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A SEPARATION THEOREM IN DIMENSION 3

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Introduction

Let M be a compact non-singular real affine algebraic variety and let A, B be open disjoint semialgebraic subsets of M. Define $Z = \overline{\bar{A} \cap \bar{B}}^Z$ (where $\overline{}^Z$ denotes the Zariski closure).

The sets A, B are said *generically separated* if there exists a proper algebraic subset $X \subset M$ and a polynomial function $p \in \mathcal{P}(M)$ (or equivalently a regular function $p \in \mathcal{R}(M)$) such that $p(A - X) > 0$ and $p(B - X) < 0$.

The sets A, B are said *separated* if there exists $p \in \mathcal{P}(M)$ such that $p(A - Z) > 0$ and $p(B - Z) < 0$.

Very general results on the problem of polynomial separation for semialgebraic sets are known, for instance Bröcker (cf. [Br 1], [Br 2]) solves the problem of the separation of constructible sets in a space of orderings. A detailed exposition of this subject can be found in [AnBrRz], where, in particular, general criterions for the separation of closed semialgebraic sets are given, by applying powerful tools of real algebra and quadratic forms theory.

We are interested in finding a finite number of geometric conditions equivalent to the separation of two open semialgebraic sets going towards an algorithmic solution of the problem. In this article we consider the case of a compact non-singular algebraic variety M of dimension 3.

The paper is structured as follows. Section 1 contains some general separation results for compact semialgebraic subsets of $\mathbf{R}^n$. Geometric obstructions to separation are found in Section 2, but the proof that this finite set of conditions is equivalent to the separation is postponed to Section 4. In Section 3 we discuss the relations between separation and generic separation in dimension 3: if A and B can be separated outside X, then they can be separated outside a set W which is

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"the best possible", in the sense that any polynomial function generically separating A from B must vanish on W. Finally in Section 5 we give a criterion of separation working essentially when the Zariski boundaries of the sets A and B have only non-singular normal crossings components. So, up to desingularization, this criterion reduces the separation problem in dimension 3 to a separation problem on the Zariski boundaries of the sets, hence to a finite number of tests, It is a first step to prove the "decidability" of this problem.

1. A separation tool

Recall the following result which makes it possible to pass from local to global separation; it can be found in [F].

PROPOSITION 1.1. *Let F, G be compact semialgebraic subsets of $\mathbf{R}^n$ such that $F \cap G = \{O\}$. Assume there exist a neighbourhood U of O and a polynomial function p such that*

$$p(F \cap U - \{O\}) > 0 \text{ and } p(G \cap U - \{O\}) < 0.$$

Then F and G can be separated.

As a consequence we have:

PROPOSITION 1.2. *Let F, G be compact semialgebraic subsets of $\mathbf{R}^n$, $\mathring{F} \cap \mathring{G} = \emptyset$ and let X be an algebraic subset of $\mathbf{R}^n$ such that $F \cap G \subseteq X$. Assume there exist a neighbourhood U of X and a polynomial function p such that*

$$p(F \cap U - X) > 0 \quad \text{and} \quad p(G \cap U - X) < 0.$$

Then there exists a polynomial function q such that

$$q(F - X) > 0 \quad \text{and} \quad q(G - X) < 0.$$

Proof. Let $\pi : \mathbf{R}^n \to N$ be the topological contraction of X to a point, say O. It is known (see [BoCRy]) that N admits an affine algebraic structure such that π becomes a regular function and $\pi|_{\mathbf{R}^n - X} : \mathbf{R}^n - X \to N - \{O\}$ a biregular isomorphism. The sets $\pi(F)$ and $\pi(G)$ are compact semialgebraic sets and $\pi(F) \cap \pi(G) = \{O\}$, since $F \cap G \subseteq X$. The function $p \circ (\pi|_{\mathbf{R}^n - X})^{-1} : N - \{O\} \to \mathbf{R}$ is regular, so it can be written as $\dfrac{\varphi}{\psi}$, with $\varphi, \psi \in \mathscr{P}(N)$, ψ never vanishing on $N - \{O\}$.

Then $\varphi \cdot \psi$ is a polynomial function on N which separates $\pi(F)$ from $\pi(G)$ in the neighbourhood $\pi(U)$ of O.

N is affine, say $N \subset \mathbf{R}^m$; $\varphi \cdot \psi$ is the restriction of a polynomial function q which verifies the hypothesis of Proposition 1.1. Hence there exists $f \in \mathscr{P}(N)$ such that

$$f(\pi(F) - \{O\}) > 0 \quad \textit{and} \quad f(\pi(G) - \{O\}) < 0.$$

Then $f \circ \pi$ is a regular function separating F from G outside X. If $f \circ \pi = \frac{q_1}{q_2}$, with $q_1, q_2 \in \mathscr{P}(\mathbf{R}^n)$, then $q_1 \cdot q_2$ is the polynomial function we looked for. □

PROPOSITION 1.3. *Let F, G be compact semialgebraic subsets of $\mathbf{R}^n$, $\mathring{F} \cap \mathring{G} = \emptyset$ and let $X \subset \mathbf{R}^n$ be an algebraic set such that $F \cap G \subseteq X$. Denote by $X_1, \ldots, X_r$ the irreducible components of X and assume that, for each $i \in \{1, \ldots, r\}$, there exist a neighbourhood U_i of X_i and a polynomial function p_i such that*

$$p_i(F \cap U_i - X) > 0 \quad \textit{and} \quad p_i(G \cap U_i - X) < 0.$$

Then there exists $q \in \mathscr{P}(\mathbf{R}^n)$ that X-separates F from G, meaning by this that

$$q(F - X) > 0 \quad \textit{and} \quad q(G - X) < 0.$$

Proof. By Proposition 1.2, it is enough to prove that there exist a neighbourhood U of X and $p \in \mathscr{P}(\mathbf{R}^n)$ such that $p(F \cap U - X) > 0$ and $p(G \cap U - X) < 0$.

This result will be achieved in some steps.

Define $X^1 = \cup_{i \neq j}(X_i \cap X_j)$.

Step 1. *Construction of a polynomial function X-separating F from G in a neighbourhood W of $X - X^1$.*

For each $i \in \{1, \ldots, r\}$, let f_i be a positive equation of X_i (i.e. $f_i \geq 0$ on $\mathbf{R}^n$, $V(f_i) = X_i$). Up to shrink it, we can assume that U_i is a closed semialgebraic set. For each i, on $(F \cup G) \cap U_i$ the zero-set $V(p_i)$ is contained in X, which is the zero-set of $f_1 \cdot \ldots \cdot f_r$. By Lojasiewicz inequality there exists an integer n_i such that the rational function $\frac{(f_1 \cdot \ldots \cdot f_r)^{n_i}}{p_i}$, extended to 0 on $V(p_i) \cap (F \cup G) \cap U_i$, is continuous on $(F \cup G) \cap U_i$. Take $m > n_i$, for each $i \in \{1, \ldots, r\}$. Then the function $\frac{(f_1 \cdot \ldots \cdot f_r)^m}{p_i}$ is continuous and vanishes on $X \cap U_i$. We want to prove that the polynomial function

$$P_m = f_1^m \cdot \ldots \cdot f_r^m \cdot \left(\sum_{i=1}^{r} \frac{p_i}{f_i^m}\right)$$

X-separates F from G in a suitable neighbourhood W of $X - X^1$.

In fact, take $x_0 \in X_i - X^1$. Since $\dfrac{(f_1 \cdot \ldots \cdot f_r)^m}{p_i}(x_0) = 0$ and for all $j \neq i$ $f_j(x_0) \neq 0$, then $\lim_{x \to x_0} \dfrac{|p_i(x)|}{f_i^m(x)} = +\infty$. On the contrary $\sum_{j \neq i} \dfrac{p_j}{f_j^m}$ is bounded locally at x_0. So there exists a neighbourhood $U(x_0)$ of x_0 such that, on $U(x_0)$, P_m has the same sign as p_i. If we take $W_i = \cup_{x_0 \in X_i - X^1} U(x_0)$, which is a neighbourhood of $X_i - X^1$, we have that P_m has the same sign as p_i on W_i; so P_m X-separates F from G in W_i. It is then enough to take $W = \cup_{i=1}^{r} W_i$.

