that $|P(r)|$ has order of magnitude $\sqrt{R_\mu - r}$. We actually show that

$$e^{\tilde{P}_j(r)} - 1 = \int (\varphi_r - \varphi_{R_\mu}) \, dm_j + o\big(\sqrt{R_\mu - r}\big).$$

We then combine this with Lemma 5.2 to complete the proof of Proposition 5.1.

Lemma 5.16. *We have*

$$\int (e^{\varphi_r - \varphi_{R_\mu}} - 1) \frac{1}{p_j} \left(\sum_{i=0}^{p_j-1} h_j^{(i)} - \tilde{h}_{j,r}^{(i)} \right) d\nu_j^{(i)} = o\big(\sqrt{R_\mu - r}\big).$$

Proof. According to Theorem 3.3, the functions $h_j^{(i)}$ are bounded away from zero and infinity on the support of $\nu_j^{(i)}$. We can thus replace $\nu_j^{(i)}$ with m_j, which is itself dominated by the measure m. Thus, we just need to show that for every i,

$$\frac{1}{\sqrt{R_\mu - r}} \int \big| e^{\varphi_r - \varphi_{R_\mu}} - 1 \big| \big| h_j^{(i)} - \tilde{h}_{j,r}^{(i)} \big| \, dm \xrightarrow[r \to R_\mu]{} 0.$$

Let r_n be a sequence converging to R_μ such that

$$\frac{1}{\sqrt{R_\mu - r_n}} \int \big| e^{\varphi_{r_n} - \varphi_{R_\mu}} - 1 \big| \big| h_j^{(i)} - \tilde{h}_{j,r_n}^{(i)} \big| \, dm \xrightarrow[n \to +\infty]{} \alpha \in [0, +\infty].$$

According to Corollary 4.13,

$$\int \big| h_j^{(i)} - \tilde{h}_{j,r_n}^{(i)} \big| \, dm \xrightarrow[n \to +\infty]{} 0,$$

so up to taking a subsequence, $\big| h_j^{(i)} - \tilde{h}_{j,r_n}^{(i)} \big|$ converges to zero m-almost everywhere.

We now focus on $1/\sqrt{R_\mu - r_n} \big| e^{\varphi_{r_n} - \varphi_{R_\mu}} - 1 \big|$. We show that

$$\frac{1}{\sqrt{R_\mu - r}} \int \big| e^{\varphi_r - \varphi_{R_\mu}} - 1 \big| \, dm \lesssim 1. \tag{30}$$

Differentiating the expression $e^{\varphi_r - \varphi_{R_\mu}}(x)$, we obtain

$$\left(\frac{d}{dr} \varphi_r(x) \right) e^{\varphi_r - \varphi_{R_\mu}}(x).$$

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Weak relative Ancona inequalities yield

$$e^{\varphi_r - \varphi_{R_\mu}}(x) = \frac{H(e, x|r)/H(x_1, x|r)}{H(e, x|R_\mu)/H(x_1, x|R_\mu)} \lesssim \frac{H(e, x_1|r)}{H(e, x_1|R_\mu)} \lesssim 1. \tag{31}$$

Thus, Lemma 4.8 shows that

$$\left| \frac{d}{dr}(e^{\varphi_r - \varphi_{R_\mu}}(x)) \right| \lesssim \frac{1}{\sqrt{R_\mu - r}} \varphi(x). \tag{32}$$

Integrating this inequality, we obtain

$$\left| e^{\varphi_r - \varphi_{R_\mu}}(x) - 1 \right| \lesssim \varphi(x)\sqrt{R_\mu - r}. \tag{33}$$

Integrating with respect to m, we finally obtain (30). In addition,

$$\frac{1}{\sqrt{R_\mu - r_n}} \left| e^{\varphi_{r_n} - \varphi_{R_\mu}} - 1 \right| \left| h_j^{(i)} - \tilde{h}_{j,r_n}^{(i)} \right| \lesssim \varphi \left| h_j^{(i)} - \tilde{h}_{j,r_n}^{(i)} \right|,$$

so that

$$\frac{1}{\sqrt{R_\mu - r_n}} \left| e^{\varphi_{r_n} - \varphi_{R_\mu}} - 1 \right| \left| h_j^{(i)} - \tilde{h}_{j,r_n}^{(i)} \right|$$

converges to zero m-almost everywhere.

Finally, we deduce from Corollary 4.13 that $\left| h_j^{(i)} - \tilde{h}_{j,r_n}^{(i)} \right|$ is uniformly bounded, hence

$$\frac{1}{\sqrt{R_\mu - r_n}} \left| e^{\varphi_{r_n} - \varphi_{R_\mu}} - 1 \right| \left| h_j^{(i)} - \tilde{h}_{j,r_n}^{(i)} \right| \lesssim \varphi.$$

We apply the dominated convergence theorem, so that

$$\frac{1}{\sqrt{R_\mu - r_n}} \int \left| e^{\varphi_{r_n} - \varphi_{R_\mu}} - 1 \right| \left| h_j^{(i)} - \tilde{h}_{j,r_n}^{(i)} \right| dm \xrightarrow[n \to +\infty]{} 0.$$

In other words, $\alpha = 0$, which concludes the proof. $\qquad\qquad\square$

LEMMA 5.17. *We have*

$$\int \left((e^{\varphi_r - \varphi_{R_\mu}} - 1) - (\varphi_r - \varphi_{R_\mu}) \right) dm = o\big(\sqrt{R_\mu - r} \big).$$

Proof. Differentiating the integrand, we obtain

$$\left(\frac{d}{dr} \varphi_r \right) (e^{\varphi_r - \varphi_{R_\mu}} - 1).$$

Fix $R \leq R_\mu$ and let

$$\overline{\varphi}_R = \sup_{R \leq r \leq R_\mu} \left| e^{\varphi_r - \varphi_{R_\mu}} - 1 \right|.$$

For every $R \leq r \leq R_\mu$, according to Lemma 4.8,

$$\left| \frac{d}{dr} \left[(e^{\varphi_r - \varphi_{R_\mu}} - 1) - (\varphi_r - \varphi_{R_\mu}) \right] \right| \leq \varphi \overline{\varphi}_R \frac{1}{\sqrt{R_\mu - r}}.$$

Integrating this inequality over r varying between R and R_μ, we obtain

$$\left| (e^{\varphi_R - \varphi_{R_\mu}} - 1) - (\varphi_R - \varphi_{R_\mu}) \right| \leq \varphi \overline{\varphi}_R \sqrt{R_\mu - R}.$$

It is thus enough to prove that

$$\int \varphi \overline{\varphi}_R \, dm \xrightarrow[R \to R_\mu]{} 0.$$

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Consider a sequence r_k converging to R_μ such that

$$\int \varphi \overline{\varphi}_{r_k} \, dm \xrightarrow[k \to \infty]{} \alpha.$$

According to (33), $\int \overline{\varphi}_R \, dm$ converges to zero, so up to taking a subsequence, $\overline{\varphi}_{r_k}$ converges to zero m-almost everywhere. Hence, $\varphi \overline{\varphi}_{r_k}$ converges to zero m-almost everywhere. In addition, according to (31), $\overline{\varphi}_r$ is uniformly bounded. We apply the dominated convergence theorem, so that

$$\int \varphi \overline{\varphi}_{r_k} \, dm \xrightarrow[R \to R_\mu]{} 0.$$

In other words, $\alpha = 0$, which concludes the proof. $\qquad\square$

We can now prove Proposition 5.1.

Proof. We combine Lemmas 5.16 and 5.17 and (29) to show that

$$e^{\tilde{P}_j(r)} - 1 = \int (\varphi_r - \varphi_{R_\mu}) \, dm_j + o\big(\sqrt{R_\mu - r}\,\big).$$

According to Lemma 5.2, the integral in the right member does not depend on j. Choose j' so that the pressure is maximal. Then, for every j,

$$e^{\tilde{P}_j(r)} - 1 = e^{P(r)} - 1 + o\big(\sqrt{R_\mu - r}\,\big),$$

hence,

$$e^{\tilde{P}_j(r)} - 1 = P(r) + o(P(r)) + o\big(\sqrt{R_\mu - r}\,\big).$$

We deduce from Lemma 4.5 and from (23) that $|P(r)|/\sqrt{R_\mu - r}$ is bounded away from zero and infinity. Thus,

$$e^{\tilde{P}_j(r)} - 1 = P(r) + o(P(r)).$$

Consequently, for every j, $\tilde{P}_j(r)$ converges to zero as r tends to R_μ, so that

$$\tilde{P}_j(r) \sim e^{\tilde{P}_j(r)} - 1, \quad r \to R_\mu.$$

This also proves that

$$\tilde{P}_j(r) \sim P(r), \quad r \to R_\mu,$$

which concludes the proof. $\qquad\square$

6. Asymptotic of the second derivative of the Green function

Our goal here is to prove the following proposition. We still assume that μ is not spectrally degenerate.

PROPOSITION 6.1. *When $r \to R_\mu$, we have*

$$I^{(2)}(r) = \xi I^{(1)}(r)^3 + O(I^{(1)}(r)^2),$$

for some $\xi > 0$.

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We introduce some notation. We define for $r < R_\mu$ the function

$$\Phi_r(\gamma) = \frac{\sum_{\gamma'} G(e, \gamma'|r) G(\gamma', \gamma|r)}{G(e, \gamma|r)}.$$

By definition,

$$I^{(2)}(r) = \sum_{n \geq 0} \sum_{\gamma \in \hat{S}^n} H(e, \gamma|r) \Phi_r(\gamma) = H(e, e|r) \sum_{n \geq 0} \mathcal{L}_r^n(1_{E_*} \Phi_r \circ \phi)(\emptyset).$$

According to Theorem 3.3, for any $f : \Gamma \cup \partial\hat{\Gamma} \to \mathbb{R}$ such that $f \circ \phi : \overline{\Sigma}_A \to \mathbb{R}$ is in $H_{\rho,\beta}$,

$$\sum_{n \geq 0} \mathcal{L}_r^n(1_{E_*} f \circ \phi)(\emptyset) = \sum_{j=1}^{k} \frac{c_f(j, r)}{|\tilde{P}_j(\varphi_r)|} + O(1), r \to R_\mu.$$

In addition, according to (24),

$$I^{(1)}(r) \sim \frac{\xi(r)}{P(r)}, \quad r \to R_\mu.$$

Finally, Proposition 5.1, shows that $|\tilde{P}_j(\varphi_r)|/P(r)$ converges to one. Hence,

$$\sum_{n \geq 0} \mathcal{L}_r^n(1_{E_*} f \circ \phi)(\emptyset) \sim I^{(1)}(r) c_f, \quad r \to R_\mu, \tag{34}$$

where c_f only depends on f. In other words $(1/I^{(1)}(r)) \sum_{n \geq 0} \mathcal{L}_r^n(1_{E_*} f \circ \phi)(\emptyset)$ converges. However, $\Phi_r \circ \phi \notin H_{\rho,\beta}$. The goal of the next subsection is to transform Φ_r in order to apply (34).

