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On the orbit-sizes of permutation groups containing elements separating finite subsets

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It is proved that if G is a permutation group on a set Ω every orbit of which contains more than mn elements, then any pair of subsets of Ω containing m and n elements respectively can be separated by an element of G.

1.

This note is the outcome of attempts to find a direct proof of the following result, which is a translation of a lemma of B.H. Neumann [1] into the language of permutation groups. (We shall state B.H. Neumann's lemma and indicate why the two results are equivalent at the end of this section.)

THEOREM 1 ([3, Lemma 2.3]). *If G is a group of permutations of a set Ω such that all the orbits of G are infinite, then for each finite subset Δ of Ω there is an element $g \in G$ such that $\Delta g \cap \Delta$ is empty.*

We shall give a direct proof of this theorem, and also prove the following quantitative version of it.

THEOREM 2. *If G is a group of permutations of a set Ω such that every orbit of G has more than mn elements then corresponding to each pair Γ, $\Delta \subseteq \Omega$ with $|\Gamma| = m$, $|\Delta| = n$, there exists $g \in G$ such that $\Gamma g \cap \Delta = \emptyset$.*

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We make two remarks. The first one is that the lower bound $(mn+1)$ on orbit-size ensuring separability of Γ and Δ is best possible. If G is a transitive group of degree mn having m blocks of imprimitivity each of size n then one can choose Γ, Δ, of sizes m, n respectively, such that $\Gamma g \cap \Delta \neq \emptyset$ for all $g \in G$: take Γ to have one element from each of the blocks of the imprimitivity system, and take Δ to be one of those blocks. The second remark is that the theorem can be proved using results and methods of B.H. Neumann from [1] and [2]. The proof we give here is in the more natural combinatorial and permutation group-theoretic style.

As promised, before proving the theorems (in Section 2) we sketch a proof of the equivalence of Theorem 1 with the result of B.H. Neumann [1, Lemma 4.1] that if a group K is the union $\bigcup_{i=1}^{k} H_i g_i$ of finitely many cosets $H_i g_i$ of subgroups H_i, then at least one of the H_i has finite index in K.

Thus assume the statement of Theorem 1 and suppose that $K = \bigcup_{1}^{k} H_i g_i$. Let $\Omega = \{H_i g \mid 1 \leq i \leq k;\ g \in G\}$, and let K act on Ω by multiplication on the right. Let $\Delta = \{H_1, \ldots, H_k, H_1 g_1, \ldots, H_k g_k\}$. Then Δ is a finite subset of Ω; if $g \in K$ then $g \in H_r g_r$ for some r and so $H_r g_r \in \Delta \cap \Delta g$; thus $\Delta \cap \Delta g \neq \emptyset$ for all $g \in K$. From our assumption it follows that one of the orbits of K in Ω is finite, and so $|K : H_i|$ is finite for some i. The converse follows as in [3, Lemma 2.3], from the observation that, if Δ is a subset of Ω such that $\Delta \cap \Delta g \neq \emptyset$ for all $g \in G$, then $G = \bigcup_{\Delta\times\Delta} C_{\alpha\beta}$ where $C_{\alpha\beta}$ is defined as $\{g \mid \alpha g = \beta,\ g \in G\}$. Each set $C_{\alpha\beta}$ is a right coset of the stabiliser G_α of α, so that if Δ is finite then G is covered by finitely many cosets. By B.H. Neumann's result G_α has finite index in G for some α, whence the orbit containing α is finite.

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2.

We first prove a combinatorial lemma which gives Theorem 1 directly.

LEMMA 1. *If every orbit of* G *has more than* $f(m, n)$ *elements where*

$$f(1, n) = n ,$$

$$f(m, n) \geq f\left(m-1, n^2+n\right) \quad (m > 1) ,$$

then for each pair $\Gamma, \Delta \subseteq \Omega$ *with* $|\Gamma| = m$, $|\Delta| = n$, *there exists* $g \in G$ *such that* $\Gamma g \cap \Delta = \emptyset$.

Proof. We use induction on m . If $m = 1$ the result is obvious since each orbit has more than n elements. Suppose that $m > 1$ and as inductive hypothesis that the desired conclusion holds when one set has fewer than m elements, and the other has any (finite) number, and suppose further that every orbit has more than $f(m, n)$ elements. Since every orbit has more than n elements, without loss of generality we may assume that there exists γ in $\Gamma - \Delta$. Let $\gamma h_1, \ldots, \gamma h_k$ be the distinct transforms of γ which lie in Δ . Note that $0 \leq k \leq n$. Since every orbit has more than $f(m, n)$ elements, every orbit must have more than $f\left(m-1, n^2+n\right)$ elements and so by inductive hypothesis there is $h \in G$ such that

$$(\Gamma-\{\gamma\})h \cap \left(\Delta \cup \Delta h_1 \cup \ldots \cup \Delta h_k\right) = \emptyset .$$

If $\gamma h \notin \Delta$ then put $g = h$. If $\gamma h \in \Delta$ then $\gamma h = \gamma h_i$ for some i , in which case put $g = hh_i^{-1}$. In both cases $\gamma g \notin \Delta$, and also $(\Gamma-\{\gamma\})g \cap \Delta = \emptyset$. Therefore $\Gamma g \cap \Delta = \emptyset$ as required.

Now we set out on the proof of Theorem 2. Unfortunately, we are unable to prove it directly - we have to use Theorem 1. First we will prove a second combinatorial lemma, which gives the right numbers but which 'begs the question' by making a strong finiteness assumption.

LEMMA 2. *If* $\Gamma, \Delta \subseteq \Omega$ *with* $|\Gamma| = m$, $|\Delta| = n$ *and each element of* Γ *is in a finite orbit of* G *with more than* mn *elements, then there exists* $g \in G$ *with* $\Gamma g \cap \Delta = \emptyset$.

Proof. Let the distinct orbits of G that contain elements of Γ be

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$\Omega_1, \ldots, \Omega_r$; for each $i = 1, \ldots, r$, suppose that $|\Omega_i \cap \Delta| = v_i$, $|\Omega_i| = t_i$ so that $\sum v_i \leq n$ and $t_i > mn$. Let the number of distinct translates Γg , for $g \in G$, be s (which from our assumptions is finite). Each element of Ω_i occurs the same number of times in the translates Γg , so at most msv_i/t_i of these translates contain an element of $\Omega_i \cap \Delta$. Since

$$\sum_{i=1}^{r} \frac{msv_i}{t_i} < \sum_{i=1}^{r} \frac{sv_i}{n} \leq s$$

and $\Gamma G \cap \Delta \subseteq \bigcup(\Omega_i \cap \Delta)$, the lemma follows.

Now we complete the proof of Theorem 2. Let Γ_0 be the subset of Γ consisting of elements that are contained in a finite orbit of G . By Lemma 2, there exists $g_1 \in G$ such that $\Gamma_0 g_1 \cap \Delta = \emptyset$. Now let H be the subgroup of G that leaves the elements of $\Gamma_0 g_1$ fixed; H has finite index in G , so the orbits of H containing elements of $(\Gamma-\Gamma_0)g_1$ are still infinite. So by Theorem 1, there exists $g_2 \in H$ such that $(\Gamma-\Gamma_0)g_1 g_2 \cap \Delta = \emptyset$. So $\Gamma g_1 g_2 \cap \Delta = \emptyset$.

References

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