Step 2. *Proof of the statement in the case* $\dim X^1 = 0$.

In this case X^1 is a finite set of points $\{Q_1, \ldots, Q_{r(1)}\}$ and, for each $j = 1, \ldots, r(1)$, there exist a bounded neighbourhood V_j of Q_j and a polynomial function q_j X-separating $F \cap V_j$ from $G \cap V_j$; of course, we can suppose the neighbourhoods V_j pairwise disjoint. Moreover, by Step 1, we have a neighbourhood W of $X - \{Q_1, \ldots, Q_{r(1)}\}$ and $p \in \mathcal{P}(\mathbf{R}^n)$ that X-separates $F \cap W$ from $G \cap W$.

By suitable manipulations of p and q_j's, we will iteratively find a neighbourhood W_j of $X - \{Q_{j+1}, \ldots, Q_{r(1)}\}$ and $p^j \in \mathcal{P}(\mathbf{R}^n)$ X-separating $F \cap W_j$ from $G \cap W_j$. Then $p^{r(1)}$ will X-separate F from G in a neighbourhood of X.

Take $j = 1$; let f be a positive equation of X and r_1 a positive equation of Q_1 such that $\{r_1 \leq 1\} \subseteq V_1$. Define $\overline{q_1} = \sup_{(F \cup G) \cap (\bar{W} - V_1)} |q_1|$. Up to shrink W a little, we have that on $(F \cup G) \cap (\bar{W} - V_1)$

$$V\left(\frac{p r_1}{\overline{q_1}}\right) \subseteq V(f).$$

So by Lojasiewicz inequality there exists an integer n such that, by taking a sufficiently small neighbourhood W_0 of $X - \{Q_1, \ldots, Q_{r(1)}\}$ one has

$$f^n \leq \frac{r_1}{\overline{q_1}} |p| \quad \text{on} \quad (F \cup G) \cap (\overline{W_0} - V_1),$$

and therefore, for any $m \in \mathbf{N}$,

$$|q_1| f^n \leq r_1 |p| \leq r_1^m |p| \quad \text{on} \quad (F \cup G) \cap (\overline{W_0} - V_1).$$

Then, for any positive integer m, the polynomial function $r_1^m p + f^n q_1$ has the same sign as p on $(F \cup G) \cap (\overline{W_0 - V_1})$.

Now consider the set $(F \cup G) \cap (\overline{V_1 - W_0})$, on which $V\left(\frac{f^n q_1}{\bar{p}}\right) \subseteq V(r_1)$, where $\bar{p} = \sup_{(F \cup G) \cap (\overline{V_1 - W_0})} |p|$. So there exists $m \in \mathbf{N}$ (depending on n) such that, by taking a sufficiently small neighbourhood V_1' of Q_1, on $(F \cup G) \cap (\overline{V_1' - W_0})$ we have $r_1^m \leq \frac{f^n}{\bar{p}} |q_1|$, and therefore $r_1^m |p| \leq f^n |q_1|$. Hence $p^1 = r_1^m p + f^n q_1$ has the same sign as q_1 on $\overline{V_1' - W_0}$. Since p^1 clearly X-separates F and G on $V_1 \cap W$, then it X-separates F and G in $(\overline{W_0 - V_1}) \cup (\overline{V_1' - W_0}) \cup (V_1 \cap W)$, which is a neighbourhood of $X - \{Q_2, \ldots, Q_r\}$.

By the same argument we can find the polynomials $p^2, \ldots, p^{r(1)}$ as planned above.

Step 3. *Proof of the Proposition in the general case.*

Consider the decreasing sequence of algebraic sets

$$X \supset X^1 \supset X^2 \supset \cdots \supset X^s$$

where $X = X_1 \cup X_2 \cup \cdots \cup X_r$, $X^1 = \cup_{i \neq j} (X_i \cap X_j)$ and recursively if $X_1^\beta \cup \cdots \cup X_{r(\beta)}^\beta$ is the decomposition into irreducible components of X^β, $X^{\beta+1} = \cup_{i \neq j} (X_i^\beta \cap X_j^\beta)$.

Clearly $\dim X^\beta < \dim X^\alpha$ if $\beta > \alpha$, so we can assume $X^s \neq \emptyset$ and $X^{s+1} = \emptyset$. We will recursively find neighbourhoods W^β of $X - X^{\beta+1}$ and polynomial functions p_β such that $p_\beta(F \cap W^\beta - X) > 0$ and $p_\beta(F \cap W^\beta - X) < 0$. Clearly p_s will X-separate F from G in a neighbourhood of X and the thesis will be a consequence of Proposition 1.2.

By Step 1, we know that F and G are X-separated by $p \in \mathscr{P}(\mathbf{R}^n)$ in a neighbourhood W of $X - X^1$.

From the hypothesis, it follows that for each $j \in \{1, \ldots, r(1)\}$ there exist a neighbourhood V_j of X_j^1 and $q_j \in \mathscr{P}(\mathbf{R}^n)$ such that $q_j(F \cap V_j - X) > 0$ and $q_j(G \cap V_j - X) < 0$. Let f be a positive equation of X and r_1 a positive equation of X_1^1 such that $\{r_1 \leq 1\} \subseteq V_1$.

Define $\overline{q_1} = \sup_{(F \cup G) \cap (\overline{W} - V_1)} |q_1|$. By the same argument used in Step 2, there exists $n \in \mathbf{N}$ such that, for any $m \in \mathbf{N}$, $f^n q_1 + r_1^m p$ has the same sign as p on $(F \cup G) \cap (\overline{W_0 - V_1})$, where W_0 is a sufficiently small neighbourhood of $X - X^1$.

Consider now a neighbourhood V_1' of $X_1^1 - X^2$, $V_1' \subset V_1$ and such that $V_1' \cap X_j^1 = \emptyset$ for each $j \neq 1$. On $(F \cup G) \cap (\overline{V_1' - W_0})$ we have that $V\left(\frac{f^n q_1}{\bar{p}}\right) \subseteq$

$V(r_1)$, where $\bar{p} = \sup_{(F \cup G) \cap (\overline{V_1'} - W_0)} |p|$.

So there exists m (depending on n) such that, possibly after shrinking V_1', on $(F \cup G) \cap (\overline{V_1'} - W_0)$ we have $r_1^m \leq \dfrac{f^n |q_1|}{\bar{p}}$, and therefore $r_1^m |p| \leq f^n |q_1|$.

So $p^1 = r_1^m p + f^n q_1$ X-separates F from G in $\overline{V_1'} - W_0$. Since p and q_1 have the same sign on $(F \cup G) \cap (W_0 - V_1)$, we get that p^1 X-separates F from G in a neighbourhood W_1 of $(X - X^1) \cup (X_1^1 - X^2)$.

We can repeat the above argument replacing W by W_1, X_1^1 by X_2^1, V_1 by V_2 and q_1 by q_2. So we find a neighbourhood W_2 of $(X - X^1) \cup (X_1^1 \cup X_2^1 - X^2)$ and a polynomial function p^2 which X-separates $F \cap W_2$ from $G \cap W_2$.

Repeating this procedure, eventually we find a neighbourhood W^1 of $X - X^2$ and $p_1 \in \mathscr{P}(\mathbf{R}^n)$ such that $p_1(F \cap W^1 - X) > 0$ and $p_1(G \cap W^1 - X) < 0$.