6.1 A partition of unity

We start with a rough study of Φ_r. Let $\gamma \in \Gamma$ and let $[e, \gamma] = (e, \gamma_1, \ldots, \gamma_{\hat{d}(e,\gamma)-1}, \gamma)$ be a relative geodesic from e to γ. For every $k \leq \hat{d}(e, \gamma)$, denote by Γ_k the set of $\gamma' \in \Gamma$ whose projection on $[e, \gamma]$ is at γ_k. If there are more than one such projections, we choose the closest to γ. Also denote by $\tilde{\gamma}_k$ the projection of γ' on the union $\mathcal{H}_k$ of parabolic subgroups containing $\gamma_{k-1}^{-1} \gamma_k$. Lemma 2.5 shows that any relative geodesic from e to γ' passes within a bounded distance of γ_{k-1}. In addition, [Sis13, Lemma 1.13 (1)] shows that the exit point from $\mathcal{H}_k$ of any such relative geodesic is within a bounded distance of $\tilde{\gamma}_k$. Thus, any relative geodesic from e to γ' passes first within a bounded distance of γ_{k-1} and then within a bounded distance of $\tilde{\gamma}_k$. In addition, any relative geodesic from γ' to γ passes within a bounded distance of $\tilde{\gamma}_k$, then of γ_k. Weak relative Ancona inequalities imply that for every $\gamma' \in \Gamma_k$,

$$\frac{G(e, \gamma'|r) G(\gamma', \gamma|r)}{G(e, \gamma|r)}$$

$$\asymp \frac{G(e, \gamma_{k-1}|r) G(\gamma_{k-1}, \tilde{\gamma}_k|r) G(\tilde{\gamma}_k, \gamma'|r) G(\gamma', \tilde{\gamma}_k|r) G(\tilde{\gamma}_k, \gamma_k|r) G(\gamma_k, \gamma|r)}{G(e, \gamma_{k-1}|r) G(\gamma_{k-1}, \gamma_k|r) G(\gamma_k, \gamma|r)}.$$

We thus obtain that

$$\frac{G(e, \gamma'|r) G(\gamma', \gamma|r)}{G(e, \gamma|r)} \asymp \frac{G(\gamma_{k-1}, \tilde{\gamma}_k|r) G(\tilde{\gamma}_k, \gamma_k|r)}{G(\gamma_{k-1}, \gamma_k|r)} G(\tilde{\gamma}_k, \gamma'|r) G(\gamma', \tilde{\gamma}_k|r).$$

We then sum over all $\gamma' \in \Gamma_k$. Let $\mathcal{H}_{\gamma_k}$ be the union of all parabolic subgroups in Ω_0 containing $\gamma_{k-1}^{-1} \gamma_k$. Then $\gamma_{k-1}^{-1} \tilde{\gamma}_k \in \mathcal{H}_{\gamma_k}$. We can decompose the sum over $\gamma' \in \Gamma_k$ according to the projection on $\tilde{\gamma}_k \in \mathcal{H}_{\gamma_k}$. Bounding $\sum G(\tilde{\gamma}_k, \gamma'|r) G(\gamma', \tilde{\gamma}_k|r)$ by $I^{(1)}(r)$, where $\tilde{\gamma}_k$ is fixed and the sum is over

all γ' projecting on $\tilde{\gamma}_k$, we finally obtain

$$\sum_{\gamma' \in \Gamma_k} \frac{G(e, \gamma'|r)G(\gamma', \gamma|r)}{G(e, \gamma|r)} \lesssim I_1(r) \sum_{\tilde{\gamma}_k \in \mathcal{H}_{\gamma_k}} \frac{G(\gamma_{k-1}, \tilde{\gamma}_k|r)G(\tilde{\gamma}_k, \gamma_k|r)}{G(\gamma_{k-1}, \gamma_k|r)}.$$

We then construct a function Υ_r as follows. For every $\gamma \in \Gamma$, we choose a relative geodesic from e to γ, using the automaton $\mathcal{G}$. Let γ_1 be the first point after e on this relative geodesic. Note that γ_1 coincides with the first increment σ_1 on this relative geodesic. In general, we denote by $\sigma_k = \gamma_{k-1}^{-1}\gamma_k$ the kth increment. In addition, let $\mathcal{H}_{\sigma_1}$ be the union of all parabolic subgroups in Ω_0 containing σ_1. We set

$$\Upsilon_r(\gamma) = \sum_{\sigma \in \mathcal{H}_{\sigma_1}} \frac{G(e, \sigma|r)G(\sigma, \sigma_1|r)}{G(e, \sigma_1|r)}.$$

This function Υ_r only depends on the first element of the relative geodesic $[e, \gamma]$. In other words, the function $\Upsilon_r \circ \phi(x)$ only depends on the first symbol of $x \in \overline{\Sigma}_A$. The estimate above yields

$$\Phi_r(\gamma) \lesssim I_1(r) \sum_{k=0}^{\hat{d}(e,\gamma)-1} \Upsilon_r \circ T^k([e, \gamma]), \tag{35}$$

where T is the left shift on relative geodesic, that is, $T([e, \gamma]) = (e, \gamma_1^{-1}\gamma_2, \ldots, \gamma_1^{-1}\gamma)$. Note that Υ_r is not bounded. Indeed, assuming that σ_1 is only in one parabolic subgroup $\mathcal{H}_1$ to simplify, then $\Upsilon_r(\gamma)$ is essentially given by

$$\sum_{\sigma \in \mathcal{H}_{\sigma_1}} \frac{G_{1,r}^{(1)}(e, \sigma_1)}{G(e, \sigma_1|r)}.$$

This quantity is not bounded.

However, to prove Proposition 6.1, we need to estimate $\mathcal{L}_r(\Upsilon_r \circ \phi)(x)$. Recall that X_x^1 is the set of symbols σ that can precede x in Σ_A. Seeing x and σ as elements of Γ,

$$\mathcal{L}_r(\Upsilon_r \circ \phi)(x) = \sum_{\sigma \in X_\gamma^1} \frac{H(e, \sigma x|r)}{H(e, x|r)} \sum_{\sigma' \in \mathcal{H}_\sigma} \frac{G(e, \sigma'|r)G(\sigma', \sigma|r)}{G(e, \sigma|r)}.$$

Therefore, weak relative Ancona inequalities show that

$$\mathcal{L}_r(\Upsilon_r \circ \phi)(x) \lesssim \sum_j \sum_{\sigma, \sigma' \in \mathcal{H}_j} G(e, \sigma'|r)G(\sigma', \sigma|r)G(\sigma, e|r).$$

We rewrite this as

$$\mathcal{L}_r(\Upsilon_r \circ \phi)(x) \lesssim \sum_j I_j^{(2)}(r).$$

As μ is not spectrally degenerate, we obtain

$$\mathcal{L}_r(\Upsilon_r \circ \phi)(x) \lesssim 1. \tag{36}$$

We only gave a rough estimate. To obtain a precise asymptotic, we replace the decomposition of Γ into subsets Γ_k as previously by a continuous decomposition, using a partition of unity. We construct such a partition of unity adapting the arguments of [GL13, Lemma 8.5].

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We first introduce the following terminology. Consider a relative geodesic α that we write $\alpha = (\alpha_{-m}, \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_n)$. Let $\hat{l}(\alpha) = \hat{d}(\alpha_{-m}, \alpha_n)$ be the relative length of α (here $n + m$) and let $l(\alpha)$ be its total length, defined by

$$l(\alpha) = \sum_{k=-m+1}^{n} d(\alpha_{k-1}, \alpha_k).$$

Note that if α, α' are two relative geodesic with the same endpoints, we do not have $l(\alpha) = l(\alpha')$ in general. However, the distance formula given by [Sis13, Theorem 3.1] shows that

$$\frac{1}{\lambda_1} d(\alpha_{-m}, \alpha_n) - c_1 \leq l(\alpha) \leq \lambda_1 d(\alpha_{-m}, \alpha_n) + c_1$$

and so

$$\frac{1}{\lambda_2} l(\alpha) - c_2 \leq l(\alpha') \leq \lambda_2 l(\alpha) + c_2.$$

Our goal is to construct a partition of unity κ_α associated to such a relative geodesic α. Write $\alpha = (\alpha_{-m}, \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_n)$ and suppose that $\alpha_0 = e$. To simplify, let α_- and α_+ be the endpoints of α, that is, $\alpha_- = \alpha_{-m}$ and $\alpha_+ = \alpha_n$. Denote by α_l the sub-relative geodesic of α from α_- to e and by α_r the sub-relative geodesic from α_1 to α_+.

Consider two constants K_1 and K_2 that we choose later. Assume that $l(\alpha_l) \geq 2K_1$ and $l(\alpha_r) \geq 2K_1$. Denote by $A(K_1)$ the set of $\gamma \in \Gamma$ such that:

(i) there either exists a relative geodesic from γ to α_+ whose distance from α_1 is at least K_1; or

(ii) there exists a relative geodesic from α_- to γ whose distance from e is at least K_1.

Also denote by $B(K_2)$ the set of $\gamma \in \Gamma$ such that:

(i) any relative geodesic from γ to α_+ passes within K_2 of α_1;
(ii) and any relative geodesic from α_- to γ passes within K_2 of e.

In other words, $A(K_1) = B(K_1)^c$. Note that $B(K_2)$ is not empty and that $A(K_1)$ is not empty, for $l(\alpha_l) \geq 2K_1$ and $l(\alpha_r) \geq 2K_1$.

The following is a simple consequence of the fact that triangles are thin along transition points [DG20, Lemme 2.4] and that transition points are within a bounded distance of points on a relative geodesic [Hru10, Proposition 8.13].

LEMMA 6.2. *For fixed K_2, if K_1 is large enough, then the closures of $A(K_1)$ and $B(K_2)$ in the Bowditch compactification $\overline{\Gamma}_B$ are disjoint.*

As $\overline{\Gamma}_B$ is compact, there exists a continuous function f_α on $\overline{\Gamma}_B$ taking values in $[0, 1]$, that vanishes on $A(K_1)$ and which is equal to one on $B(K_2)$.

We now finish the construction of the partition of unity associated with α. Let $n_1 = n_1(\alpha)$ be the largest integer such that translating n_1 times the relative geodesic α, on the right, the length on the left is still at least K_1. Formally,

$$n_1(\alpha) = \sup\{k \geq 0, l((T^{-k}\alpha)_l) \geq K_1\}. \tag{37}$$

Similarly, let $n_2 = n_2(\alpha)$ be the largest integer such that translating n_2 times α on the left, the length on the right is at least K_1. That is,

$$n_2(\alpha) = \sup\{k \geq 0, l((T^k\alpha)_r) \geq K_1\}. \tag{38}$$

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Let $A'(K_1)$ be the set of γ such that for every $k \in [-n_1, n_2 - 1]$:

(i) there either exists a relative geodesic from γ to α_+ whose distance from α_{k+1} is at least K_1; or
(ii) there exists a relative geodesic from α_- to γ whose distance from α_k is at least K_1.

Let $B'(K_2)$ be the set of γ such that there exists $k \in [-n_1, n_2 - 1]$ such that:

(i) any relative geodesic from γ to α_+ passes within K_2 of α_{k+1};
(ii) and any relative geodesic from α_- to γ passes within K_2 of α_k.

Again, if K_1 is large enough, the closures of $A'(K_1)$ and $B'(K_2)$ in the Bowditch compactification are disjoint.

