COMPOSITIO MATHEMATICA

Tate modules of universal p-divisible groups

Eike Lau

Compositio Math. **146** (2010), 220–232.

doi:10.1112/S0010437X09004242

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

Compositio Math. 146 (2010) 220–232
doi:10.1112/S0010437X09004242

Tate modules of universal p-divisible groups

Eike Lau

ABSTRACT

A p-divisible group over a complete local domain determines a Galois representation on the Tate module of its generic fibre. We determine the image of this representation for the universal deformation in mixed characteristic of a bi-infinitesimal group and for the p-rank strata of the universal deformation in positive characteristic of an infinitesimal group. The method is a reduction to the known case of one-dimensional groups by a deformation argument based on properties of the stratification by Newton polygons.

1. Introduction

Let G be a p-divisible group of dimension d and height $c+d$ over an algebraically closed field k of characteristic p. Its universal deformation $\mathcal{G}$ is defined over a $W(k)$-algebra R isomorphic to an algebra of power series in cd variables. For every point $x \in \operatorname{Spec} R$ we have a natural Galois representation, also referred to as local p-adic monodromy,

$$\rho_x : \operatorname{Gal}(\bar{x}/x) \to \operatorname{GL}(T_p\mathcal{G}(\bar{x})) \cong \operatorname{GL}_{e(x)}(\mathbb{Z}_p)$$

where $e(x)$ is the étale rank of the fibre $\mathcal{G}_x$. Note that $e(x) = c+d$ if x is a point of characteristic zero and $e(x) \leqslant c$ if x is a point of characteristic p. Let U_e be the locally closed subset of $\operatorname{Spec} R$ where $e(x) = e$. If G has positive dimension and $e \leqslant c$ then U_e lies in $\operatorname{Spec} R/pR$.

THEOREM 1.1. *If G is bi-infinitesimal and x is the generic point of* $\operatorname{Spec} R$ *then the image of ρ_x is the subgroup of all elements whose determinant is a dth power.*

THEOREM 1.2. *If G is infinitesimal and x is a generic point of U_e for some $e \leqslant c$ then ρ_x is surjective.*

This is consistent with the general expectation that the monodromy of a universal family should be as large as possible, where the restriction in Theorem 1.1 is caused by a well-known result of Raynaud [Ray74] saying that the determinant of ρ_x is the dth power of the cyclotomic character. The present article was motivated by recent work of Tian [Tia07] and Strauch [Str07]. Instead of any attempt for a complete review of the literature on p-adic monodromy of p-divisible groups and abelian varieties we refer the reader to [AN06, Cha00, Cha08] and the references given therein.

If G is one-dimensional, Theorems 1.1 and 1.2 are proved in [Str07] using the theory of Drinfeld level structures. This result actually applies to one-dimensional formal modules over the ring of integers in a local field. Previously, a number of cases were established by different methods.

Received 27 March 2008, accepted in final form 17 February 2009, published online 23 November 2009.
2000 Mathematics Subject Classification 14L05.
Keywords: p-divisible groups, local p-adic monodromy, Newton stratification, deformation theory.

Theorem 1.1 for one-dimensional p-divisible groups is proved by Rosen and Zimmermann [RZ89, Zim90]. Theorem 1.2 for the one-dimensional group of slope $1/2$ is a classical result of Igusa [Igu68], see [Kat73, Theorem 4.3]. Theorem 1.2 for the one-dimensional group of slope $1/2$ is a classical result of Igusa [Igu68], see [Kat73, Theorem 4.3]. When x is the generic point of Spec R/pR, Y. Tian proved that ρ_x is surjective for the one-dimensional group of slope $1/3$ and conjectured the surjectivity of ρ_x for elementary p-divisible groups of arbitrary dimension; see the first version of [Tia07]. Recently Tian has extended his methods to cover all p-divisible groups of a-number one, see [Tia07].

The present proof of Theorems 1.1 and 1.2 is a reduction to the one-dimensional case using the results of Oort and de Jong [JO00, Oor00, Oor01] on the Newton stratification of Spec R/pR. More precisely, to prove Theorem 1.2 we pass to the complete local ring (denoted S) of R/pR at a generic point of a suitable Newton stratum chosen so that $\mathcal{G} \otimes S$ is the extension of a one-dimensional infinitesimal p-divisible group $\mathcal{H}$ and a group of multiplicative type. Since the Tate module of the latter is trivial, we only have to observe that $\mathcal{H}$ over S is necessarily the universal deformation of its special fibre, see Lemma 3.1 and Proposition 6.1.

The proof of Theorem 1.1 is more complicated because p-divisible groups of multiplicative type over a field of characteristic zero have non-trivial Tate modules. This leads us to consider different complete local rings of R and their contributions to the image of the Galois representation at the same time. Notably, we need the following observation, proved in Proposition 7.1: if A denotes the complete local ring of R at the prime pR and if F' is an algebraic closure of the residue field of A, then the set of ring homomorphisms $A \to W(F')$ lifting the given homomorphism $A \to F'$ is bijective to the set of deformations over $W(F')$ of the fibre $\mathcal{G} \otimes F'$. As a consequence, the contribution of A to the Galois representation is sufficiently large, see Lemma 4.3.

Below we first explain the proof of Theorem 1.1 in §§ 3 and 4 and postpone the required Lemmas 3.1 and 4.3 until §§ 6 and 7. They are straightforward applications of the deformation theory of p-divisible groups developed in [Ill85, Mes72]. An alternative proof of Lemma 3.1 in the case where G has a-number one is given in [Tia07].

2. Newton strata

For reference let us recall the results on Newton strata we need. Let $\bar{R} = R/pR$.

The Newton polygon of a p-divisible group H is denoted $\mathcal{N}(H)$. Newton polygons are normalized so that slope 0 corresponds to étale groups and slope 1 to groups of multiplicative type. The set of Newton polygons carries a partial order such that $\beta \preceq \gamma$ if and only if β and γ have the same endpoints and no point of β lies strictly below γ. The subset V_β of Spec $\bar{R}$ where the Newton polygon of the universal deformation is $\preceq \beta$ is closed. We denote by V_β° the open subset of V_β where the polygon is equal to β. Let $\operatorname{codim}(\beta)$ be the number of lattice points that lie strictly below β and on or above the unique ordinary Newton polygon with the same endpoints as β (ordinary polygons are those whose slopes are all 0 or 1).

THEOREM 2.1 [Oor01, Theorem 2.10]. *The set V_β is non-empty if and only if $\mathcal{N}(G) \preceq \beta$. In that case all irreducible components of V_β have codimension* $\operatorname{codim}(\beta)$ *in* Spec $\bar{R}$, *and consequently V_β is the closure of V_β°. Generically on V_β the a-number is at most one.*

At those points where the a-number is one the strata are nested nicely.