By iterating this argument, we construct successively the polynomials $p_2, \ldots, p_s$ as described above. □

2. Obstructions

Let M be a compact, non-singular, real affine algebraic variety, $\dim M = 3$, and let A, B be open disjoint semialgebraic subsets of M.

We will denote by Y the algebraic set $\overline{\partial A}^Z \cup \overline{\partial B}^Z$, by $Y_1, \ldots, Y_k$ the irreducible components of Y of dimension 2 and by Z the set $\overline{\bar{A} \cap \bar{B}}^Z$.

DEFINITION 2.1.

a) We say that $p \in \mathscr{R}(M)$ *changes its sign at* $x \in M$ if, for every neighbourhood V of x, there exist $y_1, y_2 \in V$ such that $p(y_1)p(y_2) < 0$.

b) Let $X \subset M$ be a 2-dimensional algebraic set and let $p \in \mathscr{R}(M)$. We say that p *changes its sign across* X if it changes its sign at any point $x \in X$ such that $\dim X_x = 2$.

DEFINITION 2.2. We say that an irreducible component Y_i of Y, $i \in \{1, \ldots, k\}$, is *odd* (resp. *even*) if there exists an open set $\Omega \subseteq M$ such that $\dim(Y_i \cap \Omega) = 2$, $A \cap \Omega$ and $B \cap \Omega$ can be generically separated and every $p \in \mathscr{R}(M)$ generically separating them changes (resp. does not change) its sign across Y_i. An irreducible component Y_i of Y will be called a 2-*obstruction* if it is both odd and even.

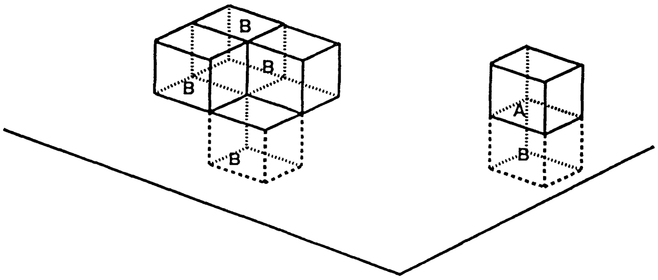

Fig. 1. An example of a 2-obstruction

Remark 2.3. In Definition 2.2 we can suppose that $\mathcal{J}(Y_i)\mathcal{R}(\Omega)$ is a principal ideal, since this is true on a suitable Zariski open set $M - X$. Let g be a generator. Then if Y_i is odd (resp. even), any regular function p generically separating $A \cap \Omega$ from $B \cap \Omega$ can be written as $p = g^m q$, with $q \notin \mathcal{J}(Y_i)\mathcal{R}(\Omega)$, and m odd (resp. even, possibly zero). It is also clear that the parity of m does not depend on the choice of the Zariski open set and of the generator.

NOTATION 2.4. Let A and B be open semialgebraic sets and g be a regular function on M. Denote by A_g and B_g the sets

$$A_g = (A \cap \{g > 0\}) \cup (B \cap \{g < 0\})$$
$$B_g = (A \cap \{g < 0\}) \cup (B \cap \{g > 0\}).$$

LEMMA 2.5. *Let g be a regular function on M such that*
– *for any $\alpha \in \{1, \ldots, r\}$, $g \in \mathcal{J}(Y_\alpha)$ and g changes its sign across Y_α*
– *for any $\alpha \in \{r+1, \ldots, k\}$, $g \notin \mathcal{J}(Y_\alpha)$.*

Then
– *for any $\alpha \in \{1, \ldots, r\}$, Y_α is odd (resp. even) with respect to A, $B \Leftrightarrow Y_\alpha$ is even (resp. odd) with respect to A_g, B_g*
– *for any $\alpha \in \{r+1, \ldots, k\}$, Y_α is odd (resp. even) with respect to A, $B \Leftrightarrow Y_\alpha$ is odd (resp. even) with respect to A_g, B_g.*

Proof. If $p \in \mathcal{R}(M)$ generically separates $A \cap \Omega$ from $B \cap \Omega$, that is

$$p(A \cap \Omega - X) > 0 \quad and \quad p(B \cap \Omega - X) < 0,$$

then

$$pg(A_g \cap \Omega - X) > 0 \quad and \quad pg(B_g \cap \Omega - X) < 0,$$

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i.e. pg generically separates $A_g \cap \Omega$ from $B_g \cap \Omega$.

Moreover, for any p' generically separating $A_g \cap \Omega$ from $B_g \cap \Omega$, we have

$$p'(A_g \cap \Omega - X) > 0 \quad \text{and} \quad p'(B_g \cap \Omega - X) < 0,$$

then

$$p'g(A \cap \Omega - (X \cup V(g))) > 0 \quad \text{and} \quad p'g(B \cap \Omega - (X \cup V(g))) < 0.$$

Hence $p'g$ generically separates $A \cap \Omega$ from $B \cap \Omega$.

Assume, for instance, Y_i is odd with respect to A, B. Then, for any p' generically separating $A_g \cap \Omega$ from $B_g \cap \Omega$, $p'g$ changes its sign across Y_i.

Since by hypothesis g changes its sign across $Y_1, \dots, Y_r$ and does not change it across $Y_{r+1}, \dots, Y_k$, then:

if $i \in \{1, \dots, r\}$, p' does not change its sign across Y_i, i.e. Y_i is even with respect to A_g, B_g.

if $i \in \{r+1, \dots, k\}$, p' changes its sign across Y_i, i.e. Y_i is even with respect to A_g, B_g.

Arguing in the same way, one easily complete the proof. □

NOTATION 2.6. We will denote by Y^c the union of the odd components of Y (with respect to A, B).

Since any regular function separating A from B must vanish on Y^c, if $Y^c \cap (A \cup B)$ is not contained in Z, evidently A and B cannot be separated in the sense of the classical definition.

Now we can state a result which will be proved in Section 4.

THEOREM 2.7. *Let M be a compact, non-singular, real affine algebraic variety, $\dim M = 3$, and let A, B be open disjoint semialgebraic subsets of M. Define $Y = \overline{\partial A}^Z \cup \overline{\partial B}^Z$ and $Z = \overline{\bar{A} \cap \bar{B}}^Z$.*

Then A and B can be separated if and only if the following conditions hold:

1) *No 2-dimensional irreducible component Y_i of Y, $i \in \{1, \dots, k\}$, is a 2-obstruction.*

2) *For every T_j, $j \in \{1, \dots, s\}$, irreducible component of* Sing Y, *there exists an open semialgebraic neighbourhood U_j of T_j such that $A \cap U_j$ and $B \cap U_j$ can be separated.*

3) *$Y^c \cap (A \cup B) \subseteq Z$.*

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EXAMPLES 2.8. In the example in Fig. 2 condition 2) fails; in the example in Fig. 3 (taken from [Br1]) neither condition 1) nor condition 2) are verified.

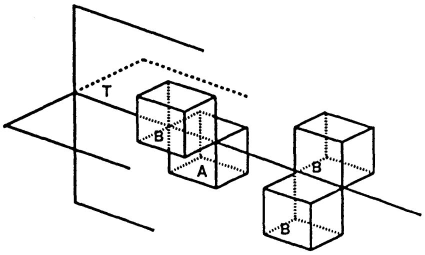

Fig. 2

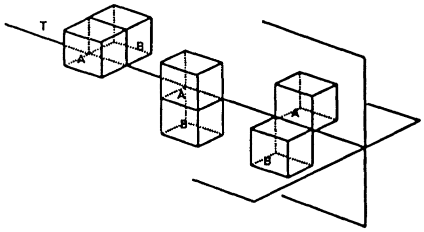

Fig. 3

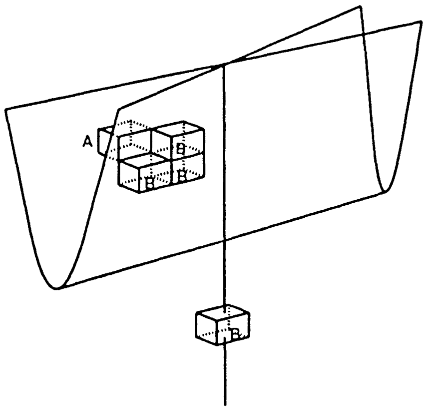

Fig. 4

In the example in Fig. 4 condition 3) fails, because Y^c is the whole Whitney umbrella while Z is a 1-dimensional algebraic subset of Y^c not containing the stick of the umbrella.