For technical reasons, we further need to truncate relative geodesics. Letting β be any relative geodesic with $\beta_0 = e$, denote by $\beta_{(2K_1)}$ the shortest sub-relative geodesic of β such that $l((\beta_{(2K_1)})_l) \geq 2K_1$ and $l((\beta_{(2K_1)})_r) \geq 2K_1$. In other words, we truncate β on the left (respectively, on the right) as soon as the length on the left (respectively, on the right) is at least $2K_1$. Note that if K_1 is large enough, whenever $\gamma \in B'(K_2)$, we have

$$\Sigma = \sum_{k=-n_1}^{n_2} f_{(T^k \alpha)_{(2K_1)}}(\alpha_k^{-1}\gamma) \geq 1. \tag{39}$$

Thus, there exists a function g_α, which is continuous on the Bowditch compactification, which is equal to one on $A'(K_1)$, which is equal to the sum Σ above on $B'(K_2)$ and whose values are between one and Σ. We set

$$\kappa_\alpha = \frac{f_{\alpha_{(2K_1)}}}{g_\alpha}.$$

Denote by α'_+ the right endpoint of $\alpha_{(2K_1)}$ and by α'_- the left endpoint of $\alpha_{(2K_1)}$. According to Lemma 2.5, the fact that a relative geodesic from γ to α'_+ passes within K_2 of α_1 and a relative geodesic from α'_- to γ passes within K_2 of α_0 means that γ projects on α approximately between $\alpha_0 = e$ and α_1. More precisely, the projections on $\alpha_{(2K_1)}$ which are the closest to α'_+ and to α'_- are within a bounded distance of α_1 and e, respectively. We deduce that the sum Σ is bounded by some constant that only depends on K_2 and K_1. Roughly speaking, κ_α is a continuous function whose successive images by the shift mimics the decomposition of Γ into the subsets Γ_k.
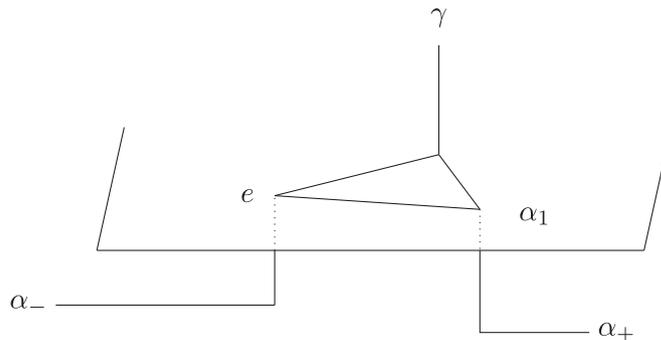
Let $\alpha = (\alpha_{-k}, \ldots, \alpha_0, \alpha_1, \ldots, \alpha_l)$ be a relative geodesic, with $\alpha_0 = e$. Whenever $l(\alpha_l) < 2K_1$ or $l(\alpha_r) < 2K_1$, we set $\Psi_r(\alpha) = 0$. Otherwise, we set

$$\Psi_r(\alpha) = \frac{1}{I^{(1)}(r)} \sum_{\gamma \in \Gamma} \kappa_\alpha(\gamma) \frac{G(\alpha_-, \gamma | r) G(\gamma, \alpha_+ | r)}{G(\alpha_-, \alpha_+ | r)}. \tag{40}$$

This defines a function Ψ_r on the set of relative geodesics α with $\alpha_0 = e$.

According to the previous discussion, $\kappa_\alpha(\gamma) \neq 0$ can only happen if the projection of γ on $\alpha_{(2K_1)}$ lies between e and α_1, up to a bounded distance. Choosing K_1 large enough, the same is true replacing $\alpha_{(2K_1)}$ with α. Indeed, let α_- and α_+ be the endpoints of α and let α'_- and α'_+ the endpoints of $\alpha_{(2K_1)}$. If K_1 is large enough, then a relative geodesic from γ to α'_- passes within a bounded distance of e if and only if a relative geodesic from γ to α_- also passes within a bounded distance of e. Similarly, a relative geodesic from γ to α_+ passes within a bounded distance of α_1 if and only if the same is true for a relative geodesic from γ to α'_+. Weak relative Ancona inequalities thus show that for any k large enough so that $\Psi_r(T^k[e,\gamma]) \neq 0$,

$$\Psi_r(T^k[e,\gamma]) \asymp \Upsilon_r(T^k\gamma). \tag{41}$$

Hence, the function Ψ_r will replace the function Υ_r in (35) to obtain a more accurate estimate.

PROPOSITION 6.3. *Let $\gamma \in \Gamma$ and let α_γ be the relative geodesic from e to γ given by the automaton $\mathcal{G}$. Then,*

$$\Phi_r(\gamma) = I^{(1)}(r) \sum_{k=0}^{\hat{d}(e,\gamma)-1} \Psi_r(T^k\alpha_\gamma) + O(I^{(1)}(r)).$$

Proof. Consider $\gamma \in \Gamma$. To simplify things, denote by α the relative geodesic from e to γ. Let m_1 be the smallest integer such that $l((T^{m_1}\alpha)_l) \geq K_1$ and m_2 the largest integer such that $l((T^{m_2}\alpha)_r) \geq K_1$. By the definition of Ψ_r,

$$\sum_{k=0}^{\hat{d}(e,\gamma)-1} \Psi_r(T^k\alpha) = \sum_{k=m_1}^{m_2} \Psi_r(T^k\alpha)$$

$$= \frac{1}{I^{(1)}(r)} \sum_{k=m_1}^{m_2} \sum_{\gamma' \in \Gamma} \kappa_{(T^k\alpha)}(\gamma') \frac{G(\alpha_k^{-1},\gamma'|r)G(\gamma',\alpha_k^{-1}\gamma|r)}{G(\alpha_k^{-1},\alpha_k^{-1}\gamma|r)}.$$

Translating on the left by α_k^{-1} and replacing γ' with $\alpha_k^{-1}\gamma'$ in the sum, we obtain

$$\sum_{k=0}^{\hat{d}(e,\gamma)-1} \Psi_r(T^k\alpha) = \frac{1}{I^{(1)}(r)} \sum_{\gamma' \in \Gamma} \left(\sum_{k=m_1}^{m_2} \kappa_{(T^k\alpha)}(\alpha_k^{-1}\gamma') \right) \frac{G(e,\gamma'|r)G(\gamma',\gamma|r)}{G(e,\gamma|r)}.$$

Recall that

$$\Phi_r(\gamma) = \sum_{\gamma' \in \Gamma} \frac{G(e,\gamma'|r)G(\gamma',\gamma|r)}{G(e,\gamma|r)}.$$

We are thus looking for the elements $\gamma' \in \Gamma$ such that

$$\sum_{k=m_1}^{m_2} \kappa_{(T^k\alpha)}(\alpha_k^{-1}\gamma') \neq 1. \tag{42}$$

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Suppose there exists $i \in [m_1, m_2 - 1]$ such that any relative geodesic from γ' to γ passes within K_2 of α_{i+1} and that any relative geodesic from e to γ' passes within K_2 of α_i. Fix $k \in [m_1, m_2 - 1]$ and consider the translated relative geodesic $T^k \alpha$. Then, any relative geodesic from $\alpha_k^{-1} \gamma'$ to the right endpoint of $T^k \alpha$ passes within K_2 of $\alpha_k^{-1} \alpha_{i+1}$ and any relative geodesic from the left endpoint of $T^k \alpha$ to $\alpha_k^{-1} \gamma'$ passes within K_2 of $\alpha_k^{-1} \alpha_i$.

In particular, $\alpha_k^{-1} \gamma' \in B'(K_2)$, where $B'(K_2)$ is the set constructed as above, using the relative geodesic $T^k \alpha$. Hence,

$$g_{T^k \alpha}(\alpha_k^{-1} \gamma') = \sum_{j=-n_1}^{n_2} f_{(T^j(T^k \alpha))_{(2K_1)}}((T^k \alpha)_j^{-1} \gamma').$$

Note that $(T^k \alpha)_j = \alpha_{k+j}$, so that

$$g_{T^k \alpha}(\alpha_k^{-1} \gamma') = \sum_{j=m_1-k}^{m_2-k} f_{(T^{k+j}\alpha)_{(2K_1)}}(\alpha_{k+j}^{-1} \gamma') = \sum_{j=m_1}^{m_2} f_{(T^j \alpha)_{(2K_1)}}(\alpha_j^{-1} \gamma').$$

In particular, we see that

$$\sum_{k=m_1}^{m_2} \kappa_{T^k \alpha}(\alpha_k^{-1} \gamma') = 1.$$

We proved that whenever (42) holds, then for every $i \in [m_1, m_2 - 1]$, either a relative geodesic from γ' to γ remains at a distance at least K_2 from α_{i+1} or a relative geodesic from e to γ' remains at distance at least K_2 from α_i. We again use Lemma 2.5. Let α_{k+1} be the projection of γ' the closest to γ on α. If K_2 was chosen large enough, then we necessarily have $k \geq m_2$ or $k \leq m_1 - 1$. Also let $\mathcal{H}_{\alpha_k}$ be the union of parabolic subgroups in Ω_0 containing $\alpha_k^{-1} \alpha_{k+1}$ and let $\tilde{\alpha}_k$ be the projection of γ' on $\mathcal{H}_{\alpha_k}$. According to [Sis13, Lemma 1.13(1)], the exit point from $\mathcal{H}_{\alpha_k}$ of any relative geodesic from e to γ' is within a bounded distance of $\tilde{\alpha}_k$. Thus, any such relative geodesic passes first within a bounded distance of α_k and then within a bounded distance of $\tilde{\alpha}_k$. Similarly, any relative geodesic from γ' to γ passes first within a bounded distance of $\tilde{\alpha}_k$ and then within a bounded distance of α_{k+1}. Weak relative Ancona inequalities yield

$$\frac{G(e, \gamma'|r) G(\gamma', \gamma|r)}{G(e, \gamma|r)} \lesssim I^{(1)}(r) \frac{G(\alpha_k, \tilde{\alpha}_k|r) G(\tilde{\alpha}_k, \alpha_{k+1}|r)}{G(\alpha_k, \alpha_{k+1}|r)}.$$

Also recall that (39) is uniformly bounded. Consequently, we have

$$\left| \Phi_r(\gamma) - I^{(1)}(r) \sum_{k=0}^{\hat{d}(e,\gamma)-1} \Psi_r(T^k \alpha) \right| \lesssim I^{(1)}(r) \sum_{0 \leq k < m_1} \sum_{\sigma \in \mathcal{H}_{\alpha_k}} \frac{G(\alpha_k, \sigma|r) G(\sigma, \alpha_{k+1}|r)}{G(\alpha_k, \alpha_{k+1}|r)}$$

$$+ I^{(1)}(r) \sum_{m_2 \leq k \leq \hat{d}(e,\gamma)} \sum_{\sigma \in \mathcal{H}_{\alpha_k}} \frac{G(\alpha_k, \sigma|r) G(\sigma, \alpha_{k+1}|r)}{G(\alpha_k, \alpha_{k+1}|r)}.$$

By definition, m_1 and $\hat{d}(e, \gamma) - m_2$ are bounded by K_1 and $d(\alpha_k, \alpha_{k+1}) \leq K_1$. In particular,

$$\sum_{\sigma \in \mathcal{H}_{\alpha_k}} \frac{G(\alpha_k, \sigma|r) G(\sigma, \alpha_{k+1}|r)}{G(\alpha_k, \alpha_{k+1}|r)}$$

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is uniformly bounded. We thus have

$$\Phi_r(\gamma) - I^{(1)}(r) \sum_{k=0}^{\hat{d}(e,\gamma)-1} \Psi_r(T^k\alpha) \lesssim I^{(1)}(r),$$

which concludes the proof. □

6.2 Truncating Ψ_r

We say that a function f defined on relative geodesic α satisfying $\alpha_0 = e$ is locally Hölder if for every $n \geq 1$, as soon as α and α' coincide in the relative ball (for the distance $\hat{d}$) of center e and radius n, $|f(\alpha) - f(\alpha')| \leq C\rho^n$, for some $C \geq 0$ and some $0 < \rho < 1$.