PROPOSITION 2.2. *Let $x \in$ Spec $\bar{R}$ be given such that $a(\mathcal{G}_x) = 1$. Then every $V_{\beta,x} = V_\beta \cap$ Spec $\bar{R}_x$ is regular and thus irreducible. If $V_{\beta,x}$ and $V_{\gamma,x}$ are non-empty, in other words if $\mathcal{N}(\mathcal{G}_x) \preceq \beta$ and $\mathcal{N}(\mathcal{G}_x) \preceq \gamma$, then $V_{\beta,x} \subseteq V_{\gamma,x}$ if and only if $\beta \preceq \gamma$.*

If x is the maximal ideal of $\bar{R}$ this is [Oor00, Theorem 3.2]. The general case can be reduced to this case as is certainly well known, but for completeness a proof is recalled in § 5 below.

We need the following supplement to Theorem 2.1. We write $x \leqslant z$ if x lies in the closure of z.

Corollary 2.3. *Assume that Newton polygons $\beta \preceq \gamma \preceq \delta$, a point $x \in V_\beta$, and a generic point $z \in V_\delta$ are given such that $x \leqslant z$. Then there is a generic point $y \in V_\gamma$ such that $x \leqslant y \leqslant z$.*

Proof. By a change of β we may assume that x lies in V_β°. Let Z be the irreducible component of V_δ that contains z and let x' be a generic point of $Z \cap V_\beta$ such that $x \leqslant x'$. Then x' also lies in V_β°. If n denotes the codimension of x' in Z, the purity theorem [JO00, Theorem 4.1] implies that $n \leqslant \operatorname{codim}(\beta) - \operatorname{codim}(\delta)$. By Theorem 2.1 we have equality and x' is a generic point of V_β, thus $a(\mathcal{G}_{x'}) = 1$. By Proposition 2.2 there is a unique generic point y of V_γ between x' and z. □

3. Reduction to one-dimensional groups

Let $\bar{R} = R/pR$ as before. We begin with the proof of Theorem 1.2 in the case where x is the generic point of Spec $\bar{R}$, or equivalently $e = c$.

Proof of Theorem 1.2 if $e = c$. We may assume that $d \geqslant 1$. Let $\beta = \mathcal{N}(G)$ and let γ be the Newton polygon given by the following slope sequence.

$$\gamma = \bigg(\underbrace{\frac{1}{c+1}, \ldots, \frac{1}{c+1}}_{c+1}, \underbrace{1, \ldots, 1}_{d-1} \bigg).$$

Then $\beta \preceq \gamma$ because G is assumed to be infinitesimal. Choose a generic point $\mathfrak{p}$ of the Newton stratum V_γ of Spec $\bar{R}$, let S be the completion of the local ring $\bar{R}_\mathfrak{p}$, and let K be its residue field. Since $\bar{R}$ is regular and regularity is preserved under localizations and completions, S is a complete regular local ring. By Theorem 2.1, the dimension of S is c and the Newton polygon of $\mathcal{G} \otimes K$ is γ.

Let K' be an algebraic closure of K and let $S' = K'[[t_1, \ldots, t_c]]$. The projection $S \to K$ admits a section $K \to S$ in the category of k-algebras because K is formally smooth over k as k is perfect. Hence there is an isomorphism of k-algebras $S \cong K[[t_1, \ldots, t_c]]$ compatible with the projections to K; we choose one such isomorphism. Then S becomes a subring of S' by $t_i \mapsto t_i$ and S' becomes an R-algebra by the composition $R \to S \to S'$.

By the choice of γ, the fibre $\mathcal{G} \otimes K'$ has a unique p-divisible subgroup M' of multiplicative type and dimension $d-1$. The quotient $H' = (\mathcal{G} \otimes K')/M'$ is a one-dimensional infinitesimal p-divisible group over K' of height $c+1$. There is a unique lift of M' to a p-divisible subgroup of multiplicative type $\mathcal{M}' \subset \mathcal{G} \otimes S'$ (see [Gro74, chapitre 3.1] and [deJ95, Lemma 2.4.4]), and the quotient $\mathcal{H}' = (\mathcal{G} \otimes S')/\mathcal{M}'$ is a deformation of H' over S'.

Lemma 3.1. *In this situation $\mathcal{H}'$ is a universal deformation of H'.*

We postpone the proof until § 6. If $f : \operatorname{Spec} S' \to \operatorname{Spec} \bar{R}$ denotes the chosen morphism and $y \in \operatorname{Spec} S'$ is the generic point, then $f(y) = x$ because f is flat. The natural homomorphism

$\mathcal{G}(\bar{x}) \cong \mathcal{G}(\bar{y}) \to \mathcal{G}'(\bar{y})$ is bijective since $\mathcal{M}(\bar{y})$ is the zero group. Hence we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Gal}(\bar{y}/y) & \xrightarrow{\rho_y} & \mathrm{GL}(T_p\mathcal{H}'(\bar{y})) \\ \downarrow & & \Big| \cong \\ \mathrm{Gal}(\bar{x}/x) & \xrightarrow{\rho_x} & \mathrm{GL}(T_p\mathcal{G}(\bar{x})) \end{array}$$

Here ρ_y is surjective by [Str07, Theorem 2.1], so ρ_x is surjective as well. □

A modification of the argument gives Theorem 1.2 in general.

Proof of Theorem 1.2. By Theorem 2.1 combined with Proposition 2.2 the generic points of U_e are precisely the generic points of the Newton stratum V_ε where ε is the lowest Newton polygon with exactly e zeros,

$$\varepsilon = \Big(\underbrace{0, \dots, 0}_{e}, \underbrace{\frac{1}{c-e+1}, \dots, \frac{1}{c-e+1}}_{c-e+1}, \underbrace{1, \dots, 1}_{d-1} \Big).$$

Let γ, $\mathfrak{p}$, S', $\mathcal{H}'$, and $f : \operatorname{Spec} S' \to \operatorname{Spec} \bar{R}$ be chosen exactly as before with the additional requirement that $\mathfrak{p} \leqslant x$. This is possible by Corollary 2.3 applied to the points $\mathfrak{m}_{\bar{R}} \leqslant x$. The inverse image $f^{-1}(U_e)$ is equal to the locus U_e' in $\operatorname{Spec} S'$ where the étale rank of $\mathcal{G}'$ is equal to e. Let $y \in U_e'$ be the unique generic point. Since f is flat and since $\mathfrak{p} \leqslant x$ we have $f(y) = x$. Now the proof continues as before, using again [Str07, Theorem 2.1]. □

4. Galois action in characteristic zero

Assume now that x is the generic point of Spec R. Let $\mathbb{T} = T_p\mathcal{G}(\bar{x})$ and let

$$\chi : \mathrm{Gal}(\bar{x}/x) \to \mathbb{Z}_p^*$$

be the cyclotomic character. By [Ray74, Theorem 4.2.1 and its proof], $\mathrm{Gal}(\bar{x}/x)$ acts on $\Lambda^{c+d}(\mathbb{T})$ by χ^d. Let $\mathrm{GL}'(\mathbb{T})$ denote the subgroup of all elements of $\mathrm{GL}(\mathbb{T})$ whose determinant is a dth power and let $\mathrm{Gal}^\circ(\bar{x}/x)$ be the kernel of χ. The homomorphism ρ_x induces the following homomorphisms ρ' and ρ°:

$$\begin{array}{rcl} \rho' : \mathrm{Gal}(\bar{x}/x) & \longrightarrow & \mathrm{GL}'(\mathbb{T}) \\ \cup & & \cup \\ \rho^\circ : \mathrm{Gal}^\circ(\bar{x}/x) & \longrightarrow & \mathrm{SL}(\mathbb{T}) \end{array}$$