Remark 2.9. If there are no 2-obstructions, then $\dim Y^c \cap (A \cup B) \leq 1$, therefore Y^c can intersect $A \cup B$ only with its "tails". For instance, if Y is a union of non-singular irreducible components and condition 1) holds, then $Y^c \cap (A \cup B) = \emptyset$.

3. Separation and generic separation in dimension 3

First, let us recall two results we shall use later on.

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THEOREM 3.1 (Bröcker-Lojasiewicz, [BoCRy] 7.7.10). *Let S be a closed semi-algebraic subset of a real algebraic variety V and let f, g be regular functions on V. Then there exists a non-negative regular function ε such that*:

— $(f + \varepsilon g)(x)$ *has the same sign as* $f(x)$, *for any* $x \in S$

— $V(\varepsilon) \subseteq \overline{V(f) \cap S}^Z$

THEOREM 3.2 (Ruiz, [Rz]). *Let U be a 1-dimensional open semialgebraic subset of a real algebraic variety V. Then there exists $h \in \mathcal{P}(V)$ such that*:

$$U = \{x \in V \mid h(x) > 0\} \text{ and } \bar{U} = \{x \in V \mid h(x) \geq 0\}.$$

It is well known that generic separation and separation are equivalent in dimension 2 (as one can prove using Theorems 3.1 and 3.2): Fig. 4 shows this is not true in dimension 3.

As we remarked before, any regular function generically separating A from B must vanish on $Y^c \cup Z$. In this section we will prove that this "lower bound" for $V(f) \cap (\bar{A} \cup \bar{B})$ can always be attained:

THEOREM 3.3. *If A and B can be generically separated, then there exists $f \in \mathcal{R}(M)$ such that*

$$f(A - (Z \cup Y^c)) > 0, f(B - (Z \cup Y^c)) < 0 \text{ and}$$
$$V(f) \cap (\bar{A} \cup \bar{B}) = (Z \cup Y^c) \cap (\bar{A} \cup \bar{B}).$$

Proof. By hypothesis, there exist an algebraic subset X of M, $\dim X \leq 2$, and $p \in \mathcal{R}(M)$ such that $p(A - X) > 0$ and $p(B - X) < 0$. Clearly we can assume $X = \overline{X \cap (A \cup B)}^Z$; in particular no irreducible 2-dimensional component of X lies in Y^c.

Let X' denote the union of the irreducible components of X of dimension 2. Since p does not change its sign across any component of X', $p \in \mathcal{J}(X')^2$ (for a proof see [AcBg]). So we can write $p = g^k p'$, where g is a generator of $\mathcal{J}(X')^2$, $p' \in \mathcal{R}(M)$ and $p' \notin \mathcal{J}(X')^2$. The function p' does not change its sign across X', so $p' \notin \mathcal{J}(X')$, i.e. $p'|_{X'} \not\equiv 0$. Then, up to replace p by p', we can suppose $\dim X \leq 1$.

Consider now all the 2-dimensional irreducible components of Y, say $Y_1, \ldots, Y_l$, which do not lie in Y^c and on which p identically vanishes (after the first reduction we have made, such components can intersect $A \cup B$ only in dimension 1). For any $\alpha \in \{1, \ldots, l\}$, since A and B can be generically separated and Y_α is not odd, there exists $q_\alpha \in \mathcal{R}(M)$ generically separating A from B which does not change its sign across Y_α. We can suppose that q_α does not vanish on Y_α; in fact if $q_{\alpha|Y_\alpha} \equiv 0$, then $q_\alpha \in \mathscr{I}(Y_\alpha)^2$, which enables us to use the same factorization argument as above.

Then the regular function $\sum_{\alpha=1}^{l} q_\alpha$ separates $A - \bigcap_{\alpha=1}^{l} V(q_\alpha)$ from $B - \bigcap_{\alpha=1}^{l} V(q_\alpha)$ and does not vanish identically on $Y_1 \cup \cdots \cup Y_l$. Hence $p + \sum_{\alpha=1}^{l} q_\alpha$ separates $A - X$ from $B - X$ and does not vanish identically on $Y_1 \cup \cdots \cup Y_l$. Therefore, up to replace p by $p + \sum_{\alpha=1}^{l} q_\alpha$, we can assume that $V(p) \cap (\bar{A} \cup \bar{B}) - (Z \cup Y^c) \leq 1$.

Consider now the semialgebraic set

$$L = V(p) \cap \bar{A} - (Z \cup Y^c).$$

We know that $\dim L \leq 1$, so assume first that $\dim L = 1$. Then there exists a finite set $\Gamma \subset L$ such that $L - \Gamma$ is open in $\bar{L}^Z$. By Theorem 3.2, we can find $h \in \mathscr{P}(M)$ such that

$$L - \Gamma = \{x \in \bar{L}^Z \mid h(x) > 0\} \quad \text{and} \quad \overline{L - \Gamma} = \{x \in \bar{L}^Z \mid h(x) \geq 0\}.$$

In particular, h is strictly negative on $V(p) \cap \bar{B} - (Z \cup Y^c)$, because $\overline{L - \Gamma} \subset \bar{A}$ and $\bar{A} \cap \bar{B} \subseteq Z$.

Consider the closed semialgebraic set

$$S = (\bar{A} \cap \{h \leq 0\}) \cup (\bar{B} \cap \{h \geq 0\})$$

and apply Theorem 3.1 to p, h and S. We get $\varepsilon \in \mathscr{R}(M)$, $\varepsilon \geq 0$, such that $\varphi = p + \varepsilon h$ has the same sign as p on S and $V(\varepsilon) \subseteq \overline{V(p) \cap S}^Z$. In particular $\varphi(\bar{A}) \geq 0$ and $\varphi(\bar{B}) \leq 0$. Moreover,

$$V(\varphi) \cap (\bar{A} \cup \bar{B}) = (V(\varphi) \cap S) \cup (V(\varphi) \cap (\bar{A} \cup \bar{B}) - S)\,;$$

but

$$\begin{aligned} V(\varphi) \cap S = V(p) \cap S = (V(p) \cap \bar{A} \cup \{h \leq 0\}) \cup (V(p) \cap \bar{B} \cap \{h \geq 0\}) \\ \subseteq \Gamma \cup Z \cup Y^c \end{aligned}$$

and

$$V(\varphi) \cap (\bar{A} \cup \bar{B}) - S \subseteq V(\varepsilon) \subseteq \overline{V(p) \cap S}^Z \subseteq \Gamma \cup Z \cup Y^c.$$

So

$$V(\varphi) \cap (\bar{A} \cup \bar{B}) \subseteq \Gamma \cup Z \cup Y^c.$$

In order to remove the 0-dimensional set Γ, it is enough to apply two more times Theorem 3.1: the first time to the functions φ and 1 with respect to $\bar{B}$ to obtain a function ψ which does not vanish any more on the points of $\Gamma \cap (\bar{A} - \bar{B})$; the second time to ψ and -1 with respect to $\bar{A}$ to obtain a function f such that

$$V(f) \cap (\bar{A} \cup \bar{B}) \subseteq Z \cup Y^c.$$

The last argument can be used also when $\dim L = 0$. □

Remark 3.4. The irreducible components of Y of dimension ≤ 1 have no influence on the possibility of separating A from B. To see this, denote their union by H and consider the sets $A' = \overset{\circ}{\widehat{A \cup H}}$ and $B' = \overset{\circ}{\widehat{B \cup H}}$. We easily see that $Z' = Z$, $Y = Y' \cup H$ and all the irreducible components of Y' have dimension 2. Remark that A and B can be separated if and only if A' and B' can be separated. In fact one implication is obvious since $A \subseteq A'$ and $B \subseteq B'$; conversely if A and B are separated by p, then p separates $A' - H$ from $B' - H$, so by Theorem 3.3 A' and B' can be separated outside $Z \cup Y^c$. This is the reason why in Theorem 2.7, in order to obtain the separation of A from B, it is enough to impose some conditions only on Sing Y and the 2-dimensional components of Y, without assuming anything on the lower dimensional ones.

