IRREDUCIBLE REPRESENTATIONS OF THE GENERALIZED SYMMETRIC GROUP B_n^m

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Introduction. This paper is devoted to the determining of the irreducible linear representations of the generalized symmetric group B_n^m (elsewhere written as $C_m^n S_n$, $C_m \wr S_n$ or $G(m, 1, n)$) by considering the conjugacy classes of B_n^m and then constructing the same number of inequivalent irreducible linear representations of B_n^m. These have previously been determined by Kerber [**2**, Section 5] using Clifford's theory applied to wreath products.

An independent approach is given here which does not use Clifford's theory, and some of the results of Kerber [**2**], Puttaswamaiah [**4**] and Osima [**3**] are obtained in a much easier and more elementary way. The analogous problem of determining the irreducible projective representations of the generalized symmetric group has been treated in [**6**].

Elementary knowledge of representation theory is assumed. The symbol $\mathbb{C}^*$ will denote the multiplicative group of non-zero complex numbers, $\mathbb{N}$ the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

2. The group B_n^m and its conjugacy classes. A set of generators and relations for B_n^m is given by

$$B_n^m = \{r_1, \ldots, r_n : r_i^2 = 1 = r_n^m,\ i = 1, \ldots, n-1;\ (r_i r_{i+1})^3 = 1,\ i = 1, \ldots, n-2;$$
$$(r_{n-1} r_n)^2 = (r_n r_{n-1})^2,\ (r_i r_j)^2 = 1,\ i, j = 1, \ldots, n,\ j \neq i, i+1\}$$

(see Coxeter [**1**]).

We may identify r_i $(i = 1, \ldots, n-1)$ with the transposition $(i, i+1)$ and therefore the group generated by $r_1, \ldots, r_{n-1}$ is the symmetric group S_n.

The generator r_n may be identified with the mapping

$$\begin{pmatrix} n \\ \xi n \end{pmatrix} : \{1, \ldots, n\} \to \mathbb{C}^*$$

defined by

$$j \to j,\ j = 1, \ldots, n-1 \quad \text{and} \quad n \to \xi n,$$

where ξ is some primitive mth root of unity.

Consequently an element

$$r_i \ldots r_{n-1} r_n r_{n-1} \ldots r_i,\ i = 1, \ldots, n-1$$

corresponds to the mapping

$$\begin{pmatrix} i \\ \xi i \end{pmatrix} : \{1, \ldots, n\} \to \mathbb{C}^*$$

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defined by

$$j \to j,\ j = 1, \ldots, i-1, i+1, \ldots, n \quad \text{and} \quad i \to \xi i.$$

An arbitrary element $\sigma \in B_n^m$ may be expressed uniquely as the product of disjoint cycles $\sigma = \theta_1 \ldots \theta_t$, where

$$\theta_i = \begin{pmatrix} b_{i1} & b_{i2} & \ldots & b_{it_i} \\ \xi^{k_{i1}} b_{i2} & \xi^{k_{i2}} b_{i3} & \ldots & \xi^{k_{it_i}} b_{i1} \end{pmatrix},$$

$b_{ij} \in \{1, \ldots, n\}$, $k_{ij} \in \{1, \ldots, m\}$ and t_i is the length of the cycle θ_i, $i = 1, \ldots, t$. (See Read [**5**] for more details.)

Definition 2.1. Let $\sigma = \theta_1 \ldots \theta_t$ as above. Define $f(\theta_i) = \sum_{j=1}^{t_i} k_{ij}$ and put $f(\sigma) = \sum_{i=1}^{t} f(\theta_i)$. Let $a_{rs}(\sigma)$ denote the number of cycles θ_i of σ such that $f(\theta_i) \equiv r (\operatorname{mod} m)$, $1 \leqq r \leqq m$, $1 \leqq s \leqq n$. Then the $m \times n$ matrix $(a_{rs}(\sigma))$ is called the type of σ, and will be written as type(σ).

Lemma 2.2. *Two elements σ and σ' of B_n^m are conjugate if and only if* type$(\sigma) =$ type(σ').

Proof. See Kerber [**2**, 3.7].