Lemma 4.1. *If ρ° is surjective then so is ρ'. For $d = 1$ the converse also holds.*

Proof. We have a homomorphism of exact sequences.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Gal}^\circ(\bar{x}/x) & \longrightarrow & \mathrm{Gal}(\bar{x}/x) & \xrightarrow{\chi} & \mathbb{Z}_p^* & \longrightarrow & 1 \\ & & \rho^\circ \big\downarrow & & \rho' \big\downarrow & & \big\downarrow (\ )^d & & \\ 1 & \longrightarrow & \mathrm{SL}(\mathbb{T}) & \longrightarrow & \mathrm{GL}'(\mathbb{T}) & \xrightarrow{\det} & \mathbb{Z}_p^{*,d} & \longrightarrow & 1 \end{array}$$

Both assertions follow easily. □

https://doi.org/10.1112/S0010437X09004242 Published online by Cambridge University Press

Theorem 4.2. *If G is bi-infinitesimal then ρ° is surjective.*

In view of Lemma 4.1 this is a refinement of Theorem 1.1; moreover, the case $d=1$ follows from [RZ89, Zim90] or from [Str08, Theorem 2.1.2]. By duality this also gives the case $c=1$ because the Tate module of $\mathcal{G}^\vee$ is $\mathrm{Hom}(\mathbb{T}, \mathbb{Z}_p(1))$.

Proof of Theorem 4.2. We may assume that $c, d \geqslant 2$. Let $\beta = \mathcal{N}(G)$ and consider the following Newton polygons γ_i with the same endpoints as β:

$$\gamma_1 = \Bigg(\underbrace{\frac{1}{c+1}, \dots, \frac{1}{c+1}}_{c+1}, \underbrace{1, \dots, 1}_{d-1}\Bigg)$$

$$\gamma_2 = \Bigg(\underbrace{0, \dots, 0}_{c-1}, \underbrace{\frac{d}{d+1}, \dots, \frac{d}{d+1}}_{d+1}\Bigg).$$

Since G is bi-infinitesimal we have $\beta \preceq \gamma_i$. Let $\mathfrak{p}_i \in \operatorname{Spec} R$ be a generic point of the Newton stratum $V_{\gamma_i} \subseteq \operatorname{Spec} R/pR$. By Theorem 2.1 the Newton polygon of the fibre $\mathcal{G}_{\mathfrak{p}_i}$ is γ_i and the codimension of $\mathfrak{p}_i$ in $\operatorname{Spec} R$ is $c_i + 1$ where $c_1 = c$ and $c_2 = d$. The complete local ring $S_i = \widehat{R}_{\mathfrak{p}_i}$ is regular and unramified in the sense that p is part of a minimal set of generators of the maximal ideal. Let S_i' be an unramified regular complete local ring of dimension $c_i + 1$ whose residue field K_i' is an algebraic closure of the residue field K_i of S_i and choose an embedding $S_i \to S_i'$ such that $S_i' \otimes_{S_i} K_i = K_i'$.

(More explicitly, put $S_i' = W(K_i')[[t_1, \dots, t_{c_i}]]$; then choose a Cohen ring C_i in S_i, an isomorphism of C_i-algebras $S_i \cong C_i[[t_1, \dots, t_{c_i}]]$, and an embedding of C_i into $W(K_i')$; extend this to an embedding $S_i \to S_i'$ by $t_i \mapsto t_i$.)

Let $\mathfrak{q} \subset R$ and $\mathfrak{q}_i \subset S_i'$ be the prime ideals generated by p. The complete local rings $A = \widehat{R}_{\mathfrak{q}}$ and $B_i = (\widehat{S}_i')_{\mathfrak{q}_i}$ are unramified discrete valuation rings. We have the following commutative diagram of rings:

$$\begin{tikzcd}
S_1' \arrow[d] & R \arrow[l] \arrow[r] \arrow[d] & S_2' \arrow[d] \\
B_1 & A \arrow[l] \arrow[r] & B_2
\end{tikzcd}$$

The scalar extensions of $\mathcal{G}$ to these rings admit natural filtrations of different types: since the fibre $\mathcal{G}_{\mathfrak{q}}$ is ordinary, over A there is an exact sequence of p-divisible groups

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{G} \otimes A \longrightarrow \mathcal{E} \longrightarrow 0 \tag{4.1}$$

where $\mathcal{M}$ is of multiplicative type of height d and $\mathcal{E}$ is étale of height c. By the choice of the polygons γ_i, over S_1' there is an exact sequence

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{G} \otimes S_1' \longrightarrow \mathcal{H}_1 \longrightarrow 0 \tag{4.2}$$

where $\mathcal{M}'$ is isomorphic to $\mu_{p^\infty}^{d-1}$ and $\mathcal{H}_1$ is bi-infinitesimal of dimension 1 and height $c+1$, while over S_2' there is an exact sequence

$$0 \longrightarrow \mathcal{H}_2 \longrightarrow \mathcal{G} \otimes S_2' \longrightarrow \mathcal{E}' \longrightarrow 0 \tag{4.3}$$

where $\mathcal{E}'$ is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^{c-1}$ and $\mathcal{H}_2$ is bi-infinitesimal of dimension d and height $d+1$. In both cases $\mathcal{H}_i$ is the universal deformation of its special fibre over $W(K_i)$-algebras because $\mathcal{H}_i \otimes S_i/pS_i$ is the universal deformation over K_i-algebras according to Lemma 3.1 (applied to the dual if $i=2$). Since over B_i all homomorphisms from groups of multiplicative type to étale groups are trivial, as subgroups of $\mathcal{G}\otimes B_1$ and $\mathcal{G}\otimes B_2$ we have

$$\mathcal{M}'\otimes_{S_1'} B_1\subseteq \mathcal{M}\otimes_A B_1,\quad \mathcal{M}\otimes_A B_2\subseteq \mathcal{H}_2\otimes_{S_2'} B_2. \tag{4.4}$$

Let F' be an algebraic closure of the residue field of A, let $A'=W(F')$, and choose an embedding $\sigma: A\to A'$ extending the given homomorphism $A\to F'$. This time the choice makes a difference and will be fixed later. In order to relate the various Galois actions on the Tate module we choose an algebraically closed field Ω together with embeddings of A' and both B_i into Ω that coincide over A. For every subring X of Ω let $\mathrm{Gal}_X=\pi_1(\mathrm{Quot}(X),\Omega)$ and denote by Gal_X° the kernel of the cyclotomic character $\mathrm{Gal}_X\to\mathbb{Z}_p^*$.