COROLLARY 3.5. *If A and B can be generically separated and $Y^c \cap (A \cup B) \subseteq Z$, then A and B can be separated. Moreover there exists f separating A from B and such that $V(f) \cap (\bar{A} \cup \bar{B}) = (Z \cup Y^c) \cap (\bar{A} \cup \bar{B})$.*

Proof. It follows immediately from Theorem 3.3. □

Theorem 3.3 assures that A and B can be generically separated if and only if the sets $\hat{A} = A - Y^c$ and $\hat{B} = B - Y^c$ can be separated. If we consider the sets $\hat{Y}$ and $\hat{Z}$ defined in an evident way with respect to $\hat{A}$ and $\hat{B}$, it is easy to see that $\hat{Y} = Y$ and $\hat{Z} = Z$. If we use Theorems 2.7 and 3.3 as a consequence we get:

COROLLARY 3.6. *A and B can be generically separated if and only if the following conditions hold:*

1) *No 2-dimensional irreducible component Y_i of Y, $i \in \{1, \ldots, k\}$, is a 2-obstruction.*
2) *For every T_j, $j \in \{1, \ldots, s\}$, irreducible component of* Sing Y, *there exists an open semialgebraic neighbourhood U_j of T_j such that $A \cap U_j$ and $B \cap U_j$ can be generically separated.*

4. Proof of Theorem 2.7

If A and B can be separated, then obviously conditions 1), 2), 3) hold.

Conversely, assume that conditions 1), 2), 3) hold; the proof that A and B can be separated will be achieved in some steps.

Step 1. *We can assume that Y has only non-singular, normal crossings irreducible components.*

Let $\pi : \tilde{M} \to M$ be a desingularization of $Y \subset M$. This means that, if we denote by $Y_1', \ldots, Y_l'$ the strict transforms of all the irreducible components $Y_1, \ldots, Y_l$ of Y, we have that:

a) $Y_1', \ldots, Y_l'$ are non-singular and pairwise disjoint,

b) $E = \pi^{-1}$ (Sing Y) has non-singular irreducible components and $E \cup Y_1' \cup \ldots \cup Y_l'$ has only normal crossings,

c) π is surjective and induces a biregular isomorphism between $\tilde{M} - E$ and $M - \mathrm{Sing}\, Y$.

Define $\tilde{A} = \pi^{-1}(A)$, $\tilde{B} = \pi^{-1}(B)$ and $\tilde{Y} = \overline{\partial \tilde{A}}^Z \cup \overline{\partial \tilde{B}}^Z$. It is clear that $\tilde{Y} \subseteq \pi^{-1}(Y) = E \cup Y_1' \cup \ldots \cup Y_l'$.

Let us see that $\tilde{A}$ and $\tilde{B}$ verify conditions 1), 2), 3).

In fact, the algebraic set $\tilde{Y}$ is contained in $E \cup Y'$, where Y' is the strict transform of Y. So an irreducible component X of $\tilde{Y}$ is either the strict transform of a component Y_t of Y, or a component of the exceptional divisor.

In the first case, if X has dimension 2, it cannot be a 2-obstruction for the separation of $\tilde{A}$ and $\tilde{B}$ since Y_t is not a 2-obstruction and π is a biregular isomorphism outside E.

In the second one, $\pi(X) \subseteq \mathrm{Sing}\, Y$ has dimension 1 or 0. So, by condition 2), there exists a polynomial function p separating A and B in a neighbourhood of $\pi(X)$. Hence $p \circ \pi$ separates $\tilde{A}$ and $\tilde{B}$ in a neighbourhood of X.

For the same reason no irreducible component of $\mathrm{Sing}\, \tilde{Y}$ can be an obstruction, because it lies in at least one component of E. So $\tilde{A}$ and $\tilde{B}$ verify 1) and 2).

Moreover, since $\tilde{Y}$ has non-singular irreducible components, 3) is automatically verified (see Remark 2.9).

Now suppose $\tilde{A}$ and $\tilde{B}$ can be separated: then, by composition with π^{-1} (where defined), we get that A and B are generically separated, so applying Corollary 3.5 they can be separated. □

Let X' be an algebraic subset of M such that $[Y^c \cup X'] = 0$ in $H_2(M, \mathbf{Z}_2)$. Being Y^c a union of non-singular components, we can assume that X' is

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transversal to each irreducible component of Y^c and of Sing Y (see for instance [BoCRy], chap. 12).

Since $\overline{(\mathrm{Sing}\ Y) - Y^c}^Z \cap Y^c$ is a discrete set of points, we can further choose X' not passing through such points. So, if we denote $\Gamma = Y^c \cap X'$, we can assume that $\dim \Gamma \leq 1$, $\dim(\Gamma \cap \mathrm{Sing}\ Y) \leq 0$ and $\Gamma \cap \overline{(\mathrm{Sing}\ Y) - Y^c}^Z = \emptyset$.

Similarily there exists an algebraic subset X'' of M such that $[Y^c \cup X''] = 0$, transversal to each irreducible component of Y^c and of Sing Y, and "avoiding" the points of $\Gamma \cap \mathrm{Sing}\ Y$. More precisely we can assume $\dim(\Gamma \cap X'') \leq 0$ and $\Gamma \cap X'' \cap \mathrm{Sing}\ Y = \emptyset$. So the set $\Gamma \cap X''$ consists of a finite number of points $Q_1, \ldots, Q_s$ lying in Y^c and each of them is a non-singular point for Y. We can suppose that each Q_j is non-singular for X'' too.

Now let g'' be a generator of the ideal $\mathcal{I}(Y^c \cup X'')$ which exists since $[Y^c \cup X''] = 0$.

Consider the sets $A_{g''}$ and $B_{g''}$, which for simplicity we will denote respectively A'' and B''; define $Y'' = \overline{\partial A''}^Z \cup \overline{\partial B''}^Z$ and $Z'' = \overline{\overline{A''} \cap \overline{B''}}^Z$. It is easy to check that $Y'' \subseteq Y \cup X''$. Moreover we claim that

$$(*) \qquad Z'' \cap (A'' \cup B'') = Z \cap (A'' \cup B'').$$

In fact, since $\overline{A''} \cap \overline{B''} \subseteq (\bar{A} \cap \bar{B}) \cup V(g'')$, we get $Z'' \subseteq Z \cup V(g'')$; in particular $Z'' \cap (A'' \cup B'') \subseteq Z \cap (A'' \cup B'')$.

Conversely, let $x \in Z \cap (A'' \cup B'')$ and assume H is an irreducible component of Z passing through x. H contains an open subset U of $\bar{A} \cap \bar{B}$ of maximal dimension such that $H = \bar{U}^Z$. Since $g(x) \neq 0$, $g_{|H} \not\equiv 0$ and also $g_{|\bar{U}} \not\equiv 0$; so $U \subseteq \overline{A''} \cap \overline{B''}$ and therefore $H \subseteq Z''$. Then $x \in Z'' \cap (A'' \cup B'')$.

