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A Fenchel-Rockafellar type duality theorem for maximization

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We prove that $\sup(f-h)(E) = \sup(h^*-f^*)(E^*)$, where f is a proper lower semi-continuous convex functional on a real locally convex space E, $h : E \to \overline{R} = [-\infty, +\infty]$ is an arbitrary functional and f^*, h^* are their convex conjugates respectively. When $h = \delta_G$, the indicator of a bounded subset G of E, this yields a formula for $\sup f(G)$.

1.

A functional f on a real locally convex space E is said to be *proper*, if $f(E) \subset (-\infty, +\infty]$ and f is not identically $+\infty$. For any functional $f : E \to \overline{R} = [-\infty, +\infty]$, the *convex conjugate* f^* and the *concave conjugate* f^+ of f are the functionals on E^* (the space of all continuous linear functions on E) defined respectively by

(1) $$f^*(\Phi) = \sup(\Phi-f)(E) \quad (\Phi \in E^*) ,$$

(2) $$f^+(\Phi) = -(-f)^*(-\Phi) = \inf(\Phi-f)(E) \quad (\Phi \in E^*) .$$

We recall the following duality theorem of Fenchel-Rockafellar for minimization (see, for example, [2], Theorem 1, or [1], p. 68). *Let f and $-h$ be proper convex functionals on a real locally convex space E, such that one of them is continuous at some point of*

$$\{x \in E \mid f(x) < +\infty, -h(x) < +\infty\} .$$

Then

(3) $$\inf(f-h)(E) = \sup(h^+-f^*)(E^*) ,$$

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and there exists $\Phi_0 \in E^*$ *for which the* sup *in* (3) *is attained.* This theorem has various applications in optimization theory (see, for example, [3] and the references therein); for example, it yields a formula for $\inf f(G)$, where G is a convex subset of E and f is a proper convex functional on E , continuous at some point of $\{g \in G \mid f(g) < +\infty\}$, by taking in (3), $h = -\delta_G$, where δ_G is the indicator of the set G (that is $\delta_G(x) = 0$ if $x \in G$ and $\delta_G(x) = +\infty$ if $x \in E \backslash G$).

In the present paper we shall show, generalizing the methods of our previous paper [4], that a Fenchel-Rockafellar type duality theorem holds also for $\sup(f-h)(E)$, under a convenient assumption on f and with arbitrary h (Theorem 1 below). In the particular case when $h = \delta_G$, the indicator of a bounded subset G of E , this theorem yields the main quasi-lagrangian duality formula of [4] for $\sup f(G)$ (Remark 3 below).

2.

We shall denote by $\Gamma_0(E)$ the set of all proper lower semi-continuous convex functionals on a real locally convex space E .

LEMMA 1. *Let* E *be a real locally convex space,* $f \in \Gamma_0(E)$ *and* $x_0 \in E$. *Then*

$$f(x_0) = \sup_{\Phi \in E^*} \{\inf(f-\Phi)(E)+\Phi(x_0)\} . \tag{4}$$

Proof. By (1), for any $\Phi \in E^*$, we have

$$\Phi(x_0) - f^*(\Phi) = \Phi(x_0) - \sup(\Phi-f)(E) = \inf(f-\Phi)(E) + \Phi(x_0) . \tag{5}$$

But since $f \in \Gamma_0(E)$, we have $f^{**}(x_0) = f(x_0)$ (see, for example, [1], p. 46). Consequently,

$$\begin{aligned} f(x_0) = f^{**}(x_0) &= \sup_{\Phi \in E^*} \{\Phi(x_0)-f^*(\Phi)\} \\ &= \sup_{\Phi \in E^*} \{\inf(f-\Phi)(E)+\Phi(x_0)\} , \end{aligned}$$

which completes the proof of Lemma 1.

REMARK 1. In general, the sup in (4) need not be attained (since,

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obviously, it is attained for some $\Phi_0 \in E^*$ if and only if $\Phi_0 \in \partial f(x_0)$), but, if f is also finite and continuous at x_0, then the sup in (4) is attained for some $\Phi_0 \in E^*$.