If we write $\mathbb{T}=T_p\mathcal{G}(\Omega)$ by a harmless change of notation, we have to show that the natural homomorphism $\rho_R^\circ:\mathrm{Gal}_R^\circ\to\mathrm{SL}(\mathbb{T})$ is surjective. Let

$$\begin{aligned}\mathbb{T}_1&=T_p\mathcal{H}_1(\Omega), & \mathbb{E}&=T_p\mathcal{E}(\Omega), & \mathbb{E}'&=T_p\mathcal{E}'(\Omega),\\ \mathbb{T}_2&=T_p\mathcal{H}_2(\Omega), & \mathbb{M}&=T_p\mathcal{M}(\Omega), & \mathbb{M}'&=T_p\mathcal{M}'(\Omega).\end{aligned}$$

From (4.2), (4.1) and (4.3) in that order we obtain the following exact sequences of free $\mathbb{Z}_p$-modules with actions of the designated groups Gal_X where the action on $\mathbb{T}$ is induced from the action of Gal_R by the natural homomorphism $\mathrm{Gal}_X\to\mathrm{Gal}_R$. The vertical arrows exist by (4.4).

$$\begin{tikzcd}
\mathrm{Gal}_{S_1'} & 0 \arrow[r] & \mathbb{M}' \arrow[r] \arrow[d] & \mathbb{T} \arrow[r] \arrow[d, equal] & \mathbb{T}_1 \arrow[r] \arrow[d] & 0\\
\mathrm{Gal}_{A'} & 0 \arrow[r] & \mathbb{M} \arrow[r] \arrow[d] & \mathbb{T} \arrow[r] \arrow[d, equal] & \mathbb{E} \arrow[r] \arrow[d] & 0\\
\mathrm{Gal}_{S_2'} & 0 \arrow[r] & \mathbb{T}_2 \arrow[r] & \mathbb{T} \arrow[r] & \mathbb{E}' \arrow[r] & 0
\end{tikzcd}$$

Here $\mathrm{Gal}_{S_1'}^\circ$ acts trivially on $\mathbb{M}'$ and $\mathrm{Gal}_{S_2'}$ acts trivially on $\mathbb{E}'$. By the known cases $d=1$ and $c=1$ of Theorem 4.2, the induced homomorphisms $\mathrm{Gal}_{S_i'}^\circ\to\mathrm{SL}(\mathbb{T}_i)$ are surjective. In many cases this already implies that ρ_R° is surjective, but in order to conclude in general we also need the action of $\mathrm{Gal}_{A'}^\circ$. Let $U\subseteq\mathrm{SL}(\mathbb{T})$ be the unipotent subgroup that acts trivially on $\mathbb{M}$ and on $\mathbb{E}$; thus $U\cong\mathbb{Z}_p^{cd}$. Then $\mathrm{Gal}_{A'}^\circ$ acts on $\mathbb{T}$ by a homomorphism

$$\rho_{A'}^\circ:\mathrm{Gal}_{A'}^\circ\to U.$$

LEMMA 4.3. *For a suitable choice of the embedding $\sigma: A\to A'$, the homomorphism $\rho_{A'}^\circ$ is surjective.*

We postpone the proof of Lemma 4.3 until § 7 and continue in the proof of Theorem 4.2. Let U_1 denote the group of all elements of $\mathrm{SL}(\mathbb{T})$ that act trivially on $\mathbb{M}'$ and on $\mathbb{T}_1$, let $U_2\subseteq\mathrm{SL}(\mathbb{T})$ be the group that acts trivially on $\mathbb{T}_2$ and on $\mathbb{E}'$, and let H be the image of $\mathrm{Gal}_R^\circ\to\mathrm{SL}(\mathbb{T})$. Then H contains U by Lemma 4.3, so $H\cap U_1$ contains $U\cap U_1$. Since $\mathrm{Gal}_{S_1'}^\circ\to\mathrm{SL}(\mathbb{T}_1)$ is surjective, $H\cap U_1$ is invariant under the conjugation action of $\mathrm{SL}(\mathbb{T}_1)$ on U_1. Thus $H\cap U_1=U_1$ and similarly $H\cap U_2=U_2$. It follows that H contains the (pointwise) stabilizers of $\mathbb{M}'\subset\mathbb{T}$ and of $\mathbb{E}'^\vee\subset\mathbb{T}^\vee$. These generate $\mathrm{SL}(\mathbb{T})$ as is easily shown by straightforward considerations of matrices. □

The following example shows that the use of Lemma 4.3 cannot be avoided.

Example 4.4. There is a subgroup H of $G=\mathrm{GL}_4(\mathbb{F}_2)$ of index 8 such that, for any parabolic subgroup $P\subset G$ of type $(3,1)$ or $(1,3)$, the projection $\pi: H\cap P\to\mathrm{GL}_3(\mathbb{F}_2)$ is bijective.

Proof (communicated by V. Paskunas). For any isomorphism $\alpha : A_8 \cong G$, which exists by [Con71, Theorem 6], we take $H = \alpha(A_7)$. Let $B \subset P$ be a Borel subgroup. Since B is a 2-Sylow subgroup of G and since $[G : H] = 2^3$, we have $[B : H \cap B] = 2^3$; hence $[P : H \cap P] = 2^3$, and thus $|H \cap P| = 7 \cdot 6 \cdot 4 = |\mathrm{GL}_3(\mathbb{F}_2)|$. Therefore, it suffices that $U = \mathrm{Ker}(\pi)$ is trivial. Since U is contained in the unipotent radical of P, it is an $\mathbb{F}_2$-vector space of dimension at most 3. However, any 2-Sylow subgroup of A_7 is non-commutative with 8 elements, so $|U| < 8$. Let $\sigma \in H \cap P$ be of order 7. Since in A_7 an element of order 7 and an element of order 2 cannot commute, all σ-orbits in $U \backslash \{1\}$ have seven elements. Hence $|U| = 1$. □

5. Deformations of p-divisible groups

Before proving Lemmas 3.1 and 4.3 let us recall some aspects of the deformation theory of p-divisible groups. Let G be a p-divisible group over an arbitrary ring R in which p is nilpotent and write $\Lambda_G = \mathrm{Hom}_R(\mathrm{Lie}\, G, \omega_{G^\vee})$. Here $\mathrm{Lie}\, G$ is the Lie algebra of G, which is a finite projective R-module, and $\omega_{G^\vee}$ is the dual of the Lie algebra of the Serre dual $G^\vee$. If $R = S/I$ where S is an I-adically complete ring, let $\mathrm{Def}_{S/R}(G)$ denote the set of isomorphism classes of lifts of G to S.

