Feedback

The Editor writes: I apologise to Martin Lukarevski for omitting some material from the end of his Note 107.23 on 'The location of the inarc circle and its point of contact with the circumcircle', which appeared in the July 2023 issue of the *Gazette*. The following should have been inserted between the end of the published text and the Acknowledgements.

One can consider the related notion of exarc circles and their radii [1,2]. The exarc radii are much larger than the inarc radii and for their sums we have established the inequality

$$8R - 4r \leqslant R_A + R_B + R_C \leqslant \frac{4R^2}{r} - 2R$$

where in the proof we have used Kooi's inequality $s^2 \leqslant \dfrac{R(4R + r)^2}{2(2R - 2)}$ [3, 4, 5]. After much effort to find a linear upper bound, we posed the following conjecture:-

The sum of the exarc radii $R_A + R_B + R_C$ cannot be bounded from above by a linear expression $\lambda R + \mu r$.

1. G. Leversha, The *geometry of the triangle*, UKMT (2013)
2. M. Lukarevski, Exarc radii and the Finsler-Hadwiger inequality, *Math. Gaz.* **106** (March 2022) pp. 138-143.
3. M. Lukarevski, A simple proof of Kooi's inequality, *Math. Mag.* **93**(3) (2022) p. 225.
4. M. Lukarevski, D.S.Marinescu, A refinement of Kooi's inequality, Mittenpunkt and applications, *J. Inequal.Appl.* **13**(3) (2019) pp. 827-832.
5. M. Lukarevski, Wolstenholme's inequality and its relation to the Barrow and Garfunkel-Bankoff inequalities, *Math. Gaz.* **107** (March 2023) pp.70-75.

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On 'Sums of the first *n* integers': Paul Stephenson writes: In [1], Chris Sangwin found twenty remarkably different ways to prove that the sum of the first n odd positive integers is n^2. (For 'number' read 'positive integer' in what follows.) This note discusses three pictorial variants.

In Example 4, the stack of odd numbers is repeated by a fourfold rotation about the centre of a square of side $2n$; it is therefore necessary to divide by 4 to obtain n^2. As an alternative, the stack can be made part of a 'centred square' figure, representing the number $(n - 1)^2 + n^2$, $n \geqslant 1$, by adding the light grey piece to achieve symmetry about the midline of the $(2n - 1)$ column (Figure 1). The circles pick out n^2. If the right-hand part is reflected in the dashed line, each circle marks a unit of the stack, as required.

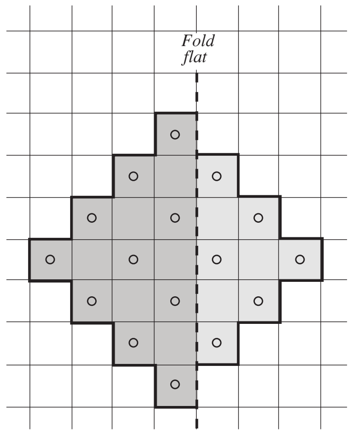
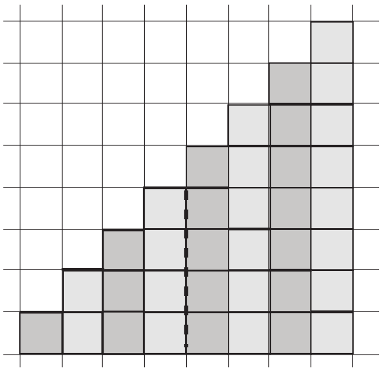


FIGURE 1

Figure 2 shows the first $2n$ numbers in the canonical block form, odds in dark grey, evens in light. In [1, Example 10] the first n odd numbers represent the difference between the first $2n$ numbers and the first n even numbers. The latter sum to twice the total of the first n numbers, duplicated pictorially in Figure 2 by the back-to-back staircases. The algebraic rearrangement of Example 10 is realised here geometrically by the displacement of the columns. We can appeal to Figure 1 and, treating the dark grey region (the stack) as a block of n^2 grid units now defined only by its outline, (that is to say, not inquiring what sort of numbers the constituent columns represent), read Figure 2 backwards to show that n^2 is the sum of the first n odd numbers. In terms of the grid, the individual displacements required to dissect out the odd columns from the undifferentiated block are:



$$1 : \begin{pmatrix} -n \\ -n \end{pmatrix}$$
$$3 : \begin{pmatrix} 1 - n \\ 1 - n \end{pmatrix}$$
$$5 : \begin{pmatrix} 2 - n \\ 2 - n \end{pmatrix}$$
.
.
.
$$2n - 1 : \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

FIGURE 2

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Figure 3 is an attempt to create a dissection distinct from Example 4. (Unfortunately, as readers can judge, the result is less clear, both visually and algebraically.) We have built a spiral from the odd numbers $a, b, c, \ldots,$ $v, w, x, \ldots$. The rectangle with circled vertices has dimensions $2n - 1$ and $2n + 1$. We have dissected this into opposed regions of area x and $3x$. (The '3' arises as $x - v - 1$.) We combine x and $3x$ as $4x$. Noting that the coefficient of a is 3, not 4, we have:

$$(2n + 1)(2n - 1) = 4n^2 - 1 = 4\sum_{k=1}^{k=n}(2k - 1) - 1$$

and the required result.

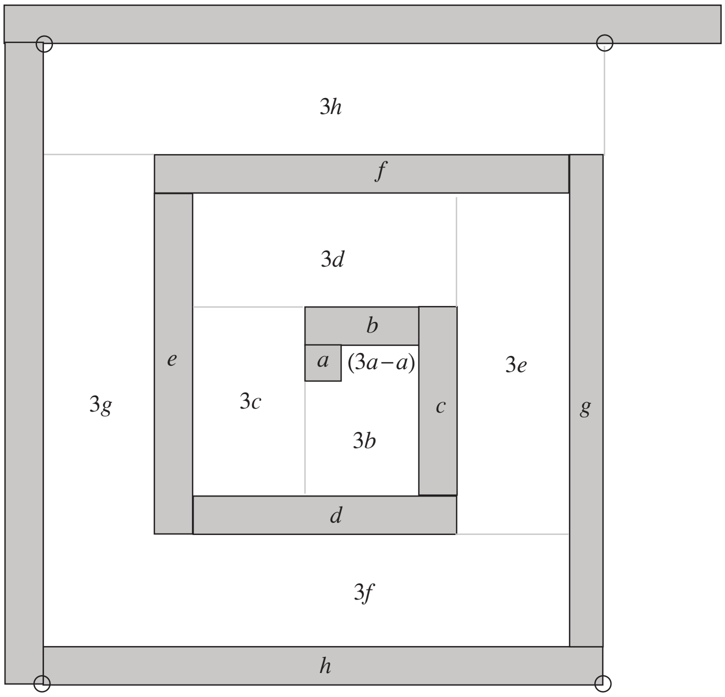

FIGURE 3

Reference

1. C. Sangwin, Sums of the first n odd integers, *Math. Gaz.* **107** (March 2023), pp. 10-24.

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