A similar function Ψ_r is defined in [Gou14] for hyperbolic groups. It is proved there that for every r, Ψ_r is continuous, locally Hölder and that Ψ_r uniformly converges to a locally Hölder function Ψ_{R_μ}, as r tends to R_μ. However, in our situation, such properties do not hold, so we cannot directly apply the strategy in [Gou14]. We are going instead to truncate Ψ_r such that our new function only depends on a finite number of symbols.

Precisely, fix a constant $N \in \mathbb{N}$ and denote by $\alpha^{(N)}$ the relative geodesic α restricted to $[-N, N]$. If $\hat{d}(e, \alpha_-) \leq N$ and $\hat{d}(e, \alpha_+) \leq N$, then $\alpha^{(N)} = \alpha$. We set $\Psi_r^{(N)}(\alpha) = \Psi_r(\alpha^{(N)})$. Let us fix another constant D and define $\Psi_r^{(D,N)}(\alpha)$ as follows. If one of the increments $\alpha_{k-1}^{-1}\alpha_k$ of α satisfies $d(\alpha_{k-1}^{-1}\alpha_k) \geq D$, for some k between $-N+1$ and N, then we set $\Psi_r^{(D,N)}(\alpha) = 0$. Otherwise, we set $\Psi_r^{(D,N)}(\alpha) = \Psi_r^{(N)}(\alpha)$. To simplify notation, we use the following convention. Whenever $\hat{d}(e, \alpha_-) \leq N$, we set $\alpha_{-N} = \alpha_-$ and, similarly, whenever $\hat{d}(e, \alpha_+) \leq N$, we set $\alpha_N = \alpha_+$.

Our goal is to prove estimates that will allow us to replace Ψ_r with $\Psi_r^{(D,N)}$. We start with the following lemma.

LEMMA 6.4. *If N is large enough, depending on K_1 and K_2, then for every relative geodesic α, we have $\kappa_\alpha = \kappa_{\alpha^{(N)}}$.*

Proof. First note that if $N \geq 2K_1$, $\alpha_{(2K_1)}^{(N)} = \alpha^{(2K_1)}$, so that $f_{\alpha_{(2K_1)}}(\gamma) = 0$ if and only if $f_{\alpha_{(2K_1)}^{(N)}}(\gamma) = 0$. In particular, $\kappa_\alpha(\gamma) = 0$ if and only if $\kappa_{\alpha^{(N)}}(\gamma) = 0$. Hence, we can assume that $f_{\alpha_{(2K_1)}}(\gamma) \neq 0$. We want to prove that if N is large enough, then the sum Σ defined by (39) is the same for α and for $\alpha^{(N)}$. By the definition of f, the fact that $f_{\alpha_{(2K_1)}}(\gamma) \neq 0$ implies that any relative geodesic from γ to the right endpoint of $\alpha_{(2K_1)}$ passes within a bounded distance of α_1 and any relative geodesic from γ to the left endpoint of $\alpha_{(2K_1)}$ passes within a bounded distance of e. Thus, the number of k in the sum defining Σ such that $f_{(T^k\alpha)_{(2K_1)}}(\alpha_k^{-1}\gamma) \neq 0$ is finite, with a bound depending only on K_1 and K_2. If N is large enough, the same holds replacing α with $\alpha^{(N)}$ and for any such k, $f_{(T^k\alpha)_{(2K_1)}}(\alpha_k^{-1}\gamma) = f_{(T^k\alpha^{(N)})_{(2K_1)}}(\alpha_k^{-1}\gamma)$. This concludes the proof. □

Thus,

$$\Psi_r(\alpha^{(N)}) = \frac{1}{I^{(1)}(r)} \sum_{\gamma \in \Gamma} \kappa_\alpha(\gamma) \frac{G(\alpha_-^{(N)}, \gamma|r) G(\gamma, \alpha_+^{(N)}|r)}{G(\alpha_-^{(N)}, \alpha_+^{(N)}|r)}.$$

In other words, when replacing $\Psi_r(\alpha)$ with $\Psi_r(\alpha^{(N)})$, we do not have to replace κ_α with $\kappa_{\alpha^{(N)}}$. This will be very convenient in the following.

PROPOSITION 6.5. *Let $\epsilon > 0$. Then for N large enough and for D large enough (depending on N), for every $r \leq R_\mu$ and for every n,*

$$\left| \sum_{k=0}^{n-1} \sum_{\gamma \in \hat{S}^n} H(e, \gamma|r)(\Psi_r \circ T^k([e, \gamma]) - \Psi_r^{(D,N)} \circ T^k([e, \gamma])) \right|$$

$$\leq \epsilon \sum_{k=0}^{n-1} \sum_{\gamma \in \hat{S}^n} H(e, \gamma|r)\Psi_r \circ T^k([e, \gamma]).$$

Proof. We first show that we can replace Ψ_r by $\Psi_r^{(N)}$, that is, we prove that if N is large enough, then

$$\left| \sum_{k=0}^{n-1} \sum_{\gamma \in \hat{S}^n} H(e, \gamma|r)(\Psi_r \circ T^k([e, \gamma]) - \Psi_r^{(N)} \circ T^k([e, \gamma])) \right|$$

$$\lesssim \epsilon \sum_{k=0}^{n-1} \sum_{\gamma \in \hat{S}^n} H(e, \gamma|r)\Psi_r \circ T^k([e, \gamma]). \tag{43}$$

Let n and let $k \leq n - 1$. Set $\alpha = T^k([e, \gamma])$. Then, according to Lemma 6.4,

$$\sum_{\gamma \in \hat{S}^n} H(e, \gamma|r)(\Psi_r \circ T^k([e, \gamma]) - \Psi_r^{(N)} \circ T^k([e, \gamma]))$$

$$= \frac{1}{I_1(r)} \sum_{\gamma \in \hat{S}^n} H(e, \gamma|r)$$

$$\times \sum_{\gamma' \in \Gamma} \kappa_\alpha(\gamma') \left(\frac{G(\alpha_-, \gamma'|r)G(\gamma', \alpha_+|r)}{G(\alpha_-, \alpha_+|r)} - \frac{G(\alpha_{-N}, \gamma'|r)G(\gamma', \alpha_N|r)}{G(\alpha_{-N}, \alpha_N|r)} \right).$$

We rewrite

$$\left| \frac{G(\alpha_-, \gamma'|r)G(\gamma', \alpha_+|r)}{G(\alpha_-, \alpha_+|r)} - \frac{G(\alpha_{-N}, \gamma'|r)G(\gamma', \alpha_N|r)}{G(\alpha_{-N}, \alpha_N|r)} \right|$$

$$= \left(\frac{G(\alpha_-, \gamma'|r)G(\gamma', \alpha_+|r)}{G(\alpha_-, \alpha_+|r)} \right) \left| 1 - \frac{G(\alpha_{-N}, \gamma'|r)G(\gamma', \alpha_N|r)G(\alpha_-, \alpha_+|r)}{G(\alpha_{-N}, \alpha_N|r)G(\alpha_-, \gamma'|r)G(\gamma', \alpha_+|r)} \right|$$

and

$$\left| 1 - \frac{G(\alpha_{-N}, \gamma'|r)G(\gamma', \alpha_N|r)G(\alpha_-, \alpha_+|r)}{G(\alpha_{-N}, \alpha_N|r)G(\alpha_-, \gamma'|r)G(\gamma', \alpha_+|r)} \right|$$

$$= \left| 1 - \frac{G(\alpha_{-N}, \gamma'|r)G(\alpha_-, \alpha_+|r)}{G(\alpha_{-N}, \alpha_+|r)G(\alpha_-, \gamma'|r)} \frac{G(\alpha_{-N}, \alpha_+|r)G(\gamma', \alpha_N|r)}{G(\alpha_{-N}, \alpha_N|r)G(\gamma', \alpha_+|r)} \right|.$$

We now show that

$$\left| 1 - \frac{G(\alpha_{-N}, \gamma'|r)G(\alpha_-, \alpha_+|r)}{G(\alpha_{-N}, \alpha_+|r)G(\alpha_-, \gamma'|r)} \frac{G(\alpha_{-N}, \alpha_+|r)G(\gamma', \alpha_N|r)}{G(\alpha_{-N}, \alpha_N|r)G(\gamma', \alpha_+|r)} \right| \tag{44}$$

is arbitrary small when N is large enough. Let

$$u_N(\gamma') = \frac{G(\alpha_{-N}, \gamma'|r)G(\alpha_-, \alpha_+|r)}{G(\alpha_{-N}, \alpha_+|r)G(\alpha_-, \gamma'|r)}$$

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and

$$v_N(\gamma') = \frac{G(\alpha_{-N}, \alpha_+|r)G(\gamma', \alpha_N|r)}{G(\alpha_{-N}, \alpha_N|r)G(\gamma', \alpha_+|r)},$$

so that (44) can be written as $|1 - u_N(\gamma')v_N(\gamma')|$. Assume that $\kappa_\alpha(\gamma') \neq 0$. Then $f_{\alpha_{(2K_1)}}(\gamma') \neq 0$ and so any relative geodesic from γ' to the left endpoint of $\alpha_{(2K_1)}$ passes within K_1 of e. This implies that relative geodesics from α_{-N} to γ' and from α_- to α_+ fellow-travel for a time at least N', where N' tends to infinity as N tends to infinity. Strong relative Ancona inequalities show that

$$\left|1 - u_N(\gamma')\right| \leq C\rho^{N'}.$$

Similarly, one can prove that

$$\left|1 - v_N(\gamma')\right| \leq C\rho^{N'}.$$

In addition, weak relative Ancona inequalities imply that $v_N(\gamma')$ is bounded. This yields

$$\left|1 - u_N(\gamma')v_N(\gamma')\right| \leq v_N(\gamma')|1 - u_N(\gamma')| + |1 - v_N(\gamma')| \leq C'\rho^{N'}.$$

Hence,

$$\left| \sum_{\gamma \in \hat{S}^n} H(e, \gamma|r)(\Psi_r \circ T^k([e, \gamma]) - \Psi_r^{(N)} \circ T^k([e, \gamma])) \right|$$

$$\leq C'\rho^{N'} \frac{1}{I^{(1)}(r)} \sum_{\gamma \in \hat{S}^n} H(e, \gamma|r) \sum_{\gamma' \in \Gamma} \kappa_\alpha(\gamma') \frac{G(\alpha_-, \gamma'|r)G(\gamma', \alpha_+|r)}{G(\alpha_-, \alpha_+|r)}$$

$$= C'\rho^{N'} \sum_{\gamma \in \hat{S}^n} H(e, \gamma|r)\Psi_r \circ T^k([e, \gamma]).$$

Thus, if N is large enough, then (43) holds.