Theorem 5.1. *If $I^2 = 0$ then $\mathrm{Def}_{S/R}(G)$ is naturally a torsor under the R-module*

$$\mathrm{Hom}_R(\omega_{G^\vee}, I \otimes \mathrm{Lie}\, G) = \mathrm{Hom}_R(\Lambda_G, I).$$

This is classical and follows either from [Ill85, théorème 4.4 and corollaire 4.7] or (except for the existence of lifts) from the crystalline deformation theorem [Mes72, V, Theorem 1.6], because the set of lifts to $\mathbb{D}(G)_S$ of the Hodge filtration

$$0 \longrightarrow \omega_{G^\vee} \xrightarrow{i} \mathbb{D}(G)_R \xrightarrow{\pi} \mathrm{Lie}\, G \longrightarrow 0$$

is a torsor under $\mathrm{Hom}_R(\omega_{G^\vee}, I \otimes \mathrm{Lie}\, G)$; here $\mathbb{D}(G)$ is the covariant Dieudonné crystal defined in *loc. cit.* Both constructions give the same action of $\mathrm{Hom}_R(\Lambda_G, I)$ on $\mathrm{Def}_{S/R}(G)$ but we could not find a reference for this fact (and will not use it).

Let $\Omega_R = \Omega^1_{R/\mathbb{Z}}$ be the absolute module of differentials of R. Theorem 5.1 implies formally that for every p-divisible group G over R as above there is a ‘Kodaira–Spencer’ homomorphism

$$\kappa'_G : \omega_{G^\vee} \to \Omega_R \otimes \mathrm{Lie}\, G \quad \text{or equivalently } \kappa_G : \Lambda_G \to \Omega_R,$$

uniquely determined by the following property. For any ring homomorphism $f : R \to A$ where $A = B/I$ such that $I^2 = 0$, denote by $\mathrm{Lift}_{B/A}(f)$ the set of ring homomorphisms $R \to B$ lifting f, which is either the empty set or a torsor under the A-module $\mathrm{Hom}_R(\Omega_R, I)$. Then the obvious map

$$\mathrm{Lift}_{B/A}(f) \to \mathrm{Def}_{B/A}(G \otimes_R A) \tag{5.1}$$

is equivariant with respect to the homomorphism $\mathrm{Hom}_R(\Omega_R, I) \to \mathrm{Hom}_R(\Lambda_G, I)$ induced by κ_G. The homomorphism κ_G is functorial in R in the obvious sense. If one uses the crystalline construction of the torsor structure in Theorem 5.1, then κ'_G can be written down directly in terms of the connection $\nabla : \mathbb{D}(G)_R \to \Omega_R \otimes \mathbb{D}(G)_R$, namely $\kappa'_G = (\mathrm{id} \otimes \pi) \circ \nabla \circ i$.

A homomorphism of p-divisible groups $G \to H$ over R induces a homomorphism of arrows (a commutative square) $\kappa'_G \to \kappa'_H$. In the special case of an exact sequence of p-divisible groups $0 \to M \to G \to H \to 0$ where M is of multiplicative type, and thus $\omega_{G^\vee} \cong \omega_{H^\vee}$, this translates into the following commutative triangle with split injective λ.

$$\begin{tikzcd} \Lambda_H \arrow[rr, hook, "\lambda"] \arrow[dr, "\kappa_H"'] & & \Lambda_G \arrow[dl, "\kappa_G"] \\ & \Omega_R & \end{tikzcd} \tag{5.2}$$

If G is a p-divisible group over a field k of characteristic p and $\mathcal{G}$ is a deformation of G over a complete local noetherian k-algebra R with residue field k, we may consider the following composite homomorphism $\bar{\kappa}_{\mathcal{G}}$:

$$\bar{\kappa}_{\mathcal{G}} : \Lambda_G \cong \Lambda_{\mathcal{G}} \otimes k \xrightarrow{\kappa_{\mathcal{G}} \otimes \mathrm{id}} \Omega_R \otimes k \to \widehat{\Omega}_{R/k} \otimes k \cong \mathfrak{m}_R/\mathfrak{m}_R^2.$$

The deformation $\mathcal{G}$ is universal if and only if R is regular and $\bar{\kappa}_{\mathcal{G}}$ is bijective; let us call $\mathcal{G}$ versal if R is regular and $\bar{\kappa}_{\mathcal{G}}$ is injective. In the universal case $\kappa_{\mathcal{G}}$ induces an isomorphism $\Lambda_{\mathcal{G}} \cong \widehat{\Omega}_{R/k}$ because both modules are free over R. If $\mathcal{G}$ is universal and k is perfect then $\kappa_{\mathcal{G}}$ is an isomorphism $\Lambda_{\mathcal{G}} \cong \Omega_R$ because in that case $\Omega_R \cong \widehat{\Omega}_{R/k}$.

As announced earlier, we conclude this section with a proof of Proposition 2.2.

Proof of Proposition 2.2. We use a variant of the transitivity argument of [Oor01, Proposition 2.8]. Let K be the residue field of $\bar{R}$ at x, let $S = \bar{R}[[t_1, \ldots, t_{cd}]]$, and let $S(x) = K[[t_1, \ldots, t_{cd}]]$. There is a deformation $\mathcal{H}$ over S of $\mathcal{G} \otimes \bar{R}$ so that $\mathcal{H}(x) = \mathcal{H} \otimes_S S(x)$ is the universal deformation in equal characteristic of its special fibre $\mathcal{G} \otimes_R K$. Indeed, let $J = (t_1, \ldots, t_{cd})$ as an ideal of S and choose an isomorphism of $\bar{R}$-modules $u : \Lambda_{\mathcal{G} \otimes \bar{R}} \cong J/J^2$. Let $\mathcal{H}_2$ over S/J^2 be the deformation of $\mathcal{G} \otimes \bar{R}$ that, under the torsor structure of Theorem 5.1, differs from the trivial deformation by u. The required $\mathcal{H}$ is any deformation over S of $\mathcal{H}_2$, which exists by Theorem 5.1 again.

Universality of $\mathcal{G}$ gives a homomorphism $\varphi : \bar{R} \to S$ such that $\mathcal{H} \cong \mathcal{G} \otimes_\varphi S$ as deformations of G. Since $\bar{R} \to S \to \bar{R}$ is the identity, the inverse image of the maximal ideal under $\bar{R} \to S \to S(x)$ is x, so there is a local homomorphism $\psi : \bar{R}_x \to S(x)$ making the following commute.

$$\begin{array}{ccc} \bar{R} & \xrightarrow{\varphi} & S = \bar{R}[[t_1, \ldots, t_{cd}]] \\ \downarrow & & \downarrow \\ \bar{R}_x & \xrightarrow{\psi} & S(x) = K[[t_1, \ldots, t_{cd}]] \end{array}$$

It follows that the universal group $\mathcal{H}(x)$ is isomorphic to $(\mathcal{G} \otimes \bar{R}_x) \otimes_\psi S(x)$. In particular, the inverse images of the various $V_{\beta,x}$ under $\operatorname{Spec} \psi : \operatorname{Spec} S(x) \to \operatorname{Spec} \bar{R}_x$ form the Newton stratification given by $\mathcal{H}(x)$. Since that stratification has the required properties by [Oor00, Theorem 3.2], the proposition follows if we show that $S(x)$ is isomorphic via ψ to a power series ring over the completion of $\bar{R}_x$.

