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COMPACT WEIGHTED COMPOSITION OPERATORS ON H^p-SPACES

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Abstract

Let u and φ be two analytic functions on the unit disc D such that $\varphi(D) \subset D$. A weighted composition operator uC_φ induced by u and φ is defined by $uC_\varphi f := u \cdot f \circ \varphi$ for every f in H^p, the Hardy space of D. We investigate compactness of uC_φ on H^p in terms of function-theoretic properties of u and φ.

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1. Introduction

Let u and φ be two analytic functions on the unit disc D such that $\varphi(D) \subset D$. They induce a *weighted composition operator* uC_φ from the Hardy space H^p $(1 \le p \le \infty)$ into the linear space of all analytic functions on D by

$$uC_\varphi(f)(z) := u(z)f(\varphi(z)) \quad \text{for every } f \in H^p \text{ and } z \in D.$$

When $u \equiv 1$ (respectively $\varphi(z) = z$ for all $z \in D$), the corresponding operator, denoted by C_φ (respectively M_u), is known as a *composition operator* (respectively a *multiplication operator*). It is well-known that C_φ is always bounded on H^p. However, this is not necessarily true for the weighted operator. If uC_φ maps H^p into itself, an appeal to the closed graph theorem yields its boundedness. In this case, we say uC_φ is a weighted composition operator on H^p.

There has been an extensive study of weighted composition operators on H^p (and on other analytic function spaces) in the last two decades. In this paper, we investigate compact weighted composition operators on H^p. The problem of characterising these operators has been considered via different approaches in the literature. It was shown in [5, Theorem 2.1] that uC_φ is compact on H^∞ if and only if the closure of the set $\varphi(\{z \in D : |u(z)| \ge \varepsilon\})$ is contained in D for every $\varepsilon > 0$. When $1 \le p < \infty$ and $u \in H^p$, Contreras and Hernández-Díaz [1, Theorems 3.4 and 3.5] characterised the compactness of uC_φ with the condition

$$\lim_{r\to 0^+} \sup_{\zeta\in T} \frac{m_p(S(\zeta, r))}{r} = 0,$$

where $S(\zeta, r) := \{z \in \overline{D} : |z - \zeta| \le r\}$ and m_p is the measure given by

$$m_p(E) := \int_{\varphi^{-1}(E)\cap T} |u|^p \, dm$$

for all measurable subsets E of $\overline{D}$. Others, such as [2, Theorem 5], studied this problem using a generalised Berezin transform

$$B(z) := \int_0^{2\pi} \frac{1 - |z|^2}{|1 - \overline{z}\varphi(e^{i\theta})|^2} |u(e^{i\theta})|^p \, dm \quad \text{for all } z \in D.$$

These characterisations, however, are rather implicit and somewhat intractable. Motivated by the work in [3] and [4], we obtain necessary conditions and sufficient conditions for the compactness of uC_φ in terms of function-theoretic properties of u and φ. These results are also illustrated with examples.

2. Preliminaries

Let D be the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and T be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. The Hardy space H^p of D, where $1 \le p < \infty$, consists of all analytic functions f on D such that

$$\sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty.$$

We define H^∞ to be the set of all functions f which are analytic and bounded on D.

Let m be the normalised Lebesgue measure on T, that is, $dm := d\theta/2\pi$, and write $L^p = L^p(m)$. Norms of H^p and L^p are both denoted by $\| \cdot \|_p$. If $f \in H^p$ for $1 \le p \le \infty$, its radial limit

$$\hat{f}(e^{i\theta}) := \lim_{r \to 1^-} f(re^{i\theta})$$

exists m-a.e. on T and $\hat{f} \in L^p$ with $\|\hat{f}\|_p = \|f\|_p$. In addition, when $f \not\equiv 0$, we have $\hat{f} \neq 0$ m-a.e. on T. It is often useful to consider the extension of f to $\overline{D} := \{z \in \mathbb{C} : |z| \le 1\}$, also denoted by f, such that $f|_T = \hat{f}$.

We assume $1 \le p < \infty$ in the remaining sections of the paper. Our goal is to relate the compactness of weighted composition operators on H^p with the function theory of analytic maps. For the class of composition operators, this property is intimately related to the notion of angular derivatives of symbol functions. We recall:

(a) a function $f : D \to \mathbb{C}$ is said to have a *nontangential limit* l at $\omega \in T$ if $f(z) \to l$ as z approaches ω in any region between two straight lines of D that meet at ω and are symmetric about the radius to ω;

(b) an analytic function $\varphi : D \to D$ has an *angular derivative* at $\omega \in T$ if there exists some $\eta \in T$ such that the difference quotient $(\eta - \varphi(z))/(\omega - z)$ has a (finite) nontangential limit at ω.