Let us compare $\Psi_r^{(D,N)} \circ T^k([e, \gamma])$ and $\Psi_r^{(N)} \circ T^k([e, \gamma])$ now. Let $\alpha = T^k[e, \gamma]$. Then, $\Psi_r^{(D,N)} \circ T^k([e, \gamma]) - \Psi_r^{(N)} \circ T^k([e, \gamma])$ is non-zero only for elements γ such that there exists j between $-N + 1$ and N such that $d(\alpha_{j-1}, \alpha_j) \geq D$. Denote by $\gamma_0 = e, \gamma_1, \ldots, \gamma_n = \gamma$ successive elements on $[e, \gamma]$, so that $\alpha_j = \gamma_k^{-1}\gamma_{j+k}$. Hence, $\Psi_r^{(D,N)} \circ T^k([e, \gamma]) - \Psi_r^{(N)} \circ T^k([e, \gamma])$ is non-zero only for elements γ such that there exists j between $-N + k + 1$ and $N + k$ such that $d(\gamma_{j-1}, \gamma_j) \geq D$.

Let $\hat{S}^n_{\geq D}$ be the set of $\gamma \in \hat{S}^n$ such that one of the increments of the relative geodesic $[e, \gamma]$ between $-N + k + 1$ and $N + k$ has length at least D. In addition, for a fixed j between $-N + k + 1$ and $N + k$, let $\hat{S}^{n,j}_{\geq D}$ be the subset of $\hat{S}^n_{\geq D}$ of elements γ such that the first such increment is at step j. Then,

$$\sum_{\gamma \in \hat{S}^n} H(e, \gamma|r)(\Psi_r^{(N)} \circ T^k([e, \gamma]) - \Psi_r^{(D,N)} \circ T^k([e, \gamma]))$$

$$= \frac{1}{I^{(1)}(r)} \sum_{\gamma \in \hat{S}^n_{\geq D}} H(e, \gamma|r) \sum_{\gamma' \in \Gamma} \kappa_\alpha(\gamma') \frac{G(\alpha_{-N}, \gamma'|r)G(\gamma', \alpha_N|r)}{G(\alpha_{-N}, \alpha_N|r)}$$

$$= \frac{1}{I^{(1)}(r)} \sum_{j=-N+k+1}^{N+k} \sum_{\gamma \in \hat{S}^{n,j}_{\geq D}} H(e, \gamma|r) \sum_{\gamma' \in \Gamma} \kappa_\alpha(\gamma') \frac{G(\alpha_{-N}, \gamma'|r)G(\gamma', \alpha_N|r)}{G(\alpha_{-N}, \alpha_N|r)}.$$

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Fix j. For $\gamma \in \hat{S}^n$, we write $\gamma = \gamma_1 \sigma \gamma_2$, where $\gamma_1 \in \hat{S}^{j-1}$, σ is in a factor $\mathcal{H}'_k$ and $\gamma' \in \hat{S}^{n-j}$. If $\gamma \in \hat{S}^{n,j}_{\geq D}$, then $d(e, \sigma) \geq D$. Weak relative Ancona inequalities show that

$$H(e, \gamma|r) \lesssim H(e, \gamma_1|r) H(e, \sigma|r) H(e, \gamma_2|r).$$

In addition, using (41), we can replace Ψ_r with Υ_r in the right member of the sum above. We obtain

$$\sum_{\gamma \in \hat{S}^n} H(e, \gamma|r) (\Psi_r^{(N)} \circ T^k([e, \gamma]) - \Psi_r^{(D,N)} \circ T^k([e, \gamma]))$$

$$\lesssim \sum_{j=-N+k+1}^{N+k} \sum_{\gamma \in \hat{S}^{n,j}_{\geq D}} H(e, \gamma_1|r) H(e, \sigma|r) H(e, \gamma_2|r) \Upsilon_r(T^k\alpha). \tag{45}$$

Recall that X^1_x is the set of symbols that can precede x in Σ_A. More generally, X^m_x is the set of words of length m that can precede x. Decompose the sum over γ as follows:

$$\sum_{\gamma \in \hat{S}^n} H(e, \gamma|r) (\Psi_r^{(N)} \circ T^k([e, \gamma]) - \Psi_r^{(D,N)} \circ T^k([e, \gamma]))$$

$$\lesssim \sum_{j=-N+k+1}^{N+k} \sum_{\gamma_2 \in \hat{S}^{n-j}} \sum_{\substack{\sigma \in X^1_{\gamma_2} \\ d(e,\sigma) \geq D}} \sum_{\gamma_1 \in X^{j-1}_{\sigma\gamma_2}} H(e, \gamma_1|r) H(e, \sigma|r) H(e, \gamma_2|r) \Upsilon_r(T^k\alpha).$$

Note that $\Upsilon_r(T^k\alpha)$ only depends on the kth increment of $[e, \gamma]$. In particular, for $j \neq k$, we can factorize the sum over $\gamma_1, \sigma, \gamma_2$ by $\Upsilon_r(T^k\alpha)$. Hence, we can bound the terms $j \neq k$ by

$$\Upsilon_r(T^k\alpha) \sum_{j=-N+k+1}^{N+k} \sum_{\gamma_2 \in \hat{S}^{n-j}} H(e, \gamma_2|r) \sum_{\substack{\sigma \in X^1_{\gamma_2} \\ d(e,\sigma) \geq D}} H(e, \sigma|r) \sum_{\gamma_1 \in X^{j-1}_{\sigma\gamma_2}} H(e, \gamma_1|r).$$

Corollary 2.10 shows that

$$\sum_{\sigma \in X^1_{\gamma_2}} H(e, \sigma|r)$$

is uniformly bounded. Thus, for large enough D,

$$\sum_{\substack{\sigma \in X^1_{\gamma_2} \\ d(e,\sigma) \geq D}} H(e, \sigma|r) \leq \frac{\epsilon}{2N} \sum_{\sigma \in X^1_{\gamma_2}} H(e, \sigma|r). \tag{46}$$

Let us focus on the term $j = k$. We can still factorize the sum over γ_1 by $\Upsilon_r(T^k\alpha)$. We want to bound the sum over σ. According to the definition of Υ_r, we thus need to bound

$$\sum_{\substack{\sigma \in X^1_{\gamma_2} \\ d(e,\sigma) \geq D}} H(e, \sigma|r) \sum_{\sigma' \in \mathcal{H}_\sigma} \frac{G(e, \sigma'|r) G(\sigma', \sigma|r)}{G(e, \sigma|r)}$$

$$= \sum_{\substack{\sigma \in X^1_{\gamma_2} \\ d(e,\sigma) \geq D}} \sum_{\sigma' \in \mathcal{H}_\sigma} G(e, \sigma'|r) G(\sigma', \sigma|r) G(\sigma, e|r).$$

As μ is not spectrally degenerate, the sum

$$\sum_{\sigma \in X^1_{\gamma_2}} \sum_{\sigma' \in \mathcal{H}_\sigma} G(e, \sigma'|r) G(\sigma', \sigma|r) G(\sigma, e|r)$$

is uniformly bounded. Thus, for large enough D,

$$\sum_{\substack{\sigma \in X^1_{\gamma_2} \\ d(e,\sigma) \geq D}} \sum_{\sigma' \in \mathcal{H}_\sigma} G(e, \sigma'|r) G(\sigma', \sigma|r) G(\sigma, e|r)$$

$$\leq \frac{\epsilon}{2N} \sum_{\sigma \in X^1_{\gamma_2}} \sum_{\sigma' \in \mathcal{H}_\sigma} G(e, \sigma'|r) G(\sigma', \sigma|r) G(\sigma, e|r). \tag{47}$$

When j is fixed, there is a unique way of decomposing γ as $\gamma_1 \sigma \gamma_2$. Hence, combining (45), (46), and (47), we obtain

$$\sum_{\gamma \in \hat{S}^n} H(e, \gamma|r)(\Psi_r^{(N)} \circ T^k([e, \gamma]) - \Psi_r^{(D,N)} \circ T^k([e, \gamma]))$$

$$\lesssim \sum_{j=-N+k+1}^{N+k} \frac{\epsilon}{2N} \sum_{\gamma \in \hat{S}^n} H(e, \gamma|r) \Psi_r^{(N)} \circ T^k([e, \gamma])$$

$$\leq \epsilon \sum_{\gamma \in \hat{S}^n} H(e, \gamma|r) \Psi_r^{(N)} \circ T^k([e, \gamma]). \tag{48}$$

Finally, combining (43) and (48), we obtain the desired inequality. $\qquad \square$

Recall that we want to compare $I^{(2)}(r)$ and $I^{(1)}(r)$. As we saw,

$$I^{(2)}(r) = \sum_{n \geq 0} \sum_{\gamma \in \hat{S}^n} H(e, \gamma|r) \Phi_r(\gamma).$$

Proposition 6.3 thus yields

$$I^{(2)}(r) = I^{(1)}(r) \sum_{n \geq 0} \sum_{\gamma \in \hat{S}^n} \sum_{k=0}^{n-1} H(e, \gamma|r) \Psi_r(T^k[e, \gamma]) + O(I^{(1)}(r)^2).$$

We want to prove that

$$I^{(2)}(r) = \xi I^{(1)}(r)^3 + O(I^{(1)}(r)^2),$$

so that we only have to deal with

$$I^{(1)}(r) \sum_{n \geq 0} \sum_{\gamma \in \hat{S}^n} \sum_{k=0}^{n-1} H(e, \gamma|r) \Psi_r(T^k[e, \gamma]).$$

In view of Proposition 6.5, we can replace Ψ_r with $\Psi_r^{(D,N)}$.

We now consider the set $\overline{\Sigma}_{A,\mathbb{Z}}$ of (finite or infinite) sequences $x = (x_n)$ indexed by $\mathbb{Z}$ such that $x_n \in \Sigma$ and for every n, x_n and x_{n+1} are adjacent edges in the automaton $\mathcal{G}$. The map T still defines a shift on $\overline{\Sigma}_{A,\mathbb{Z}}$.

As $\Psi^{(D,N)}(\alpha)$ only depends on the truncated geodesic $\alpha^{(N)}$, $\Psi^{(D,N)}$ can be extended to a function defined on finite or infinite relative geodesics. For any $x \in \overline{\Sigma}_{A,\mathbb{Z}}$, $(\ldots, x_{-n}, \ldots, x_0, \ldots, x_n, \ldots)$ defines such a relative geodesic, so $\Psi^{(D,N)} \circ \phi$ is a well-defined function on $\overline{\Sigma}_{A,\mathbb{Z}}$. We will omit the reference to ϕ and see $\Psi^{(D,N)}$ as a function on $\overline{\Sigma}_{A,\mathbb{Z}}$ to simplify. In addition, because $\Psi^{(D,N)}(\alpha)$

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only depends on the truncated relative geodesic $\alpha^{(N)}$ and vanishes on relative geodesics α whose increments are too long, the induced function on $\overline{\Sigma}_{A,\mathbb{Z}}$ only depends on a finite number of symbols.