Assume $Y^c = Y_1 \cup \ldots \cup Y_r$. By Lemma 2.5, the components $Y_1, \ldots, Y_r$ are even w.r.t. A'', B'', while $Y_{r+1}, \ldots, Y_k$ are not odd w.r.t. A'', B'', because they were not odd w.r.t. A, B. This means that no 2-dimensional irreducible component of Y is odd w.r.t. A'', B'' and therefore that $(\overline{A''} \cap \overline{B''}) - X'' \subseteq \mathrm{Sing}\ Y$; in other words $Z'' \subseteq (\mathrm{Sing}\ Y) \cup X''$.

Step 2. *A'' and B'' can be separated in a neighbourhood of $X'' \cap \Gamma$.*

For each $j \in \{1, \ldots, s\}$, $Q_j \notin \mathrm{Sing}\ Y$, so there exists a neighbourhood V_j of Q_j such that $Y \cap V_j$ is contained in exactly one irreducible component of Y (more precisely, of Y^c). We can assume the V_j's pairwise disjoint. Let $V = V_1 \cup \cdots \cup V_s$.

Since $\overline{A''} \cap \overline{B''} \subseteq (\mathrm{Sing}\ Y) \cup X''$, we have that $\overline{A''} \cap \overline{B''} \cap V \subseteq X''$. If the V_j's are small enough, also $X'' \cap V$ consists of non-singular points for X''. Let q

be a regular function in $\mathcal{J}(X'')$ such that $V(q) \cap V = X'' \cap V$ and q changes its sign at any point of $X'' \cap V$.

For each $j \in \{1, \ldots, s\}$, $A'' \cap V_j$ and $B'' \cap V_j$ are separated by q or $-q$. Then we can suppose that q separates $A'' \cap V$ from $B'' \cap V$ (up to multiplying q by the equation of a sphere centered in Q_i and containing V_i, for each i such that $A'' \cap V_i$ and $B'' \cap V_i$ are separated by $-q$). □

Step 3. *A'' and B'' can be separated in a neighbourhood of Γ.*

It is possible to choose a semialgebraic neighbourhood T of Γ such that $X'' \cap \bar{T} \subseteq X'' \cap V$. We want to prove that A'' and B'' can be separated in T by applying Proposition 1.3 to the compact sets $\overline{A'' \cap T}$ and $\overline{B'' \cap T}$.

Since $\overline{A''} \cap \overline{B''} \subseteq (\text{Sing } Y) \cup X''$, also $\overline{A'' \cap T} \cap \overline{B'' \cap T} \subseteq (\text{Sing } Y) \cup X''$.

As for X'', let U'' be a neighbourhood of X'' such that $U'' \cap \bar{T} \subseteq V$. By Step 2, we have

$$q(\overline{A'' \cap T} \cap U'' - X'') > 0 \quad \text{and} \quad q(\overline{B'' \cap T} \cap U'' - X'') < 0.$$

For each irreducible component T_j of Sing Y, by condition 2), there exists a regular function p_j separating $A \cap U_j$ from $B \cap U_j$, i.e.

$$p_j(A \cap U_j - Z) > 0 \qquad p_j(B \cap U_j - Z) < 0.$$

Then

$$p_j g''(A'' \cap U_j - Z) > 0 \qquad p_j g''(B'' \cap U_j - Z) < 0.$$

From ($*$) we get that $p_j g''$ separates $A'' \cap U_j$ from $B'' \cap U_j$. Recall that no irreducible component of Y is odd w.r.t. A'', B'', so $(Y'')^c \subseteq X''$. So, if we apply Corollary 3.5 to $A'' \cap T \cap U_j$ and $B'' \cap T \cap U_j$, we get that, for each j, there exists a regular function p''_j separating $A'' \cap T \cap U_j$ from $B'' \cap T \cap U_j$ and such that

$$p''_j(\overline{A'' \cap T} \cap U_j - ((\text{Sing } Y) \cup X'')) > 0$$
$$p''_j(\overline{B'' \cap T} \cap U_j - ((\text{Sing } Y) \cup X'')) < 0.$$

This allows us to apply Proposition 1.3 to the compact sets $\overline{A'' \cap T}$ and $\overline{B'' \cap T}$ relatively to the algebraic set $(\text{Sing } Y) \cup X''$: we get a function φ which separates A'' from B'' in the neighbourhood T. □

Step 4. *A and B can be separated in a neighbourhood of Γ.*

Coming back to A and B, it follows from Step 3 that

$$\varphi g''(A \cap T - ((\mathrm{Sing}\, Y) \cup X'' \cup V(g''))) > 0$$
$$\varphi g''(B \cap T - ((\mathrm{Sing}\, Y) \cup X'' \cup V(g''))) < 0.$$

that is $A \cap T$ and $B \cap T$ can be generically separated. Because of condition 3), Corollary 3.5 implies that $A \cap T$ and $B \cap T$ can be separated by a regular function, we will denote p_T. □

Let g' be a generator of the ideal $\mathcal{J}(Y^c \cup X')$ and consider the sets $A' = A_{g'}$ and $B' = B_{g'}$. Arguing as above, we can see that

$$(**) \qquad Z' \cap (A' \cup B') = Z \cap (A' \cup B').$$

Step 5. *A' and B' can be separated in a neighbourhood of Y^c.*

Let Ω be a semialgebraic neighbourhood of Y^c such that $\bar{\Omega} \cap X' \subseteq T \cap X'$. We want to prove that A' and B' can be separated in Ω by applying Proposition 1.3 to the compact sets $\overline{A' \cap \Omega}$ and $\overline{B' \cap \Omega}$.

Since $\overline{A'} \cap \overline{B'} \subseteq (\mathrm{Sing}\, Y) \cup X'$, we have also $\overline{A' \cap \Omega} \cap \overline{B' \cap T} \subseteq (\mathrm{Sing}\, Y) \cup X'$.

As for X', let U' be a neighbourhood of X' such that $U' \cap \bar{\Omega} \subseteq T$. By Step 4, p_T separates $A \cap T$ from $B \cap T$; hence

$$p_T g'((A' \cap T) - Z) > 0 \qquad p_T g'((B' \cap T) - Z) < 0.$$

From $(**)$, we get that $p_T g'$ separates $A' \cap T$ from $B' \cap T$.

As before, we see that $(Y')^c \subseteq X'$. So, if we apply Corollary 3.5 to $A' \cap T$ and $B' \cap T$, we get that there exists a regular function p' separating $A' \cap T$ from $B' \cap T$ and such that

$$p'(\overline{A' \cap T} - (\mathrm{Sing}\, Y \cup X')) > 0 \qquad p'(\overline{B' \cap T} - (\mathrm{Sing}\, Y \cup X')) < 0.$$

Now, since $U' \cap \bar{\Omega} \subseteq T$, we have

$$p'(\overline{A' \cap \Omega} \cap U' - (\mathrm{Sing}\, Y \cup X')) > 0 \quad p'(\overline{B' \cap \Omega} \cap U' - (\mathrm{Sing}\, Y \cup X')) < 0,$$

that is the hypothesis of Proposition 1.3 is fulfilled in the neighbourhood $U'$of X' with respect to the algebraic set $(\mathrm{Sing}\, Y) \cup X'$.

We have to prove that the hypothesis is satisfied also around each irreducible component T_j of $\mathrm{Sing}\, Y$.

Arguing as in Step 3, from condition 2) we get that $p_j g'$ separates $A' \cap U_j$ from $B' \cap U_j$. Since $(Y')^c \subseteq X'$, if we apply Corollary 3.5 to $A' \cap \Omega \cap U_j$ and $B' \cap \Omega \cap U_j$, we get that there exists a regular function p'_j separating $A' \cap \Omega \cap U_j$ from $B' \cap \Omega \cap U_j$ and such that

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$$p'(\overline{A' \cap \Omega} \cap U_j - (\text{Sing } Y \cup X')) > 0 \quad p'(\overline{B' \cap \Omega} \cap U_j - (\text{Sing } Y \cup X')) < 0.$$

We can therefore apply Proposition 1.3 to the compact sets $\overline{A' \cap \Omega}$ and $\overline{B' \cap \Omega}$ relatively to the algebraic set $(\text{Sing } Y) \cup X'$: we get a function ϕ which separates A' from B' in the neighbourhood Ω. □

Step 6. *A and B can be separated in a neighbourhood of* Y^c.