By the Julia–Carathéodory theorem, the following statements are equivalent:

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(i) $\liminf_{z\to\omega}(1-|\varphi(z)|)/(1-|z|)=\delta<\infty$;
(ii) φ has an angular derivative at ω;
(iii) both φ and φ' have nontangential limits at ω.

If one of these conditions holds, then $\delta>0$ and the nontangential limit of φ at ω is η, where η is defined in the definition of angular derivative. Moreover, the angular derivative of φ at ω is nonzero.

Consequently, φ has *no* angular derivative at ω when the radial limit of φ at ω (if it exists) has a modulus less than one. We state three well-known sufficient conditions for compactness and noncompactness of composition operators [7, pages 23 and 57].

(a) If $\|\varphi\|_\infty<1$, then C_φ is compact on H^p.
(b) If φ is univalent and has no angular derivative at any point of T, then C_φ is compact on H^p.
(c) If φ has an angular derivative at some point of T, then C_φ is not compact on H^p.

In his seminal paper [6, Theorem 2.3], Shapiro showed that C_φ is compact on H^p if and only if

$$\lim_{|\omega|\to1^-}\frac{N_\varphi(\omega)}{\log 1/|\omega|}=0,$$

where N_φ is the *Nevanlinna counting function* given by

$$N_\varphi(\omega):=\begin{cases}\displaystyle\sum_{z\in\varphi^{-1}\{\omega\}}\log\frac{1}{|z|} & \text{if } \omega\in\varphi(D)\backslash\{\varphi(0)\},\\ 0 & \text{if } \omega\notin\varphi(D),\end{cases}\tag{2.1}$$

and $\varphi^{-1}\{\omega\}$ denotes the sequence of φ-preimages of ω with each point occurring as many times as its multiplicity.

Recall that a bounded linear operator T from a Banach space B_1 to a Banach space B_2 is said to be *compact* if it maps bounded subsets of B_1 into relatively compact subsets of B_2. Thus, T is compact if and only if it maps every bounded sequence $\{x_n\}_{n=1}^\infty$ in B_1 onto a sequence $\{Tx_n\}_{n=1}^\infty$ in B_2 which has a convergent subsequence.

The above results, together with the following direct generalisation of [3, Lemma 1], are crucial to the study of compact weighted composition operators.

LEMMA 2.1. *Let uC_φ be a weighted composition operator on H^p. The following two statements are equivalent:*

(i) *uC_φ is compact on H^p;*
(ii) *if $\{f_n\}_{n=1}^\infty$ is a bounded sequence in H^p and $f_n\to0$ uniformly on compact subsets of D, then $\|uC_\varphi f_n\|_p\to0$.*

As an application of this lemma, we prove a result of independent interest.

PROPOSITION 2.2. *The following two statements are equivalent:*

(i) $\|\varphi\|_\infty<1$*;*

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(ii) *if $u \in H^p$, then uC_φ is compact on H^p.*

PROOF. Suppose (i) holds, that is, there is a constant M with $0 < M < 1$ for which $|\varphi| \le M$ on D. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in H^p such that $f_n \to 0$ uniformly on compact subsets of D. Choose any $\varepsilon > 0$. Since the set $\{z \in \mathbb{C} : |z| \le M\}$ is compact, there is a natural number N such that

$$|f_n(\varphi(e^{i\theta}))|^p < \frac{\varepsilon}{2\|u\|_p^p}$$

for all $n > N$ and $\theta \in [0, 2\pi]$. Hence

$$\|uC_\varphi f_n\|_p^p = \int_0^{2\pi} |u(e^{i\theta})|^p |f_n(\varphi(e^{i\theta}))|^p \, dm \le \frac{\varepsilon}{2\|u\|_p^p} \int_0^{2\pi} |u(e^{i\theta})|^p \, dm < \varepsilon.$$

This shows that uC_φ is compact.