For a continuous function $f : \overline{\Sigma}_{A,\mathbb{Z}} \to \mathbb{R}$, we define

$$\tilde{V}_n(f) = \sup\{|f(x) - f(y)|, x_{-n} = y_{-n}, \dots, x_0 = y_0, \dots, x_n = y_n\}.$$

Letting $0 < \rho < 1$, we say that f is ρ-locally Hölder if there exists $C \geq 0$ such that

$$\forall n \geq 1, \quad V_n(f) \leq C\rho^n.$$

As before, we do not ask anything on $V_0(f)$ and f can be unbounded. Say that f is locally Hölder if it is ρ-locally Hölder for some ρ. Define the Hölder norm D_ρ as

$$D_\rho(f) = \sup_n \frac{\tilde{V}_n(f)}{\rho^n}.$$

In addition, let H_ρ be the set of bounded ρ-locally Hölder functions and define the norm

$$\|\cdot\|_\rho = D_\rho + \|\cdot\|_\infty$$

on this space. Then, $(H_\rho, \|\cdot\|_\rho)$ is a Banach space.

We want to use Proposition 6.3 and apply the transfer operator to $\Psi_r^{(D,N)}$. To apply this operator, we first need to transform $\Psi_r^{(D,N)}$ into a function only depending on the future, that is, a function on $\overline{\Sigma}_A$. We start by proving the following.

LEMMA 6.6. *Fix D and N. The functions $\Psi_r^{(D,N)}$ are ρ-locally Hölder and uniformly bounded. They uniformly converge in $(H_{\rho,\beta}, \|\cdot\|_{\rho,\beta})$ to a function $\Psi_{R_\mu}^{(D,N)}$, as r tends to R_μ.*

Proof. We first show that $\Psi_r^{(D,N)}$ is uniformly bounded. Recall that

$$\Psi_r^{(D,N)}(\alpha) = \frac{1}{I^{(1)}(r)} \sum_{\gamma \in \Gamma} \kappa_\alpha(\gamma) \frac{G(\alpha_{-N}, \gamma | r) G(\gamma, \alpha_N | r)}{G(\alpha_{-N}, \alpha_N | r)}.$$

Denote by Γ_k the set of γ whose projection on $\alpha^{(N)}$ is on α_{k+1}, where we choose the projection which is the closest to α_N. In addition, let $\mathcal{H}_k$ be the union of parabolic subgroups containing $\alpha_k^{-1}\alpha_{k+1}$. Then, weak relative Ancona inequalities, together with Lemma 2.5 show that

$$\sum_{\gamma \in \Gamma_k} \frac{G(\alpha_{-N}, \gamma | r) G(\gamma, \alpha_N | r)}{G(\alpha_{-N}, \alpha_N | r)} \lesssim I^{(1)}(r) \sum_{\sigma \in \mathcal{H}_k} \frac{G(\alpha_k, \sigma | r) G(\sigma, \alpha_{k+1} | r)}{G(\alpha_k, \alpha_{k+1} | r)}.$$

As κ_α is bounded, we thus have

$$\Psi_r^{(D,N)}(\alpha) \leq C_\alpha,$$

where C_α only depends on α. Actually, because $\Psi_r^{(D,N)}(\alpha)$ is non-zero for a finite number of α which only depends on N and D, C_α also only depends on D and N. Moreover, $\Psi^{(D,N)}(\alpha)$ only depends on $\alpha^{(N)}$, so it is ρ-locally Hölder and $\|\Psi^{(D,N)}\|_{\rho,\beta}$ is bounded by some number only depending on D and N.

Finally, because $\Psi^{(D,N)}(\alpha)$ only depends on a finite number of symbols, pointwise convergence is equivalent to convergence in $(H_{\rho,\beta}, \|\cdot\|_{\rho,\beta})$. Let us fix α and prove that $\Psi_r^{(D,N)}(\alpha)$ converges to a function $\Psi_{R_\mu}^{(D,N)}(\alpha)$, as r tends to R_μ. To do so, we express $\Psi_r^{(D,N)}$ as a sum using the transfer

operator. We introduce a function ψ_r on Γ as follows. We set

$$\psi_r(\gamma) = \kappa_\alpha(\gamma) \frac{G(\alpha_{-N}, \gamma | r) G(\gamma, \alpha_N | r)}{G(\alpha_{-N}, \alpha_N | r) H(e, \gamma | r)}$$

for any relative geodesic α such that $\Psi_r^{(N,D)}(\alpha) \neq 0$. Otherwise, we set $\psi_r = 0$. Weak relative Ancona inequalities imply that $\gamma \mapsto G(\alpha_{-N}, \gamma | r) G(\gamma, \alpha_N | r)$ can be extended to $\partial \hat{\Gamma}$. As κ is defined on the whole Bowditch compactification, ψ_r can also be extended to $\Gamma \cup \partial \hat{\Gamma}$, so $\psi_r \circ \phi$ is a function on $\overline{\Sigma}_A$. Note that

$$\Psi_r^{(D,N)}(\alpha) = \frac{1}{I^{(1)}(r)} \frac{1}{H(e, e | r)} \sum_{n \geq 0} \mathcal{L}_r^n(1_{E_*} \psi_r \circ \phi)(\emptyset). \tag{49}$$

We want to apply 4.13 to prove that Ψ_r converges, so we have to transform ψ_r into a locally Hölder function. First, ψ_r is defined using the function κ_α which is only continuous. We again have to truncate ψ_r to conclude our proof.

Fix N' and let $\gamma_{N'}$ be the N'th element on the relative geodesic $[e, \gamma]$ whenever $\hat{d}(e, \gamma) \geq N'$ and $\gamma_{N'} = \gamma$ otherwise. Set then

$$\psi_r^{(N')} = \kappa_\alpha(\gamma_{N'}) \frac{G(\alpha_{-N}, \gamma | r) G(\gamma, \alpha_N | r)}{G(\alpha_{-N}, \alpha_N | r) H(e, \gamma | r)}.$$

The functions ψ_r and $\psi_r^{(N')}$ implicitly depend on α and on N and D. Actually, Lemma 6.4 shows that they do not depend on α, but only on $\alpha^{(N)}$.

LEMMA 6.7. *For every $\epsilon > 0$, for every N, and every D, there exists N_0' such that for every $N' \geq N_0'$, for every α, and for every $r < R_\mu$,*

$$\left| \psi_r - \psi_r^{(N')} \right| \leq \epsilon.$$

Proof. Let $\epsilon > 0$. The function κ_α is continuous on the Bowditch compactification. Endow this compactification with any distance d. We can extend the definition of $\gamma_{N'}$ to any infinite relative geodesic α declaring $\alpha_{N'}$ to be the N'th point on α. Then $\alpha_{N'}$ uniformly converges to the conical limit point defined by α, as N' tends to infinity. Thus, for any $\delta > 0$, if N' is large enough, then $d(\gamma, \gamma_{N'}) \leq \delta$. Note that this can be easily directly shown if one chooses the shortcut metric on the Bowditch compactification defined in [GP13, Definition 2.6]. By compactness, κ_α is uniformly continuous. Hence, for N' large enough, $\left| \kappa_\alpha(\gamma_{N'}) - \kappa_\alpha(\gamma) \right| \leq \epsilon$. Thus,

$$\left| \psi_r(\gamma) - \psi_r^{(N')}(\gamma) \right| \leq \epsilon \frac{G(\alpha_{-N}, \gamma | r) G(\gamma, \alpha_N | r)}{G(\alpha_{-N}, \alpha_N | r) H(e, \gamma | r)} \lesssim C_\alpha \epsilon,$$

where C_α only depends on α. The integer N_0' *a priori* depends on α, because of C_α in the upper-bounded above and because uniform continuity of κ_α depends on α. However, ψ_r and $\psi_r^{(N')}$ are the null function except for a finite number of relative geodesics α which only depends on N and D. This concludes the proof. $\square$

To show that $\Psi_r^{(D,N)}(\alpha)$ converges, it is enough to prove it is Cauchy, that is for every $\epsilon > 0$, there exists $r_0 < R_\mu$ such that for any $r, r' \in [r_0, R_\mu)$,

$$\left| \Psi_r^{(D,N)}(\alpha) - \Psi_{r'}^{(D,N)}(\alpha) \right| \leq \epsilon.$$

Fix $\epsilon > 0$. Let N' be given by Lemma 6.7 so that for every $r < R_\mu$,

$$\left| \psi_r - \psi_r^{(N')} \right| \leq \epsilon.$$

According to (49),

$$\left| \Psi_r^{(D,N)}(\alpha) - \Psi_{r'}^{(D,N)}(\alpha) \right|$$

$$\lesssim 2\epsilon + \frac{1}{I^{(1)}(r)} \left| \sum_{n \geq 0} \mathcal{L}_r^n(1_{E_*} \psi_r^{(N')} \circ \phi)(\emptyset) - \sum_{n \geq 0} \mathcal{L}_{r'}^n(1_{E_*} \psi_{r'}^{(N')} \circ \phi)(\emptyset) \right|.$$

We thus only need to prove that

$$\frac{1}{I^{(1)}(r)} \sum_{n \geq 0} \mathcal{L}_r^n(1_{E_*} \psi_r^{(N')} \circ \phi)(\emptyset)$$

converges, as r tends R_μ. Note that the functions $\gamma \mapsto G(\alpha_{-N}, \gamma|r)G(\gamma, \alpha_N|r)/G(\alpha_{-N}, \alpha_N|r)$ and $\gamma \mapsto \kappa_{\alpha_{(2K_1)}}(\gamma_{N'})$ are bounded and locally Hölder, so $\psi_r \circ \phi$ lies in $H_{\rho,\beta}$.

To prove that the above sum, we need to prove that $\psi_r \circ \phi$ uniformly converges to $\psi_{R_\mu} \circ \phi$. This is not obvious and so we truncate ψ_r as we truncated Ψ_r. Fix another constant D'. For $\gamma \in \Gamma$ let $[e, \gamma] = (e, \gamma_1, \ldots, \gamma_n = \gamma)$ be the relative geodesic from e to γ given by the automaton $\mathcal{G}$. If one of the increments of $[e, \gamma]$ is at least D', set $\psi_r^{(D',N')}(\gamma) = 0$. Otherwise, set $\psi_r^{(D',N')}(\gamma) = \psi_r^{(N')}(\gamma_{N'})$. As $\psi_r \circ \phi$ is bounded and locally Hölder, the same proof as the proof of Proposition 6.5 shows that for every $\eta > 0$, for large enough N' and D',

$$\frac{1}{I^{(1)}(r)} \sum_{n \geq 0} \mathcal{L}_r^n(1_{E_*} |\psi_r^{(N')} \circ \phi - \psi^{(N',D')} \circ \phi|)(\emptyset) \leq \eta. \tag{50}$$

Remark 6.1. It might seem strange that we first had to truncate κ_α when defining $\psi_r^{(N')}$, before truncating again to define $\psi_r^{(D',N')}$. However, to apply the same strategy as in Proposition 6.5, we needed to know *a priori* that our function was locally Hölder.