Lemma 2.3. *Let $t_i \in \mathbb{N}^*$ and let $p(t_i)$ be the number of partitions of t_i if $t_i \in \mathbb{N}$ and $p(0) = 1$. Then the number of conjugacy classes of B_n^m is given by*

$$\sum p(t_1) \ldots p(t_m),$$

where the summation is taken over all the m-tuples $(t_1, \ldots, t_m)$ such that $\sum_{i=1}^{m} t_i = n$.

Proof. We prove the lemma by establishing a one-to-one correspondence between the set of all the conjugacy classes of B_n^m and the set of all the m-partitions of n. By an m-partition of n we mean an m-tuple $(\pi(t_1), \ldots, \pi(t_m))$, $t_i \in \mathbb{N}^*$ such that $t_1 + \ldots + t_m = n$ and each $\pi(t_i)$ is an n-tuple $(a_{i1}, \ldots, a_{in})$, where $a_{ij} \in \mathbb{N}^*$ and $\sum_{j=1}^{n} j a_{ij} = t_i$.

If a conjugacy class is of type(a_{ij}), we associate with it an m-partition given by $(\pi(t_1), \ldots, \pi(t_m))$, where $t_i = \sum_{j=1}^{n} j a_{ij}$ and $\pi(t_i) = (a_{i1}, \ldots, a_{in})$. Clearly, this partition is uniquely defined and conjugacy classes of different types correspond to different m-partitions of n.

Conversely, let $(\pi(t_1), \ldots, \pi(t_m))$ be an m-partition of n, where $\pi(t_i) = (a_{i1}, \ldots, a_{in})$. Then it can be easily seen that the set $\{1, \ldots, n\}$ can be uniquely expressed as a disjoint union of $\sum_{i,j} a_{ij}$ subsets such that exactly $\sum_{i=1}^{n} a_{ij}$ of these subsets have j

elements. On each of these subsets define a cyclic permutation as follows. For example, if $A=\{b_1,\ldots,b_j\}$ is one of the subsets having j elements, define

$$\theta_A=\begin{pmatrix} b_1 & b_2 & \ldots & b_j \\ b_2 & b_3 & \ldots & \xi^k b_1 \end{pmatrix},$$

where $k=1$ for the first a_{1j} subsets having j elements, $k=2$ for the next a_{2j} subsets having j elements and so on. Let σ be the product of all such cycles. Clearly type$(\sigma)=(a_{ij})$. This completes the proof.

3. Generalized Young subgroups and basic representations.

DEFINITION 3.1. Let $(t_1,\ldots,t_k)$ be a k-tuple such that $t_i\in\{0,1,\ldots,n\}$ and $t_1+\ldots+t_k=n$. We shall call $(t_1,\ldots,t_k)$ a permissible k-tuple. Define $p_0=0$ and $p_i=\sum_{j=1}^{i} t_j$, $i=1,\ldots,k$. If $t_i\neq 0$, let $B_{t_i}^m$ be the generalized symmetric group on the t_i symbols

$$P_i=\{p_{i-1}+1,\ldots,p_i\},\ i=1,\ldots,k,$$

and let $B_0^m=1$, the trivial subgroup of B_n^m.

The group $B_{t_1}^m\times B_{t_2}^m\times\ldots\times B_{t_k}^m$ is called the generalized Young subgroup determined by the k-tuple $(t_1,\ldots,t_k)$. We denote this group by $B_{(t_1,\ldots,t_k)}^m$.

Let

$$\sigma=\begin{pmatrix} 1 & 2 & \ldots & n \\ \xi^{k_1}b_1 & \xi^{k_2}b_2 & \ldots & \xi^{k_n}b_n \end{pmatrix}\in B_n^m,$$

where $b_1,\ldots,b_n\in\{1,\ldots,n\}$ and the k_i are positive integers. Define $\phi:B_n^m\to S_n$ by

$$\phi(\sigma)=\begin{pmatrix} 1 & 2 & \ldots & n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}.$$

Then ϕ is an epimorphism and it can be verified by induction on i that the kernel of ϕ is an abelian group generated by

$$\{r_i\ldots r_{n-1}r_nr_{n-1}\ldots r_i : i=1,\ldots,n\}.$$

The kernel of ϕ is in fact the direct product of n cyclic groups each of order m generated by $r_i\ldots r_{n-1}r_nr_{n-1}\ldots r_i$, $i=1,\ldots,n$ (respectively).