Conversely, assume (ii) holds. In particular, uC_φ is an operator on H^p. By fixing any $f \in H^p$, we see that $C_\varphi f \in H^\infty$. Thus, C_φ maps H^p into H^∞. This operator is also bounded, so that

$$\|C_\varphi{}^* \delta_\omega\| \le \|C_\varphi{}^*\| \, \|\delta_\omega\|,$$

where $C_\varphi{}^*$ is the adjoint of C_φ and δ_ω ($\omega \in D$) is the evaluation functional on H^∞ at $z = \omega$. Note that $C_\varphi{}^* \delta_\omega = \delta_{\varphi(\omega)}$, which is in $(H^p)^*$, the dual space of H^p. With $\|\delta_\omega\| = 1$ and $\|\delta_{\varphi(\omega)}\| = 1/(1 - |\varphi(\omega)|^2)^{1/p}$,

$$|\varphi(\omega)|^2 \le 1 - \frac{1}{\|C_\varphi{}^*\|^p} < 1.$$

Since ω is arbitrary, we obtain (i). □

When $\|\varphi\|_\infty < 1$ and $u \in H^2$, it follows from [4, Theorem 9] that uC_φ is even Hilbert–Schmidt.

3. Necessary conditions for compactness

Gunatillake [3] remarked that one could begin with a noncompact composition operator C_φ and then produce a compact weighted composition operator uC_φ by choosing a suitable weight function u. In light of this observation, we prove a criterion which is applicable in constructing examples of noncompact weighted composition operators on H^p.

THEOREM 3.1. *Suppose φ is continuous up to T and has an angular derivative at $z = e^{i\alpha}$ for some $\alpha \in [0, 2\pi)$. If there exists a constant $\delta > 0$ such that u is bounded away from zero m-a.e. on $(\alpha - \delta, \alpha + \delta)$ and $|\varphi(e^{i\theta})| < 1$ off $(\alpha - \delta, \alpha + \delta)$, then uC_φ is not compact on H^p.*

PROOF. Since φ has an angular derivative at $z = e^{i\alpha}$, the operator C_φ is noncompact on H^p. There exists a bounded sequence $\{f_n\}_{n=1}^{\infty}$ in H^p such that $f_n \to 0$ uniformly on compact subsets of D and a subsequence of it, say $\{f_{n_k}\}_{k=1}^{\infty}$, satisfies

$$\|C_\varphi f_{n_k}\|_p^p \ge \varepsilon_0 \quad \text{for some } \varepsilon_0 > 0.$$

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Put $I = (\alpha - \delta, \alpha + \delta)$. As φ is continuous on the compact set $\{e^{i\theta} : \theta \in [0, 2\pi]\backslash I\}$, the set $\{\varphi(e^{i\theta}) : \theta \in [0, 2\pi]\backslash I\}$ is compact in D. Thus, there is a natural number N such that

$$|f_n(\varphi(e^{i\theta}))|^p < \frac{\varepsilon_0}{2} \quad \text{for all } n > N \text{ and } \theta \in [0, 2\pi]\backslash I.$$

If $n_k > N$,

$$\begin{aligned}\varepsilon_0 &\le \int_0^{2\pi} |f_{n_k}(\varphi(e^{i\theta}))|^p\, dm \\ &= \int_I |f_{n_k}(\varphi(e^{i\theta}))|^p\, dm + \int_{[0,2\pi]\backslash I} |f_{n_k}(\varphi(e^{i\theta}))|^p\, dm \\ &\le \int_I |f_{n_k}(\varphi(e^{i\theta}))|^p\, dm + \frac{\varepsilon_0}{2},\end{aligned}$$

which gives

$$\int_I |f_{n_k}(\varphi(e^{i\theta}))|^p\, dm \ge \frac{\varepsilon_0}{2}.$$

Let $c > 0$ be a constant for which $|u| \ge c$ m-a.e. on I. Then

$$\begin{aligned}\|uC_\varphi f_{n_k}\|_p^p &\ge \int_I |u(e^{i\theta})|^p |f_{n_k}(\varphi(e^{i\theta}))|^p\, dm \\ &= \int_{\{\theta\in I:|u(e^{i\theta})|\ge c\}} |u(e^{i\theta})|^p |f_{n_k}(\varphi(e^{i\theta}))|^p\, dm \\ &\ge c^p \int_I |f_{n_k}(\varphi(e^{i\theta}))|^p\, dm \ge \frac{c^p \varepsilon_0}{2}.\end{aligned}$$

Hence uC_φ is not compact. □

EXAMPLE 3.2. Let $\varphi(z) = \frac{1}{2}(z + 1)$ and $u(z) = z + 1$. Since $(1 - \varphi(z))/(1 - z) = \frac{1}{2}$, φ has an angular derivative at $z = 1$ (in fact, φ does *not* have angular derivatives at other points of T because $|\varphi(e^{i\theta})|^2 = \frac{1}{2}(1 + \cos\theta) < 1$ for $\theta \in (0, 2\pi)$). By choosing $\delta = \pi/3$, $|u(e^{i\theta})|^2 = 2(1 + \cos\theta) \ge 3$ on $(-\delta, \delta)$. From Theorem 3.1, uC_φ is not compact on H^p.