THEOREM 1. *Let* E *be a real locally convex space,* $f \in \Gamma_0(E)$, *and* $h : E \to \overline{R} = [-\infty, +\infty]$ *an arbitrary functional. Then*

$$\sup(f-h)(E) = \sup(h^*-f^*)(E^*) . \tag{6}$$

Proof[1]. Assume, for the moment, that $f, h : E \to \overline{R} = [-\infty, +\infty]$ are two arbitrary functionals. Then, by (1), for any $\Phi \in E^*$, we have

$$\begin{aligned} h^*(\Phi) - f^*(\Phi) &= \sup(\Phi-h)(E) - \sup(\Phi-f)(E) \\ &= \sup(\Phi-h)(E) + \inf(f-\Phi)(E) . \end{aligned} \tag{7}$$

Assume now, *a contrario*, that

$$\sup(f-h)(E) < \sup(h^*-f^*)(E^*) . \tag{8}$$

Then there exists $\Phi_0 \in E^*$ such that, using also (7),

$$\sup(f-h)(E) < (h^*-f^*)(\Phi_0) = \sup(\Phi_0-h)(E) + \inf(f-\Phi_0)(E) ,$$

and hence there exists also $x_0 \in E$ such that

$$\sup(f-h)(E) < \Phi_0(x_0) - h(x_0) + f(x_0) - \Phi_0(x_0) = f(x_0) - h(x_0) ,$$

which is absurd, so (8) can not hold. Thus,

$$\sup(f-h)(E) \geq \sup(h^*-f^*)(E^*) . \tag{9}$$

Assume now, *a contrario*, that $f \in \Gamma_0(E)$ and

$$\sup(f-h)(E) > \sup(h^*-f^*)(E^*) . \tag{10}$$

Then there exists $x_0 \in E$ such that we have, using also (7) and Lemma 1,

[1] For a shorter proof, see the addendum at the end of this note.

$$\begin{aligned} f(x_0) - h(x_0) &> \sup(h^*-f^*)(E^*) \\ &= \sup_{\Phi \in E^*} \{\sup(\Phi-h)(E)+\inf(f-\Phi)(E)\} \\ &\geq \sup_{\Phi \in E^*} \{\Phi(x_0)-h(x_0)+\inf(f-\Phi)(E)\} \\ &= f(x_0) - h(x_0) , \end{aligned}$$

which is absurd, so (10) can not hold. This, together with (9), yields (6), completing the proof of Theorem 1.

REMARK 2. As shown by the above proof, the inequality (9), that is, the inequality $\geq$ in (6), is valid for any two functionals $f, h : E \to \overline{R} = [-\infty, +\infty]$. It is well known (see, for example, [1], p. 67) that the same is also true for the inequality $\geq$ in (3).

REMARK 3. In the particular case when $h = \delta_G$, the indicator of a bounded subset G of E , Theorem 1 yields the following main "quasi-lagrangian duality theorem" of [4] (see [4], Theorem 2.1). *Let* E *be a real locally convex space,* $f \in \Gamma_0(E)$ *and* G *a bounded subset of* E *. Then*

$$\begin{aligned} \text{(11)} \qquad \sup f(G) &= \sup_{\Phi \in E^*} \{\inf(f-\Phi)(E)+\sup \Phi(G)\} \\ &= \sup_{\Phi \in E^*} \{\inf(f+\Phi)(E)-\inf \Phi(G)\} . \end{aligned}$$

Of course, in the particular case when G is a singleton $\{x_0\}$, this result reduces to Lemma 1 above. The sup in the right hand side of (11) (and hence of (6)) need not be attained, even when f is also finite and continuous on E (see [4]). Some other applications of (11) are given in [4].

ADDENDUM [Received 6 February 1979]. (1) After this paper had been submitted, Th. Precupanu has communicated to us the following simplification of the above proof of Theorem 1.

Proof. Since $f \in \Gamma_0(E)$, we have $f^{**} = f$ (see, for example, [1], p. 46). Hence

$$
\begin{aligned}
\sup(f-h)(E) &= \sup(f^{**}-h)(E) \\
&= \sup_{x\in E}\left\{\sup_{\Phi\in E^*}\{\Phi(x)-f^*(\Phi)\}-h(x)\right\} \\
&= \sup_{\Phi\in E^*}\left\{\sup_{x\in E}\{\Phi(x)-h(x)\}-f^*(\Phi)\right\} = \sup(h^*-f^*)(E^*) \ ,
\end{aligned}
$$

which completes the proof of Theorem 1. Let us observe that this argument yields also Remark 2 above, since $f \geq f^{**}$ for any functional $f : E \to \overline{R} = [-\infty, +\infty]$.

(2) Let us also observe the following corollary of Theorem 1 (which is, actually, equivalent to Theorem 1), which should be compared with (3).

COROLLARY 1. *Let* E *be a real locally convex space,* $f : E \to \overline{R} = [-\infty, +\infty]$ *an arbitrary functional, and* $h \in \Gamma_0(E)$. *Then*

(12) $$\inf(f-h)(E) = \inf(h^*-f^*)(E^*) \ .$$

Proof. By Theorem 1 we have

$$
\begin{aligned}
\inf(f-h)(E) &= -\sup(h-f)(E) = -\sup(f^*-h^*)(E^*) \\
&= \inf(h^*-f^*)(E^*) \ ,
\end{aligned}
$$

which completes the proof.

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