Let $\mathfrak{m} \subset \bar{R}_x$ and $\mathfrak{n} \subset S(x)$ be the maximal ideals. Since $\bar{R}_x$ is regular and ψ an isomorphism on residue fields, it suffices that $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$ is injective. Now $\mathfrak{m}/\mathfrak{m}^2$ is a submodule of $\Omega_{\bar{R}_x} \otimes K$, and $\mathfrak{n}/\mathfrak{n}^2$ is isomorphic to $\widehat{\Omega}_{S(x)/K} \otimes K$, so it suffices that $\Omega_{\bar{R}_x} \otimes_\psi S(x) \to \widehat{\Omega}_{S(x)/K}$ is an isomorphism. This follows because $\Lambda_{\mathcal{G} \otimes \bar{R}_x} \otimes_\psi S(x) \cong \Lambda_{\mathcal{H}(x)}$ and the Kodaira–Spencer homomorphisms $\Lambda_{\mathcal{G} \otimes \bar{R}_x} \to \Omega_{\bar{R}_x}$ and $\Lambda_{\mathcal{H}(x)} \to \widehat{\Omega}_{S(x)/K}$ are isomorphisms by the universality of $\mathcal{G}$ and of $\mathcal{H}(x)$. □

https://doi.org/10.1112/S0010437X09004242 Published online by Cambridge University Press

6. Universality over completions

In this section we prove Lemma 3.1, but we consider a more general situation. Assume that G is a p-divisible group over a perfect field k of characteristic p, let R be its universal deformation ring over k, so $R \cong k[[t_1, \dots, t_{cd}]]$, and let $\mathcal{G}$ be the universal deformation over R.

For an arbitrary prime $\mathfrak{p} \in \operatorname{Spec} R$ we consider the complete local ring $S = \widehat{R}_{\mathfrak{p}}$ with residue field $K = S/\mathfrak{m}_S$. The maximal subgroup of multiplicative type M of $\mathcal{G} \otimes K$ lifts uniquely to a subgroup of multiplicative type $\mathcal{M}$ of $\mathcal{G} \otimes S$. The quotient $\mathcal{H} = (\mathcal{G} \otimes S)/\mathcal{M}$ is a deformation over S of the p-divisible group $H = (\mathcal{G} \otimes K)/M$ over K. To ask whether $\mathcal{H}$ is a universal or versal deformation of H makes sense only after a structure of K-algebra is chosen on S, and in general the answer depends on the choice (but not in the special case of Lemma 3.1).

Denote by Σ (or $\bar{\Sigma}$) the set of all k-algebra homomorphisms $\sigma : K \to S$ (or $\bar{\sigma} : K \to S/\mathfrak{m}_S^2$) lifting the identity of K. Since k is perfect, K is formally smooth over k, so Σ is non-empty and the reduction map $\Sigma \to \bar{\Sigma}$ is surjective. The set $\bar{\Sigma}$ is a torsor under the finite-dimensional K-vector space $\operatorname{Hom}_K(\Omega_K, \mathfrak{m}_S/\mathfrak{m}_S^2)$. Hence the Zariski topology on the vector space induces a well-defined topology on $\bar{\Sigma}$.

Proposition 6.1. *There is an open subset U of $\bar{\Sigma}$ such that $\mathcal{H}$ is versal with respect to some $\sigma \in \Sigma$ if and only if its reduction $\bar{\sigma}$ lies in U. The set U is non-empty if and only if $\dim(S) \geqslant \dim_K(\Lambda_H)$. We have $U = \bar{\Sigma}$ if and only if $\kappa_H : \Lambda_H \to \Omega_K$ is zero.*

Note that if $\dim(S) = \dim_K(\Lambda_H)$ then 'versal' is equivalent to 'universal'.

Lemma 6.2. *The natural homomorphism $\Omega_R \otimes_R S \to \Omega_S$ is an isomorphism.*

Proof. Since this is true with $R_{\mathfrak{p}}$ in place of S, it suffices that $\Omega_{R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} S \to \Omega_S$ is an isomorphism. The ring S has a finite p-basis because it is isomorphic to a power series ring over the field K which has a finite p-basis because this holds for R. Thus Ω_S is a finite S-module, and hence

$$\Omega_S \cong \varprojlim_n (\Omega_S/\mathfrak{m}_S^n \Omega_S) \cong \varprojlim_n \Omega_{S/\mathfrak{m}_S^n}.$$

Since $\Omega_{R_{\mathfrak{p}}}$ is a finite $R_{\mathfrak{p}}$-module, the same reasoning shows that $\Omega_{R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} S \cong \varprojlim \Omega_{S/\mathfrak{m}_S^n}$ as well. $\square$

Proof of Proposition 6.1. Assume that $\sigma : K \to S$ is given. Since S is regular, $\mathcal{H}$ is versal with respect to σ if and only if the homomorphism $\bar{\kappa}_{\mathcal{H}}$ defined by the upper triangle of the following commutative diagram is injective. The lower triangle is (5.2) and is independent of σ. The homomorphism $\kappa_{\mathcal{G} \otimes S} : \Lambda_{\mathcal{G} \otimes S} \to \Omega_S$ is an isomorphism because it can be identified with $\kappa_{\mathcal{G}} \otimes \mathrm{id} : \Lambda_{\mathcal{G}} \otimes_R S \to \Omega_R \otimes_R S$ by Lemma 6.2 and because $\kappa_{\mathcal{G}}$ is an isomorphism as $\mathcal{G}$ is universal and k is perfect.

$$\begin{tikzcd}
\Lambda_{\mathcal{H}} \otimes_S K \arrow[r, "\bar{\kappa}_{\mathcal{H}}"] \arrow[d, hook, "\lambda"'] \arrow[dr, "\kappa_{\mathcal{H}} \otimes \mathrm{id}"] & \Omega_{S/K} \otimes_S K \\
\Lambda_{\mathcal{G} \otimes S} \otimes_S K \arrow[r, "\cong", "\kappa_{\mathcal{G} \otimes S} \otimes \mathrm{id}"'] & \Omega_S \otimes_S K \arrow[u, "v"']
\end{tikzcd}$$

https://doi.org/10.1112/S0010437X09004242 Published online by Cambridge University Press

In order to see how the kernel of v varies with σ we write down two standard exact sequences for modules of differentials. The first depends on σ, while the second does not:

$$0 \longrightarrow \Omega_K \xrightarrow{u} \Omega_S \otimes_S K \xrightarrow{v} \Omega_{S/K} \otimes_S K \longrightarrow 0,$$
$$0 \longrightarrow \mathfrak{m}_S/\mathfrak{m}_S^2 \xrightarrow{d} \Omega_S \otimes_S K \xrightarrow{\pi} \Omega_K \longrightarrow 0.$$