Coming back again to A and B, from Step 5 it follows that

$$\phi g'(\overline{A \cap \Omega} - (\text{Sing } Y \cup X' \cup V(g'))) > 0$$
$$\phi g'(\overline{B \cap \Omega} - (\text{Sing } Y \cup X' \cup V(g'))) < 0,$$

that is $A \cap \Omega$ and $B \cap \Omega$ can be generically separated. Because of condition 3), Corollary 3.5 assures that $A \cap \Omega$ and $B \cap \Omega$ can be separated by a regular function, say p_Ω. □

Step 7. *A and B can be separated.*

We want to apply Proposition 1.3 to $\bar{A}$ and $\bar{B}$ relatively to $Y^c \cup Z$. In the neighbourhood Ω of Y^c, by Corollary 3.5 we may assume that

$$p_\Omega(\bar{A} \cap \Omega - (Y^c \cup Z)) > 0 \quad p_\Omega(\bar{B} \cap \Omega - (Y^c \cup Z)) < 0.$$

As for Z, it is enough to consider its irreducible components T_j not contained in Y^c and therefore contained in Sing Y. Using condition 2) and again Corollary 3.5, we get that the hypothesis of Proposition 1.3 is verified also around T_j, and so we get a function p such that

$$p(\bar{A} - (Y^c \cup Z)) > 0 \quad \text{and} \quad p(\bar{B} - (Y^c \cup Z)) < 0.$$

Then, by condition 3),

$$p(A - Z) > 0 \quad \text{and} \quad p(B - Z) < 0.$$

□

Remark 4.1. In the proof of Theorem 2.7, we actually separate $\bar{A}$ and $\bar{B}$ up to $W = Z \cup Y^c$, which is "minimal" in the sense of Theorem 3.3 and Corollary 3.5. So if F, G are closed semialgebraic sets such that $F = \bar{\mathring{F}}$, $G = \bar{\mathring{G}}$ and verifying conditions 1), 2), 3), then there exists $p \in \mathcal{R}(M)$ such that

$$p(F - W) > 0 \quad p(G - W) < 0.$$

5. A separation criterion

In this section we look for a criterion that makes it easier to decide whether A and B can be separated.

Consider, at first, the case in which the algebraic set Y is a union of non-singular normal crossings components $Y_1, \ldots, Y_k$, each one of dimension 2. Assume also that $Y_\alpha \cap Y_\beta$ is irreducible for any $\alpha \neq \beta$.

The test we are going to describe relates the separation of A and B with the separation or their two-dimensional "traces" on each irreducible component Y_α of Y, that is the sets

$$\mathrm{tr}_\alpha A = \overbrace{\bar{A} \cap Y_\alpha}^{\circ} \qquad \mathrm{tr}_\alpha B = \overbrace{\bar{B} \cap Y_\alpha}^{\circ},$$

where the interior part is taken in Y_α.

If $f \in \mathcal{J}(Y_\alpha)$ changes its sign across Y_α, we have to consider also the traces of the sets A_f and B_f.

DEFINITION 5.1. Let C, D be open semialgebraic subsets of M. We will say that the triple (C, D, Y_α) *satisfies the property* (P) if the sets $\mathrm{tr}_\alpha C$ and $\mathrm{tr}_\alpha D$ are disjoint and can be separated in Y_α. We will say that it *satisfies the property* (P′) if (C_f, D_f, Y_α) verifies (P), where f is an element in $\mathcal{J}(Y_\alpha)$ that changes its sign across Y_α.

It is easy to verify that the property (P′) does not depend on the choice of f: suppose that both f and g change their sign across Y_α; if q separates $\mathrm{tr}_\alpha C_f$ from $\mathrm{tr}_\alpha D_f$, then qfg, reduced modulo $\mathcal{J}(Y_\alpha)^2$, generically separates $\mathrm{tr}_\alpha C_g$ from $\mathrm{tr}_\alpha D_g$, so (being in dimension 2) they can be separated.

We begin by proving the following

LEMMA 5.2. *The statements*:

i) "*Y_α is odd* (*resp. even*)"

ii) "*(A, B, Y_α) verifies* (P) (*resp.* (P′))"

cannot hold simultaneously.

Proof. Suppose, by contradiction, Y_α is odd and $\mathrm{tr}_\alpha A$, $\mathrm{tr}_\alpha B$ are disjoint and can be separated by a regular function q.

Then there exists an open semialgebraic subset Ω of M such that dim $(Y_\alpha \cap \Omega)$

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$= 2$ and $A \cap \Omega$ and $B \cap \Omega$ can be generically separated, say by $f \in \mathcal{R}(M)$, that is

$$f(A \cap \Omega - X) > 0 \quad \text{and} \quad f(B \cap \Omega - X) < 0.$$

By the same argument already used in the proof of Theorem 3.3, we can assume $\dim X \leq 1$.

The functions f and q have the same sign on $U - Y_\alpha$, where $U \subseteq \Omega$ is a suitable semialgebraic neighbourhood of $(\mathrm{tr}_\alpha A \cup \mathrm{tr}_\alpha B) \cap \Omega - X$.

Define

$$S = (\bar{A} \cup \bar{B}) \cap \bar{\Omega} - U\,;$$

it is a closed semialgebraic set and $\dim (S \cap Y_\alpha) \leq 1$.

Applying Theorem 3.1 to f, q and S, we get a regular function $p = f + \varepsilon q$ which separates $A \cap \Omega - X$ from $B \cap \Omega - X$ and does not vanish on Y_α. In fact:

- on $((A \cup B) - X) \cap \Omega \cap S$, p and f have the same sign,
- on $((A \cup B) - X) \cap \Omega - S$, which is contained in U, f and q have the same sign, so again p and f have the same sign.

Therefore p separates $A \cap \Omega - X$ from $B \cap \Omega - X$.

Moreover $p \notin \mathcal{J}(Y_\alpha)$; in fact, otherwise, ε should identically vanish on Y_α, which is impossible since $V(\varepsilon) \subseteq \overline{V(f) \cap S}^Z$ and $\dim (S \cap Y_\alpha) \leq 1$.

So p does not change its sign across Y_α and Y_α is not odd. Contradiction.

To complete the proof, let $f \in \mathcal{J}(Y_\alpha)$ be a regular function that changes its sign across Y_α. Then it is enough to remark that Y_α is even w.r.t. A, B if and only if Y_α is odd w. r. t. A_f, B_f (Lemma 2.5) and that (A, B, Y_α) verifies (P′) if and only if (A_f, B_f, Y_α) verifies (P). The first part of the proof yields the thesis. □

THEOREM 5.3. *Let A and B be open disjoint semialgebraic sets. Assume that $Y = \overline{\partial A}^Z \cup \overline{\partial B}^Z$ is a union of non-singular irreducible components $Y_1, \ldots, Y_k$ of dimention 2, simultaneously normal crossings and such that $Y_\alpha \cap Y_\beta$ is irreducible for $\alpha \neq \beta$. Then A and B can be separated if and only if for each $\alpha \in \{1, \ldots, k\}$ (A, B, Y_α) verifies at least one between the property* (P) *and the property* (P′).

Proof. (⇒) Assume that A and B can be separated and suppose, by contradiction, there exists α such that (A, B, Y_α) verifies neither (P) nor (P′).