LEMMA 3.2. *Let $k\leqq m$ and $\sigma=\sigma_1\ldots\sigma_k\in B_{(t_1,\ldots,t_k)}^m$, where $\sigma_i\in B_{t_i}^m$, $i=1,\ldots,k$. Define*

$$\chi_{(t_1,\ldots,t_k)}(\sigma)=\xi^{\sum_{i=1}^{k} if(\sigma_i)},$$

where ξ is some primitive m-th root of unity. Then

(i) *$\chi_{(t_1,\ldots,t_k)}$ is an irreducible linear representation of $B_{(t_1,\ldots,t_k)}^m$ and*

(ii) *$\chi_{(t_1,\ldots,t_k)}^g(x)\neq\chi_{(t'_1,\ldots,t'_k)}(x)$ for some $x\in\ker\phi$ and for all $g\in B_n^m$ unless $(t_1,\ldots,t_k)=(t'_1,\ldots,t'_k)$ in which case this holds for all $g\in B_n^m\setminus B_{(t_1,\ldots,t_k)}^m$.*

(We shall call $\chi_{(t_1,\ldots,t_k)}$ the basic linear representation of $B_{(t_1,\ldots,t_k)}^m$.)

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Proof. (i) Let $\sigma = \sigma_1 \ldots \sigma_k$, $\sigma' = \sigma'_1 \ldots \sigma'_k \in B^m_{(t_1,\ldots,t_k)}$ such that $\sigma_i, \sigma'_i \in B_{t_i}$, $i = 1, \ldots, k$. Then

$$\begin{aligned}\chi_{(t_1,\ldots,t_k)}(\sigma\sigma') &= \chi_{(t_1,\ldots,t_k)}(\sigma_1 \ldots \sigma_k\sigma'_1 \ldots \sigma'_k)\\ &= \chi_{(t_1,\ldots,t_k)}(\sigma_1\sigma'_1 \ldots \sigma_k\sigma'_k)\\ &= \xi^{\sum_{i=1}^k if(\sigma_i\sigma'_i)}\\ &= \xi^{\sum_{i=1}^k if(\sigma_i)+\sum_{i=1}^k if(\sigma'_i)}\\ &= \xi^{\sum_{i=1}^k if(\sigma_i)}\xi^{\sum_{i=1}^k if(\sigma'_i)}\\ &= \chi_{(t_1,\ldots,t_k)}(\sigma)\chi_{(t_1,\ldots,t_k)}(\sigma').\end{aligned}$$

Thus $\chi_{(t_1,\ldots,t_k)}$ is a homomorphism from $B^m_{(t_1,\ldots,t_k)}$ into $\mathbb{C}^*$, which proves (i).

(ii) If $(t_1, \ldots, t_k) \neq (t'_1, \ldots, t'_k)$, let i be the least index such that $t'_i \neq t_i$. We may assume, without any loss of generality, that $t_i < t'_i$, that is, $P_i \subset P'_i$. If $g \in B^m_n$ is such that $\phi(g)P_i = P_i$, let $j \in P'_i \backslash P_i$; then $\phi(g)(j) \in P_l$, $l \neq i$, and we define

$$x = \begin{pmatrix}1 & 2 & \ldots & j & \ldots & n\\ 1 & 2 & \ldots & \xi j & \ldots & n\end{pmatrix} = \begin{pmatrix} j \\ \xi j\end{pmatrix}.$$

If $\phi(g)P_i \neq P_i$ then there exists $j \in P_i \subset P'_i$ such that $\phi(g)(j) \in P$, $1 \leqq l \leqq k$, $l \neq i$, and for this j we define x as above. In each case $\chi_{(t'_1,\ldots,t'_k)}(x) = \xi^i$, but

$$\begin{aligned}\chi^g_{(t_1,\ldots,t_k)}(x) &= \chi_{(t_1,\ldots,t_k)}(gxg^{-1})\\ &= \chi_{(t_1,\ldots,t_k)}(\phi(g)x\phi(g)^{-1})\\ &= \chi_{(t_1,\ldots,t_k)}\left(\begin{pmatrix}\phi(g)(j)\\ \xi\phi(g)(j)\end{pmatrix}\right) = \begin{pmatrix} l \\ \xi l\end{pmatrix}\\ &= \xi^l,\ l \neq i.\end{aligned}$$