Once again, to prove that

$$\frac{1}{I^{(1)}(r)} \sum_{n \geq 0} \mathcal{L}_r^n(1_{E_*} \psi_r^{(N')} \circ \phi)(\emptyset)$$

converges, it is enough to prove that this quantity is Cauchy, as r tends to R_μ. In view of (50), we thus only need to prove that

$$\frac{1}{I^{(1)}(r)} \sum_{n \geq 0} \mathcal{L}_r^n(1_{E_*} \psi_r^{(N',D')} \circ \phi)(\emptyset)$$

converges. The function $\psi_r^{(N',D')} \circ \phi$ is bounded and locally Hölder. Moreover, whenever x, y, z are fixed, $r \mapsto G(x, y|r)G(y, z|r)/G(x, z|r)H(e, y|r)$ is a continuous function. It converges to $G(x, y|R_\mu)G(y, z|R_\mu)/G(x, z|R_\mu)H(e, y|R_\mu)$, as r tends to R_μ. Hence, $\psi_r^{(N',D')} \circ \phi$ converges to a function $\psi_{R_\mu}^{(N',D')} \circ \phi$. In addition, $\psi_r^{(N',D')} \circ \phi$ only depends on a finite number of symbols, so this convergence also holds in $(H_{\rho,\beta}, \|\cdot\|_{\rho,\beta})$. Now that every parameter is fixed, we set $f = 1_{E_*} \psi_{R_\mu}^{(N',D')} \circ \phi$ for convenience. We are left to proving that

$$\frac{1}{I^{(1)}(r)} \sum_{n \geq 0} \mathcal{L}_r^n f(\emptyset)$$

converges, as r tends to R_μ, which is a direct consequence of (34). $\qquad \square$

6.3 From the double-sided to the one-sided shift

As announced, to study

$$\sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}H(e,\gamma|r)\Psi_r^{(D,N)}(T^k[e,\gamma]),$$

we express this sum with the transfer operator and then use Theorem 3.3, exactly as in the proof of Lemma 6.6. However, we cannot apply the transfer operator to the function $\Psi_r^{(D,N)}$, which depends both on past and future.

We use the following trick.

LEMMA 6.8. *Let f be a ρ-locally Hölder function on $\overline{\Sigma}_{A,\mathbb{Z}}$. Then, there exist $\rho^{1/2}$-locally Hölder functions g and u on $\overline{\Sigma}_{A,\mathbb{Z}}$ such that*

$$f = g + u - u \circ T.$$

Moreover, $g(x) = g(y)$ as soon as $x_n = y_n$ for every non-negative n, so that g induces a function on $\overline{\Sigma}_A$. In addition, if f is bounded, then g and u also are bounded and the maps

$$f \in (H_\rho, \|\cdot\|_\rho) \mapsto g \in (H_{\rho^{1/2}}, \|\cdot\|_{\rho^{1/2}}), \quad f \in (H_\rho, \|\cdot\|_\rho) \mapsto u \in (H_{\rho^{1/2}}, \|\cdot\|_{\rho^{1/2}})$$

are continuous.

This is proved in [PP90, Proposition 1.2] for finite-type shifts. However, the proof does not use that the set of symbols is finite.

According to Lemma 6.6, the functions $\Psi_r^{(D,N)}$ are bounded and locally Hölder on $\overline{\Sigma}_{A,\mathbb{Z}}$ and they converge in $(H_{\rho,\beta}, \|\cdot\|_{\rho,\beta})$ to a function $\Psi_{R_\mu}^{(D,N)}$. We thus obtain from Lemma 6.8 functions $\tilde{\Psi}_r^{(D,N)}$, $r \leq R_\mu$ defined on $\overline{\Sigma}_A$ and functions $u_r^{(D,N)}$ defined on $\overline{\Sigma}_{A,\mathbb{Z}}$ such that

$$\Psi_r^{(D,N)} = \tilde{\Psi}_r^{(D,N)} + u_r^{(D,N)} - u_r^{(D,N)} \circ T.$$

For any $x \in \overline{\Sigma}_A$ of length n,

$$\sum_{k=0}^{n-1}\Psi_r^{(D,N)}(T^k x) = \sum_{k=0}^{n-1}\tilde{\Psi}_r^{(D,N)}(T^k x) + u_r^{(D,N)}(x) - u_r^{(D,N)}(T^n x).$$

The functions $u_r^{(D,N)}$ are bounded by some number that only depends on D and N, so

$$\sum_{k=0}^{n-1}\Psi_r^{(D,N)}(T^k x) = \sum_{k=0}^{n-1}\tilde{\Psi}_r^{(D,N)}(T^k x) + O_{D,N}(1).$$

Thus,

$$\sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}H(e,\gamma|r)\Psi_r^{(D,N)}(T^k[e,\gamma])$$

$$= \sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}H(e,\gamma|r)\tilde{\Psi}_r^{(D,N)}(T^k[e,\gamma]) + O_{D,N}(I^{(1)}(r)). \qquad (51)$$

As $\tilde{\Psi}_r^{(D,N)}$ only depends on the future, we rewrite this as

$$
\sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}H(e,\gamma|r)\tilde{\Psi}_r^{(D,N)}(T^k[e,\gamma])
$$

$$
= H(e,e|r)\sum_{n\geq 0}\mathcal{L}_r^n\left(1_{E_*}\sum_{k=0}^{n-1}\tilde{\Psi}_r^{(D,N)}\circ T^k\right)(\emptyset). \tag{52}
$$

6.4 Proof of Proposition 6.1: convergence of $I^{(2)}(r)/I^{(1)}(r)^3$

We first prove that the quantity (52) is asymptotic to $\xi_{D,N}I^{(1)}(r)^2$, as r tends to R_μ, where $\xi_{D,N}$ is some number only depending on D and N. As $\Psi_r^{(D,N)}$ converges in $(H_{\rho,\beta},\|\cdot\|_{\rho,\beta})$ to $\Psi_{R_\mu}^{(D,N)}$, we deduce from Lemma 6.6, up to changing ρ, that $\tilde{\Psi}_r^{(D,N)}$ converges in $(H_{\rho,\beta},\|\cdot\|_{\rho,\beta})$ to $\tilde{\Psi}_{R_\mu}^{(D,N)}$. We thus only need to prove that

$$
\sum_{n\geq 0}\mathcal{L}_r^n\left(1_{E_*}\sum_{k=0}^{n-1}\tilde{\Psi}_{R_\mu}^{(D,N)}\circ T^k\right)(\emptyset)
$$

is asymptotic to $\xi_{D,N}I^{(1)}(r)^2$. Recall that $\mathcal{L}_r(u\cdot v\circ T)=v\mathcal{L}_r(u)$, so that

$$
\sum_{n\geq 0}\mathcal{L}_r^n\left(1_{E_*}\sum_{k=0}^{n-1}\tilde{\Psi}_{R_\mu}^{(D,N)}\circ T^k\right) = \sum_{n\geq 0}\sum_{k=1}^n\mathcal{L}_r^k(\tilde{\Psi}_{R_\mu}^{(D,N)}\mathcal{L}_r^{n-k}1_{E_*})
$$

$$
= \sum_{k\geq 1}\mathcal{L}_r^k\left(\tilde{\Psi}_{R_\mu}^{(D,N)}\sum_{n\geq 0}\mathcal{L}_r^n 1_{E_*}\right). \tag{53}
$$

From Corollary 4.13, we deduce that for any r close enough to R_μ and for any $x\in\overline{\Sigma}_A$,

$$
\sum_{n\geq 0}\mathcal{L}_r^n 1_{E_*}(x) = \sum_{j=1}^k\frac{1}{\tilde{P}_j(r)}\sum_{i=1}^{p_j}\tilde{h}_{j,r}^{(i)}(x)\int 1_{E_*}d\tilde{\nu}_{j,r}^{((i-n)\bmod p_j)} + O(1). \tag{54}
$$

Let

$$
\alpha_{i,j,r} = \int 1_{E_*}d\tilde{\nu}_{j,r}^{((i-n)\bmod p_j)},
$$

so that $\alpha_{i,j,r}$ converges to $\alpha_{i,j}$, as r tends to R_μ. We now estimate

$$
\sum_{k\geq 1}\mathcal{L}_r^k\left(\tilde{\Psi}_{R_\mu}^{(D,N)}\sum_{j=1}^k\frac{1}{\tilde{P}_j(r)}\sum_{i=1}^{p_j}\alpha_{i,j,r}\tilde{h}_{j,r}^{(i)}\right)(\emptyset).
$$

According to (19), $\tilde{h}_{j,r}^{(i)}(\emptyset)$ converges to $h_j^{(i)}(\emptyset)$, as r tends to R_μ, so we can start the above sum at $k=0$. Fix j and let $1\leq i\leq p_j$. We use again Corollary 4.13 to obtain

$$
\sum_{k\geq 0}\mathcal{L}_r^k(\tilde{\Psi}_{R_\mu}^{(D,N)}\tilde{h}_{j,r}^{(i)})(\emptyset)
$$

$$
= \sum_{j'}\frac{1}{|\tilde{P}_{j'}(r)|}\sum_{i'=1}^{p_{j'}}\tilde{h}_{j',r}^{(i')}(\emptyset)\int\tilde{\Psi}_{R_\mu}^{(D,N)}\tilde{h}_{j,r}^{(i)}d\tilde{\nu}_{j',r}^{((i'-n)\bmod p_{j'})} + O(1). \tag{55}
$$

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We show that for every j', i',

$$\int \tilde{\Psi}_{R_\mu}^{(D,N)} \tilde{h}_{j',r}^{(i')} \, d\tilde{\nu}_{j',r}^{((i'-n) \bmod p_{j'})}$$

converges, as r tends to R_μ. Write

$$\int \tilde{\Psi}_{R_\mu}^{(D,N)} \tilde{h}_{j',r}^{(i')} \, d\tilde{\nu}_{j',r}^{((i'-n) \bmod p_{j'})} = \int \tilde{\Psi}_{R_\mu}^{(D,N)} (\tilde{h}_{j',r}^{(i')} - h_j^{(i)}) \, d\tilde{\nu}_{j',r}^{((i'-n) \bmod p_{j'})}$$
$$+ \int \tilde{\Psi}_{R_\mu}^{(D,N)} h_j^{(i)} \, d\tilde{\nu}_{j',r}^{((i'-n) \bmod p_{j'})}.$$

Corollary 4.13 shows that $\tilde{\nu}_{j',r}^{((i'-n) \bmod p_{j'})}$ weakly converges to $\nu_{j'}^{((i'-n) \bmod p_{j'})}$, so that the second integral in the right-hand term converges. We show that the first converges to zero. Let $m_{j',r}$ be the measure defined by $dm_{j',r} = (1/p_{j'}) \sum_{i'=1}^{p_{j'}} h_{j',r}^{(i)} d\nu_{j',r}^{(i')}$. According to [Sar99, Proposition 4], $m_{j',r}$ is Gibbs and according to [Sar99, Proposition 2], the functions $h_{j',r}^{(i)}$ are bounded away from zero and infinity on the support of $\nu_{j',r}^{(i)}$, so that

$$\nu_{j',r}^{(i')}([x_1 \ldots x_n]) \leq CH(e, x_1 \ldots x_n | r) \leq CH(e, x_1 \ldots x_n | R_\mu).$$

Using (10), we see that the measure $\nu_{j',r}^{(i')}$ is dominated by the measure m on cylinders. As $\tilde{\Psi}_{R_\mu}^{(D,N)} (\tilde{h}_{j',r}^{(i')} - h_j^{(i)})$ is locally Hölder, we have

$$\left| \int \tilde{\Psi}_{R_\mu}^{(D,N)} (\tilde{h}_{j',r}^{(i')} - h_j^{(i)}) \, d\tilde{\nu}_{j',r}^{((i'-n) \bmod p_{j'})} \right| \lesssim \int \left| \tilde{\Psi}_{R_\mu}^{(D,N)} (\tilde{h}_{j',r}^{(i')} - h_j^{(i)}) \right| dm.$$

Finally, because $\tilde{\Psi}_{R_\mu}^{(D,N)}$ is bounded, we have

$$\left| \int \tilde{\Psi}_{R_\mu}^{(D,N)} (\tilde{h}_{j',r}^{(i')} - h_j^{(i)}) \, d\tilde{\nu}_{j',r}^{((i'-n) \bmod p_{j'})} \right| \lesssim \int |\tilde{h}_{j,r}^{(i)} - h_j^{(i)}| \, dm.$$

According to Corollary 4.13, this last quantity converges to zero.