Here $v \circ d$ is an isomorphism and $\pi \circ u = \mathrm{id}$, which proves exactness on the left. If σ is changed so that $\bar{\sigma}$ is altered by $\delta : \Omega_K \to \mathfrak{m}_S/\mathfrak{m}_S^2$ then u changes by $d \circ \delta$. It follows that $\bar{\Sigma}$ is bijective to the set of homomorphisms u with $\pi \circ u = \mathrm{id}$, which reduces the proposition to linear algebra. Namely, $\bar{\kappa}_{\mathcal{H}}$ is injective if and only if the images of u and of $\kappa_{\mathcal{H}} \otimes \mathrm{id}$ in $\Omega_S \otimes K$ have zero intersection. This condition defines an open subset U of $\bar{\Sigma}$ that is non-empty if and only if $\dim_K(\Lambda_H) \leqslant \dim_K(\Omega_{S/K} \otimes K) = \dim(S)$. The intersection is zero for all choices of u if and only if the composition $\pi \circ (\kappa_{\mathcal{H}} \otimes \mathrm{id})$ is zero, but this composition is just κ_H. □

Proof of Lemma 3.1. Clearly $\mathcal{H}'$ is universal if and only if the group $\mathcal{H}$ considered above is universal with respect to the chosen section $\sigma : K \to S$. Since $\dim_K(\Lambda_H) = c = \dim(S)$, Proposition 6.1 implies that $\mathcal{H}$ is universal if $\bar{\sigma}$ lies in a dense open subset of $\bar{\Sigma}$, which is sufficient for our applications. In order for $\mathcal{H}$ to be universal for all choices of σ, we need in addition that κ_H vanishes. Since H is a one-dimensional formal group, its base change to a separable closure K^{sep} is defined over the prime field $\mathbb{F}_p$ by [Zin84, Satz 5.33]. Using the fact that $\Omega_{K^{\mathrm{sep}}} \cong \Omega_K \otimes_K K^{\mathrm{sep}}$ and $\Omega_{\mathbb{F}_p} = 0$, the vanishing of κ_H follows by its functoriality with respect to the base ring. □

An alternative proof of Lemma 3.1 in the case where $a(G) = 1$ is given in [Tia07].

7. Generic completion of the universal deformation

In this section we prove Lemma 4.3. As in the previous section let G be a p-divisible group over a perfect field k of characteristic p, but now let R again be the universal deformation ring of G over $W(k)$ and $\mathcal{G}$ the universal deformation over R.

We consider the unramified discrete valuation rings $A = \widehat{R}_{(p)}$ and $A' = W(F')$ where F' is a fixed algebraic closure of the residue field F of A. Let Σ be the set of ring homomorphisms $A \to A'$ that induce the given embedding $F \to F'$ modulo p.

PROPOSITION 7.1. *The map $\psi : \Sigma \to \mathrm{Def}_{A'/F'}(\mathcal{G} \otimes F')$ that maps a homomorphism $\sigma : A \to A'$ to the scalar extension of $\mathcal{G} \otimes A$ by σ is bijective.*

Proof. Let $\bar{R} = R/pR$. Since $\mathcal{G} \otimes \bar{R}$ is the universal deformation of G in characteristic p, the homomorphism $\kappa_{\mathcal{G}\otimes\bar{R}}$ is an isomorphism, and hence induces an isomorphism

$$\kappa_{\mathcal{G}\otimes F} : \Lambda_{\mathcal{G}\otimes F} \cong \Omega_F$$

as F is the quotient field of $\bar{R}$. The proposition is a formal consequence. Let $A'_n = A'/p^n A'$. It suffices that for every $n \geqslant 1$ and every homomorphism $\sigma : A \to A'_n$ lifting $F \to F'$, using the notation of (5.1), the obvious map

$$\mathrm{Lift}_{A'_{n+1}/A'_n}(\sigma) \to \mathrm{Def}_{A'_{n+1}/A'_n}(\sigma_*(\mathcal{G} \otimes A))$$

is bijective. Since its source is non-empty by [BM90, Proposition 1.2.6] and since $\Omega_A/p\Omega_A = \Omega_F$, this is an equivariant map of torsors with respect to the homomorphism of F'-vector spaces

$$\mathrm{Hom}_F(\Omega_F, p^nA'/p^{n+1}A') \to \mathrm{Hom}_F(\Lambda_{\mathcal{G}_F}, p^nA'/p^{n+1}A')$$

induced by $\kappa_{\mathcal{G}\otimes F}$, which is bijective. □

In order to deduce Lemma 4.3 we have to describe the Galois representation on the Tate module of an arbitrary deformation over A' of $H = \mathcal{G} \otimes F'$. Let K be the quotient field of A', choose an algebraically closed field Ω containing K, and write $\mathrm{Gal}_K = \pi_1(K, \Omega)$. We fix an isomorphism of p-divisible groups over F',

$$H \cong (\mathbb{Q}_p/\mathbb{Z}_p)^c \oplus \mu_{p^\infty}^d.$$

Lemma 7.2. *The map of sets $e : \mathrm{Def}_{A'/F'}(H) \to H^1(\mathrm{Gal}_K, \mathbb{Z}_p(1))^{cd}$ that maps a deformation $\mathcal{H}$ to the isomorphism class of the associated extension of Gal_K-modules*

$$0 \longrightarrow \mathbb{Z}_p(1)^d \longrightarrow T_p\mathcal{H}(\Omega) \longrightarrow \mathbb{Z}_p^c \longrightarrow 0$$

can be written as a composition

$$\mathrm{Def}_{A'/F'}(H) \xrightarrow{\alpha} \widehat{\mathbb{G}}_m(A')^{cd} \xrightarrow{i} (K^*)^{cd} \xrightarrow{\delta} H^1(\mathrm{Gal}_K, \mathbb{Z}_p(1))^{cd}$$

where α is bijective, i is the natural inclusion, and δ is the Kummer homomorphism.

Proof. We have an obvious bijection $\gamma : \mathrm{Def}_{A'/F'}(H) \cong \mathrm{Ext}^1_{A'}(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})^{cd}$ and an isomorphism

$$\beta : \widehat{\mathbb{G}}_m(A') \cong \mathrm{Ext}^1_{A'}(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$$

defined as the projective limit over n of the connecting homomorphisms associated to the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$ over A'_n; these are isomorphisms by [Lau08, Proposition A.1]. Put $\alpha = (-\beta^{-1})^{cd} \circ \gamma$. Let $\bar{K}$ be the algebraic closure of K in Ω. The required relation $e = \delta \circ i \circ \alpha$ translates into anti-commutativity of the following diagram, where δ_1 is induced by the exact sequence $0 \to \mu_{p^n} \to \bar{K}^* \to \bar{K}^* \to 0$ and δ_2 is induced by $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0$, while ε_1 maps an extension E to E/p^nE and ε_2 maps E to $E[p^n]$.