If $(t'_1, \ldots, t'_k) = (t_1, \ldots, t_k)$ and $g \in B^m_n \backslash B^m_{(t_1,\ldots,t_k)}$ then there exists at least one index i, $1 \leqq i \leqq k$, and an integer $j \in P_i$ such that $\phi(g)(j) \in P_l$, $l \neq i$. Once again we define $x \in \ker \phi$ as above for this particular j. Clearly

$$\chi^g_{(t_1,\ldots,t_k)}(x) = \xi^l \neq \xi^i = \chi_{(t'_1,\ldots,t'_k)}(x),$$

which completes the proof.

4. Representations of B^m_n.

It is well known that the number of inequivalent irreducible linear representations (henceforth abbreviated as i.l.r.) of a Young subgroup $S_{(t_1,\ldots,t_k)} = S_{t_1} \times \ldots \times S_{t_k}$ is equal to $p(t_1) \ldots p(t_k)$. This enables us to state our main result.

THEOREM 4.1. *A full set of inequivalent i.l.r. of B^m_n is given by*

$$\{(\chi_{(t_1,\ldots,t_m)} \otimes \mathbb{P}) \uparrow B^m_n\},$$

where $(t_1, \ldots, t_m)$ ranges over all permissible m-tuples, $\chi_{(t_1,\ldots,t_m)}$ is the basic linear *representation of $B^m_{(t_1,\dots,t_m)}$ and $\mathbb{P}$ is an i.l.r. of $B^m_{(t_1,\dots,t_m)}$ lifted from an i.l.r. P of $S_{(t_1,\dots,t_m)}$, where P ranges over a complete set of inequivalent i.l.r of $S_{(t_1,\dots,t_m)}$ and $\uparrow$ denotes induction of a representation.*

Proof. It is clear from Lemma 2.3 that the cardinality of the above set is equal to the number of inequivalent i.l.r. of B_n^m.

Let $(t_1, \dots, t_m)$ and $(t'_1, \dots, t'_m)$ be two arbitrary permissible m-tuples and $\chi_{(t_1,\dots,t_m)}$, $\chi_{(t'_1,\dots,t'_m)}$ the basic linear representations of $B^m_{(t_1,\dots,t_m)}$ and $B^m_{(t'_1,\dots,t'_m)}$ respectively. Let $\mathbb{P}$ and $\mathbb{P}'$ be two i.l.r. of $B^m_{(t_1,\dots,t_m)}$ and $B^m_{(t'_1,\dots,t'_m)}$ lifted from the i.l.r. P and P' of $S_{(t_1,\dots,t_m)}$ and $S_{(t'_1,\dots,t'_m)}$ respectively. If $\hat{\psi}$, $\hat{\psi}'$, ψ and ψ' denote the characters of $\mathbb{P}$, $\mathbb{P}'$, P and P' respectively, then we prove that

$$((\chi_{(t_1,\dots,t_m)}\hat{\psi})\uparrow B_n^m, (\chi_{(t'_1,\dots,t'_m)}\hat{\psi}')\uparrow B_n^m)_{B_n^m} = 0$$

unless $(t_1, \dots, t_m) = (t'_1, \dots, t'_m)$ and $\psi = \psi'$ in which case it is equal to 1. This will complete the proof of the theorem.