We want to see that, since (A, B, Y_α) does not verify (P), then Y_α is odd. This is clear if $\mathrm{tr}_\alpha A$ and $\mathrm{tr}_\alpha B$ are not disjoint. In the case they are disjoint, but not separated, let g be a generator of $\mathcal{J}(Y_\alpha)^2$; for any regular function p generically separating A from B, we can write $p = g^h \cdot q$, with $q \notin \mathcal{J}(Y_\alpha)^2$. Nevertheless

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$q \in \mathcal{I}(Y_\alpha)$, otherwise it would generically separate $\mathrm{tr}_\alpha A$ from $\mathrm{tr}_\alpha B$, which is impossible since in dimension 2 generic separation is equivalent to separation. The functions p and q have the same sign, so p changes its sign across Y_α and hence Y_α is odd.

Arguing as before, we see that, since (A, B, Y_α) does not verify (P′), then Y_α is even. Contradiction.

($\Leftarrow$) Assume that, for each $\alpha \in \{1, \ldots, k\}$, (A, B, Y_α) verifies (P) or (P′). Then by Lemma 5.2 there are no 2-obstructions. Since $Y^c \cap (A \cup B) = \emptyset$ (see Remark 2.9), in order to apply Theorem 2.7 we have only to show that A and B can be separated in a neighbourhood U_j of each irreducible component T_j of Sing Y.

This can be done by modifying a little the proof of Theorem 2.7; let us give a sketch of it.

Let Y_α be an irreducible component of Y containing T_j and assume, for instance, that (A, B, Y_α) verifies (P). The curve $H = \overline{\partial \mathrm{tr}_\alpha A}^Z \cup \overline{\partial \mathrm{tr}_\alpha B}^Z$ has non-singular, normal crossing, irreducible components, say $H_1, \ldots, H_q$ and for each $i = 1, \ldots, q$ there exists by hypothesis an irreducible component Y_i such that $Y_i \cap Y_\alpha = H_i$.

Denote by $H_1, \ldots, H_s$ the irreducible components of H lying in $\overline{\overline{\mathrm{tr}_\alpha A} \cap \overline{\mathrm{tr}_\alpha B}}^Z$. We can find an algebraic subset X of M such that $[Y_1 \cup \ldots \cup Y_s \cup X] = 0$ in $H_2(M, \mathbf{Z}_2)$ and such that X is transversal to each irreducible component of Y and of Sing Y.

Take a generator g of $\mathcal{I}(Y_1 \cup \ldots \cup Y_s \cup X)$ and consider the sets A_g and B_g and their traces on Y_α.

No H_i, $i \in \{1, \ldots, s\}$, can be contained in $\overline{\overline{\mathrm{tr}_\alpha A_g} \cap \overline{\mathrm{tr}_\alpha B_g}}^Z$, otherwise it would be an obstruction to the separation of $\mathrm{tr}_\alpha A$ and $\mathrm{tr}_\alpha B$. So the set

$$(\overline{\mathrm{tr}_\alpha A_g} \cap \overline{\mathrm{tr}_\alpha B_g}) - (X \cap Y_\alpha)$$

is a finite set of points $\{Q_1, \ldots, Q_h\}$ with $Q_i = Y_\alpha \cap Y_{m(i)} \cap Y_{l(i)}$.

If we denote by Γ_i the curve $Y_{m(i)} \cap Y_{l(i)}$ and if the neighbourhood U_j is small enough, we have

$$\overline{A_g \cap U_j} \cap \overline{B_g \cap U_j} \subseteq \bigcup_{i=1}^{h} \Gamma_i \cup X.$$

We want to apply Proposition 1.3 to the sets $\overline{A_g \cap U_j}$ and $\overline{B_g \cap U_j}$ and the algebraic set $\cup_{i=1}^{h} \Gamma_i \cup X$. Arguing as in Step 3 of the proof of Theorem 2.7 we get that $\overline{A_g \cap U_j}$ and $\overline{B_g \cap U_j}$ can be separated in a neighbourhood of X.

In a small neighbourhood V_i of $\Gamma_i \cap U_j$ take local equations $f_{m(i)}, f_{l(i)}$ for $Y_{m(i)}$ and $Y_{l(i)}$. If U_j is sufficiently small, in U_j the Γ_i's are pairwise disjoint, so the function $p_i = f_{m(i)} + f_{l(i)}$ (or $-p_i$) verifies

$$p_i(\overline{A_g \cap U_j} \cap V_i - \Gamma_i) > 0 \quad \text{and} \quad p_i(\overline{B_g \cap U_j} \cap V_i - \Gamma_i) < 0.$$

Since all the hypothesis of the Proposition 1.3 are fulfilled, we get that $A_g \cap U_j$ and $B_g \cap U_j$ can be separated by a regular function p_{U_j}. Then $p_{U_j} \cdot g$ generically separates $A \cap U_j$ from $B \cap U_j$ and therefore, as in Step 4, there exists a regular function separating them too. So also condition 2) of Theorem 2.7 is verified and therefore A and B can be separated. □

Remark 5.4. It is easy to see that the "only if" part of Theorem 5.3 holds even if Y is not normal crossings. Fig. 5 shows that the converse is not true in general.

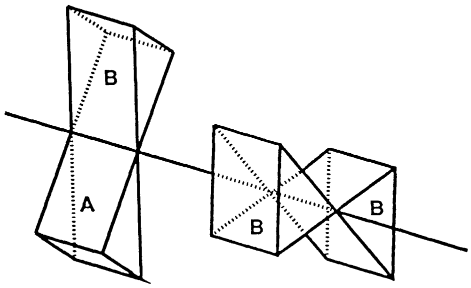

Fig. 5

Remark 5.5. If we now come back to the general situation (without the supplementary hypothesis on Y considered before), we can make use of Theorem 5.3 as follows.

First of all consider a resolution of the singularities of Y, say $:\tilde{M} \to M$. Let $\tilde{Y} = \pi^{-1}(Y)$. By performing, if necessary, some further blowing-ups, we can suppose that $\tilde{Y}_\alpha \cap \tilde{Y}_\beta$ is irreducible, for any irreducible components $\tilde{Y}_\alpha$, $\tilde{Y}_\beta$ of $\tilde{Y}$, $\alpha \neq \beta$. We can also assume that $\tilde{Y}$ satisfies the hypothesis of Theorem 5.3, because the 1-dimensional components of $\tilde{Y}$ can be "removed", as pointed out in Remark 3.4.

Of course if A and B can be separated, then $\pi^{-1}(A)$ and $\pi^{-1}(B)$ can be separated too. Conversely we know (see Step 1 in the proof of Theorem 2.7) that if $\pi^{-1}(A)$ and $\pi^{-1}(B)$ can be separated, then A and B can be generically separated; if moreover $Y^c \cap (A \cup B) \subseteq Z$, they can be separated.

Now we can test if $\pi^{-1}(A)$ and $\pi^{-1}(B)$ can be separated by means of Theorem 5.3, which therefore becomes a criterion of generic separation for A and B. So Theorem 5.3 reduces the problem to a finite number of 2-dimensional tests: the separation of the traces of $\pi^{-1}(A)$ and $\pi^{-1}(B)$. For that one can make use of the following result, analogous to Theorem 2.7:

THEOREM 5.6. *Let M be a non-singular compact surface and A and B be open semialgebraic subsets of M. Then A and B can be separated if and only if*:

a) *No irreducible component of $Y = \overline{\partial A}^{Z} \cup \overline{\partial B}^{Z}$ is both odd and even*

b) *A and B can be locally separated at any singular point of Y*

In [AcBgF] one can find a proof of this result under a supplementary condition, which can be removed arguing as in Section 4; a direct and geometric proof that such condition is not necessary can be found in [P].

It is important to remark that, when applying Theorem 5.5, one has to verify only condition a) of the theorem, because it is clear that in a normal crossings situation condition a) implies condition b).

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