Now, (19) shows that $\tilde{h}_{j',r}^{(i')}(\emptyset)$ converges and so we deduce from (55) that

$$\sum_{k \geq 0} \mathcal{L}_r^k (\tilde{\Psi}_{R_\mu}^{(D,N)} \tilde{h}_{j,r}^{(i)})(\emptyset) = \sum_{j'} \frac{\xi_{j',D,N,r}^{i,j}}{|\tilde{P}_{j'}(r)|},$$

where $\xi_{j',D,N,r}^{i,j}$ converges, as r tends to R_μ. Also recall that we proved in Proposition 5.1 that $\tilde{P}_j(r) \sim P(r)$ for every j, so (23) yields

$$O\left(\sum_{j=1}^{k} \frac{1}{\tilde{P}_j(r)} \right) = O(I^{(1)}(r)).$$

Finally, we get from (53) and (54) that

$$\sum_{n \geq 0} \mathcal{L}_r^n \left(1_{E_*} \sum_{k=0}^{n-1} \tilde{\Psi}_{R_\mu}^{(D,N)} \circ T^k \right) = \sum_{j,j'} \frac{\xi_{D,N,r}^{j,j'}}{|\tilde{P}_j(r)||\tilde{P}_{j'}(r)|} + O(I^{(1)}(r)),$$

where $\xi_{D,N,r}^{j,j'}$ converges. Consequently,

$$\sum_{n \geq 0} \mathcal{L}_r^n \left(1_{E_*} \sum_{k=0}^{n-1} \tilde{\Psi}_{R_\mu}^{(D,N)} \circ T^k \right) = \frac{\xi_{D,N,r}}{P(r)^2} + O(I^{(1)}(r)),$$

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where $\xi_{D,N,r}$ converges to some $\xi_{D,N}$. Therefore,

$$\sum_{n \geq 0} \mathcal{L}_r^n \left(1_{E_*} \sum_{k=0}^{n-1} \tilde{\Psi}_{R_\mu}^{(D,N)} \circ T^k \right)(\emptyset)$$

is asymptotic to $\xi_{D,N} I^{(1)}(r)^2$, as r tends to R_μ.

We thus deduce from (51) and (52) that

$$\sum_{n \geq 0} \sum_{\gamma \in \hat{S}^n} \sum_{k=0}^{n-1} H(e, \gamma|r) \Psi_r^{(D,N)}(T^k[e,\gamma]) = \xi_{D,N} I^{(1)}(r)^2 + o_{D,N}(I^{(1)}(r)^2). \tag{56}$$

Also note that we deduce from (53) that

$$\sum_{n \geq 0} \mathcal{L}_r^n \left(1_{E_*} \sum_{k=0}^{n-1} \tilde{\Psi}_{R_\mu}^{(D,N)} \circ T^k \right) = \sum_{k \geq 1} \mathcal{L}_r^k \left(\tilde{\Psi}_{R_\mu}^{(D,N)} \sum_{n \geq 0} \mathcal{L}_r^n 1_{E_*} \right)$$

and so according to (23), we have

$$\sum_{n \geq 0} \mathcal{L}_r^n \left(1_{E_*} \sum_{k=0}^{n-1} \tilde{\Psi}_{R_\mu}^{(D,N)} \circ T^k \right) \lesssim I^{(1)}(r) \sum_{k \geq 1} \mathcal{L}_r^k \tilde{\Psi}_{R_\mu}^{(D,N)}.$$

Thus, (36) and (41) show that

$$\sum_{n \geq 0} \mathcal{L}_r^n \left(1_{E_*} \sum_{k=0}^{n-1} \tilde{\Psi}_{R_\mu}^{(D,N)} \circ T^k \right) \lesssim I^{(1)}(r)^2.$$

Hence,

$$\xi_{D,N} \lesssim 1. \tag{57}$$

We finally conclude the proof of Proposition 6.1.

Proof. Recall that

$$I^{(2)}(r) = \sum_{\gamma \in \Gamma} H(e, \gamma|r) \Phi_r(\gamma) = \sum_{n \geq 0} \sum_{\gamma \in \hat{S}^n} \Phi_r(\gamma).$$

According to Proposition 6.3,

$$I^{(2)}(r) = I^{(1)}(r) \sum_{n \geq 0} \sum_{\gamma \in \hat{S}^n} \sum_{k=0}^{n-1} \Psi_r(T^k[e,\gamma]) + O(I^{(1)}(r)).$$

We need to prove that $I^{(2)}(r)/I^{(1)}(r)^3$ converges, as r tends to R_μ. It is thus enough to show that

$$\frac{1}{I^{(1)}(r)^2} \sum_{n \geq 0} \sum_{\gamma \in \hat{S}^n} \sum_{k=0}^{n-1} \Psi_r(T^k[e,\gamma])$$

converges.

Fix $\epsilon > 0$. Choose sequences D_l and N_l that tend to infinity, as l tends to infinity. As we want to apply Proposition 6.5, the sequence D_l will actually depend on the sequence N_l. According to (57), we can assume, up to taking a sub-sequence, that ξ_{D_l, N_l} converges to some constant ξ. We show that the above sum also converges to ξ. According to Proposition 6.5, we can choose

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N_l and D_l so that for any l large enough,

$$\frac{1}{I^{(1)}(r)^2}\left|\sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}\Psi_r(T^k[e,\gamma]) - \sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}\Psi_r^{(D_l,N_l)}(T^k[e,\gamma])\right|$$

$$\leq \epsilon\frac{1}{I^{(1)}(r)^2}\sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}\Psi_r(T^k[e,\gamma]).$$

Fix a large enough l so that this inequality is satisfied and so that $|\xi_{D_l,N_l} - \xi| \leq \epsilon$. Now that l is fixed, we set $D = D_l$ and $N = N_l$. We thus have

$$\frac{1}{I^{(1)}(r)^2}\sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}\Psi_r(T^k[e,\gamma]) - \xi$$

$$\leq \frac{1}{1-\epsilon}\frac{1}{I^{(1)}(r)^2}\sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}\Psi_r^{(D,N)}(T^k[e,\gamma]) - \xi.$$

Hence, (56) shows that whenever r is close enough to R_μ,

$$\frac{1}{I^{(1)}(r)^2}\sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}\Psi_r(T^k[e,\gamma]) - \xi \leq \frac{1}{1-\epsilon}\xi_{D,N} - \xi \leq \frac{\epsilon}{1-\epsilon} + \xi\left(\frac{1}{1-\epsilon} - 1\right).$$

Similarly

$$\xi - \frac{1}{I^{(1)}(r)^2}\sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}\Psi_r(T^k[e,\gamma])$$

$$\leq \xi - \frac{1}{1+\epsilon}\frac{1}{I^{(1)}(r)^2}\sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}\Psi_r^{(D,N)}(T^k[e,\gamma])$$

and so whenever r is close enough to R_μ,

$$\xi - \frac{1}{I^{(1)}(r)^2}\sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}\Psi_r(T^k[e,\gamma]) \leq \xi\left(1 - \frac{1}{1+\epsilon}\right) - \frac{\epsilon}{1+\epsilon}.$$

As ϵ is arbitrary, this shows that

$$\frac{1}{I^{(1)}(r)^2}\sum_{n\geq 0}\sum_{\gamma\in\hat{S}^n}\sum_{k=0}^{n-1}\Psi_r(T^k[e,\gamma]) \xrightarrow[r\to R_\mu]{} \xi.$$

Finally, we already know that $I^{(2)}(r)/I^{(1)}(r)^3$ is bounded away from zero, independently of r, so that $\xi \neq 0$. This concludes the proof. $\qquad\square$

Theorem 4.1 is a direct consequence of Proposition 6.1.

7. From the Green asymptotics to the local limit theorem

We can finally prove Theorem 1.1. We first deduce from Theorem 4.1 the following.

COROLLARY 7.1. *Let Γ be a non-elementary relatively hyperbolic group. Let μ be a finitely supported, admissible, and symmetric probability measure on Γ. Assume that the corresponding*

random walk is non-spectrally degenerate along parabolic subgroups. Then, for every γ_1, γ_2, there exists $C_{\gamma_1, \gamma_2} > 0$ such that

$$\frac{d}{dr}(G(\gamma_1, \gamma_2|r)) \underset{r \to R_\mu}{\sim} C_{\gamma_1, \gamma_2} \frac{1}{\sqrt{R_\mu - r}}.$$

Proof. For $\gamma_1 = \gamma_2 = e$, this is a direct consequence of Theorem 4.1, combined with [Dus22, Lemma 3.2] which relates the derivatives of the Green function with the sums $I^{(k)}(r)$. Note that by equivariance, we only need to prove the result with $\gamma_2 = e$. According to Lemma 2.7, an asymptotic of $d/dr(G(\gamma, e|r))$ is given by an asymptotic of

$$\sum_{\gamma' \in \Gamma} G(\gamma, \gamma'|r)G(\gamma', e|r).$$

Consider $\gamma \in \Gamma$ and set

$$f_r(\gamma') = \frac{G(\gamma, \gamma'|r)}{G(e, \gamma'|r)}.$$

Let $\tilde{f}_r = f_r \circ \phi$. Then,

$$\sum_{\gamma' \in \Gamma} G(\gamma, \gamma'|r)G(\gamma', e|r) = H(e, e|r) \sum_{n \geq 0} \mathcal{L}_r^n \tilde{f}_r(\emptyset).$$

As γ is fixed, $\tilde{f}_r$ is uniformly bounded. Strong relative Ancona inequalities also imply that $\tilde{f}_r$ can be extended to a function on $\overline{\Sigma}_A$ which lie in $H_{\rho, \beta}$. If $\tilde{f}_r$ were uniformly converging to a function $\tilde{f}$, as r tends to R_μ, then we could directly conclude the proof, using (23). However, exactly like for Ψ_r, this uniform convergence does not necessarily hold and we have to truncate $\tilde{f}_r$. We can apply the same strategy as for the proof of Proposition 6.1 to conclude. □

Theorem 1.1 follows directly from Corollary 7.1 and [GL13, Theorem 9.1]. Corollary 1.2 thus follows from [Gou14, Proposition 4.1]. Beware that the symmetry assumption on the measure μ is needed here, see the remarks in [Gou14, Section 4].

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