By Frobenius' reciprocity theorem and Mackey's subgroup theorem, the above inner product is equal to

$$((\chi_{(t_1,\dots,t_m)}\hat{\psi}), (((\chi_{(t'_1,\dots,t'_m)}\hat{\psi}')\uparrow B_n^m)\downarrow B^m_{(t_1,\dots,t_m)}))_{B^m_{(t_1,\dots,t_m)}}$$
$$= \sum_x ((\chi_{(t_1,\dots,t_m)}\hat{\psi}), ((\chi_{(t'_1,\dots,t'_m)}\hat{\psi}')^x \downarrow H_x)\uparrow B^m_{(t_1,\dots,t_m)})_{B^m_{(t_1,\dots,t_m)}}$$
$$= \sum_x ((\chi_{(t_1,\dots,t_m)}\hat{\psi})\downarrow H_x, (\chi_{(t'_1,\dots,t'_m)}\hat{\psi}')\downarrow H_x)_{H_x},$$

where $H_x = B^m_{(t_1,\dots,t_m)} \cap x^{-1}B^m_{(t'_1,\dots,t'_m)}x$ and x ranges over all representative elements of a double coset decomposition of B_n^m relative to the generalized Young subgroups $B^m_{(t_1,\dots,t_m)}$ and $B^m_{(t'_1,\dots,t'_m)}$.

We claim that each of the terms in the above summation is zero except in the case noted earlier. For, if for some x

$$(\chi_{(t_1,\dots,t_m)}\hat{\psi})\downarrow H_x \text{ and } (\chi_{(t'_1,\dots,t'_m)}\hat{\psi}')^x\downarrow H_x$$

have an irreducible component in common then so do

$$(\chi_{(t_1,\dots,t_m)}\psi)\downarrow \ker\phi \quad \text{and} \quad (\chi_{(t'_1,\dots,t'_m)}\psi')^x\downarrow \ker\phi.$$

(Note that $\ker\phi \subseteq B^m_{(t_1,\dots,t_m)} \cap x^{-1}B^m_{(t'_1,\dots,t'_m)}x$ for all x.)

But in this case these representations are multiples of

$$\chi_{(t_1,\dots,t_m)}\downarrow \ker\phi \quad \text{and} \quad \chi^x_{(t'_1,\dots,t'_m)}\downarrow \ker\phi$$

respectively. Both of these representations being irreducible, we get

$$\chi_{(t_1,\dots,t_m)} = \chi^x_{(t'_1,\dots,t'_m)} \text{ on } \ker\phi.$$

By Lemma 3.2, this implies that

$$(t_1, \dots, t_m) = (t'_1, \dots, t'_m) \quad \text{and} \quad x \in B^m_{(t_1,\dots,t_m)}.$$

Thus the possibility of getting a nonzero term arises from the inner product

$$(\chi_{(t_1,\dots,t_m)}\hat{\psi},\ \chi_{(t_1,\dots,t_m)}\hat{\psi}')_{B^m_{(t_1,\dots,t_m)}}$$

which is equal to

$$\frac{1}{|B^m_{(t_1,\dots,t_m)}|}\sum_{\sigma\in B^m_{(t_1,\dots,t_m)}}(\chi_{(t_1,\dots,t_m)}\hat{\psi})(\sigma)\overline{(\chi_{(t_1,\dots,t_m)}\hat{\psi}')(\sigma)}$$

$$=\frac{1}{|B^m_{(t_1,\dots,t_m)}|}\sum_{\sigma\in B^m_{(t_1,\dots,t_m)}}\chi_{(t_1,\dots,t_m)}(\sigma)\overline{\chi_{(t_1,\dots,t_m)}(\sigma)}\hat{\psi}(\sigma)\overline{\hat{\psi}'(\sigma)}$$

$$=\frac{1}{|B^m_{(t_1,\dots,t_m)}|}\sum_{\sigma\in B^m_{(t_1,\dots,t_m)}}\psi(\sigma)\overline{\psi(\sigma)}$$

(since $\chi_{(t_1,\dots,t_m)}(\sigma)\overline{\chi_{(t_1,\dots,t_m)}(\sigma)}=|\chi_{(t_1,\dots,t_m)}(\sigma)|^2=1$)

$$=\frac{1}{|S_{(t_1,\dots,t_m)}|}\sum_{\sigma_1\in S_{(t_1,\dots,t_m)}}\psi(\sigma_1)\overline{\psi'(\sigma_1)}$$

$$=(\psi,\ \psi')$$

and this is nonzero if and only if $\psi=\psi'$, in which case it is equal to 1 because ψ and ψ' are both irreducible.

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