$$\begin{tikzcd}
K^* = \mathrm{Hom}_{\mathrm{Gal}_K}(\mathbb{Z}, \bar{K}^*) \arrow[r, "\delta_1"] \arrow[d, "\delta_2"'] & \mathrm{Ext}^1_{\mathrm{Gal}_K}(\mathbb{Z}, \mu_{p^n}) \arrow[d, "\varepsilon_1"] \\
\mathrm{Ext}^1_{\mathrm{Gal}_K}(\mathbb{Z}/p^n\mathbb{Z}, \bar{K}^*) \arrow[r, "\varepsilon_2"] & \mathrm{Ext}^1_{\mathrm{Gal}_K}(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^n})
\end{tikzcd}$$

This is easily checked. □

Proof of Lemma 4.3. Let $K_\infty = K[\mu_{p^\infty}]$. The homomorphism $\rho^\circ_{A'} : \mathrm{Gal}^\circ_{A'} \to \mathbb{Z}_p^{cd}$ is surjective if and only if its reduction $\bar{\rho}^\circ_{A'} : \mathrm{Gal}^\circ_{A'} \to \mathbb{F}_p^{cd}$ is surjective. By Proposition 7.1 and Lemma 7.2 we have a bijection $a : \Sigma \cong \widehat{\mathbb{G}}_m(A')^{cd}$ so that $\bar{\rho}^\circ_{A'}$ is the image of $a(\sigma)$ under

$$\widehat{\mathbb{G}}_m(A') \xrightarrow{i} K^*_\infty \xrightarrow{pr} K^*_\infty/(K^*_\infty)^p \overset{\delta}{\cong} \mathrm{Hom}(\mathrm{Gal}^\circ_{A'}, \mathbb{F}_p)$$

(componentwise) where δ is induced by the Kummer sequence. Thus $\rho^\circ_{A'}$ is surjective if and only if the components of $a(\sigma)$ map to linearly independent elements in the $\mathbb{F}_p$-vector space $K^*_\infty/(K^*_\infty)^p$. Since $a(\sigma)$ is arbitrary it suffices to show that the image of $\widehat{\mathbb{G}}_m(A')$ in $K^*_\infty/(K^*_\infty)^p$ is infinite.

Let $K_n = K[\mu_{p^n}]$ and $V_n = K_n^*/(K_n^*)^p$, which for $n \geqslant 1$ is identified with $\mathrm{Hom}(\mathrm{Gal}_{K_n}, \mathbb{F}_p)$. By using the fact that K_∞ over K_2 is a $\mathbb{Z}_p$-extension it is easy to see that the kernel of $V_2 \to V_n$ is independent of n for $n \geqslant 3$. Since $V_1 \to V_3$ has finite kernel, it suffices that the image of $\widehat{\mathbb{G}}_m(A')$ in V_1 is infinite. A consideration of valuations shows that for every $x \in A'$ of valuation 1 the element $1 + x$ does not lie in $(K_1^*)^p$. In other words, the kernel of $\widehat{\mathbb{G}}_m(A') \to V_1$ is contained in the kernel of the surjection $\widehat{\mathbb{G}}_m(A') \to F'$ given by $1 + px \mapsto \bar{x}$. Since F' is infinite the assertion follows. $\square$

Acknowledgements

The author is grateful to V. Paskunas, M. Strauch, Y. Tian, E. Viehmann, T. Wedhorn and Th. Zink for interesting discussions on the subject of this article.

References

AN06 J. Achter and P. Norman, *Local monodromy of p-divisible groups*, Preprint (2006), arXiv:math/0402460v2.

BM90 P. Berthelot and W. Messing, *Théorie de Dieudonné cristalline III*, in *The Grothendieck Festschrift, Vol. I*, Progress in Mathematics, vol. 86 (Birkhäuser, Boston, 1990), 173–247.

Cha00 C.-L. Chai, *Local monodromy for deformations of one-dimensional formal groups*, J. Reine Angew. Math. **524** (2000), 227–238.

Cha08 C.-L. Chai, *Methods for p-adic monodromy*, J. Inst. Math. Jussieu **7** (2008), 247–268.

Con71 J. H. Conway, *Three lectures on exceptional groups*, in *Finite simple groups* (Academic Press, New York, 1971).

deJ95 J. A. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Publ. Math. Inst. Hautes Études Sci. **82** (1995), 5–96.

JO00 A. J. de Jong and F. Oort, *Purity of the stratification by Newton polygons*, J. Amer. Math. Soc. **13** (2000), 209–241.

Gro74 A. Grothendieck, *Groupes de Barsotti–Tate et Cristaux de Dieudonné* (Les presses de l'Université de Montréal, Montréal, 1974).

Igu68 J. Igusa, *On the algebraic theory of elliptic modular functions*, J. Math. Soc. Japan **20** (1968), 96–106.

Ill85 L. Illusie, *Déformations de groupes de Barsotti–Tate (d'après A. Grothendieck)*, in *Seminar on arithmetic bundles: the Mordell conjecture*, Astérisque **127** (1985), 151–198.

Kat73 N. M. Katz, *p-adic properties of modular schemes and modular forms*, in *Modular functions of one variable, III*, Lecture Notes in Mathematics, vol. 350 (Springer, Berlin, 1973), 69–190.

Lau08 E. Lau, *A duality theorem for Dieudonné displays*, Ann. Sci. École Norm. Sup. (4) **42** (2009), 241–259.

Mes72 W. Messing, *The crystals associated to Barsotti–Tate groups: with applications to abelian schemes*, Lecture Notes in Mathematics, vol. 264 (Springer, Berlin, 1972).

Oor00 F. Oort, *Newton polygons and formal groups: conjectures by Manin and Grothendieck*, Ann. of Math. (2) **152** (2000), 183–206.

Oor01 F. Oort, *Newton polygon strata in the moduli space of abelian varieties*, in *Moduli of abelian varieties*, Progress in Mathematics, vol. 195 (Birkhäuser, Boston, 2001), 417–440.

Ray74 M. Raynaud, *Schémas en groupes de type* $(p, \ldots, p)$, Bull. Soc. Math. France **102** (1974), 241–280.

RZ89 M. Rosen and K. Zimmermann, *Torsion points of generic formal groups*, Trans. Amer. Math. Soc. **311** (1989), 241–253.

Str07 M. Strauch, *Galois actions on torsion points of universal one-dimensional formal modules*, Preprint (2007), arXiv:0709.3542, J. Number Theory, to appear.

Str08 M. Strauch, *Deformation spaces of one-dimensional formal modules and their cohomology*, Adv. Math. **217** (2008), 889–951.

Tia07 Y. Tian, *p-adic monodromy of the universal deformation of an elementary Barsotti–Tate group*, Preprint (2007), arXiv:0708.2022.

Zim90 K. Zimmermann, *Torsion points of generic formal groups defined over* $\mathbb{Z}_2[[t_1, \ldots, t_{h-1}]]$, Arch. Math. (Basel) **54** (1990), 119–124.

Zin84 Th. Zink, *Cartiertheorie kommutativer formaler Gruppen* (Teubner, Leipzig, 1984).

Eike Lau lau@math.uni-bielefeld.de
Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany

https://doi.org/10.1112/S0010437X09004242 Published online by Cambridge University Press