We now give another necessary condition for compactness (without assuming the continuity of φ on T). This result, which was obtained for the case $p = 2$ in [4, Theorem 8], can be generalised to an arbitrary H^p-space in a similar fashion.

THEOREM 3.3. *If uC_φ is a compact weighted composition operator on H^p, then*

$$\lim_{|z|\to 1^-} \frac{|u(z)|^p (1 - |z|^2)}{1 - |\varphi(z)|^2} = 0. \tag{3.1}$$

EXAMPLE 3.4. Let $\varphi(z) = 1 - (1 - z)^{1/2}$ and $u(z) = 1/(1 - z)^{1/2p}$, where $u \in H^p$. Since

$$1 - [\varphi(r)]^2 = 1 - [1 - (1 - r)^{1/2}]^2 = (1 - r)^{1/2}[2 - (1 - r)^{1/2}],$$

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it follows that

$$\frac{[u(r)]^p(1-r^2)}{1-[\varphi(r)]^2} = \frac{1+r}{2-(1-r)^{1/2}} \to 1\ (\neq 0) \quad \text{as } r \to 1^-.$$

By Theorem 3.3, uC_φ is not compact on H^p. However, C_φ is compact on H^p. This follows from the univalence of φ and the nonexistence of the angular derivative at each point of T. To justify the latter fact, write

$$(1-e^{i\theta})^{1/2} = |1-e^{i\theta}|^{1/2}e^{i(\theta-\pi)/4} \qquad \text{for } \theta \in [0, 2\pi].$$

Let $\Re(z)$ denote the real part of z. Then

$$\begin{aligned} 1-|\varphi(e^{i\theta})|^2 &= 2\,\Re(1-e^{i\theta})^{1/2} - |1-e^{i\theta}| \\ &= |1-e^{i\theta}|^{1/2}\Big[2\cos\Big(\frac{\theta}{4}-\frac{\pi}{4}\Big) - |1-e^{i\theta}|^{1/2}\Big] \\ &= |1-e^{i\theta}|^{1/2}\Big[\sqrt{2}\cos\frac{\theta}{4} + \sqrt{2}\sin\frac{\theta}{4} - \sqrt{2}\Big(\sin\frac{\theta}{2}\Big)^{1/2}\Big] \\ &= \sqrt{2}|1-e^{i\theta}|^{1/2}\Big[\Big(\Big(\cos\frac{\theta}{4}\Big)^{1/2} - \Big(\sin\frac{\theta}{4}\Big)^{1/2}\Big)^2 + (\sqrt{2}-1)\Big(\sin\frac{\theta}{2}\Big)^{1/2}\Big]. \end{aligned}$$

The function in the square bracket of the last equality is continuous on $[0, 2\pi]$. As it never vanishes, there exist constants $m, M > 0$ such that

$$m|1-e^{i\theta}|^{1/2} \le 1-|\varphi(e^{i\theta})|^2 \le M|1-e^{i\theta}|^{1/2}.$$

Thus, $|\varphi(e^{i\theta})| < 1$ for $\theta \in (0, 2\pi)$, so that φ has no angular derivative on $T\setminus\{1\}$. Moreover,

$$\frac{1-\varphi(z)}{1-z} = \frac{1}{(1-z)^{1/2}},$$

which is unbounded near $z = 1$. The angular derivative of φ at $z = 1$ does not exist either.

The converse of Theorem 3.3 is not necessarily true, even for composition operators. For example, it is possible to construct a Blaschke product B with an angular derivative at *no* point of T [7, page 185], that is, $\lim_{|z|\to 1^-}(1-|B(z)|)/(1-|z|) = \infty$. It follows from [4, Theorem 10] that C_B is not compact on H^p. In the next section, we prove that the condition in (3.1) does characterise compactness of weighted composition operators under additional assumptions on the symbol functions u and φ.

4. Sufficient conditions for compactness

From Example 3.4 (in the previous section), the compactness of C_φ may not guarantee that of uC_φ. A natural question is that if C_φ is compact, how can we choose u so that uC_φ is also compact? One such condition is presented below.

THEOREM 4.1. *Suppose $u \in H^p$ and C_φ is compact on H^p. If there is a constant c with $0 < c < 1$ such that u is essentially bounded on the set $\{e^{i\theta} \in T : |\varphi(e^{i\theta})| > c\}$, then uC_φ is compact on H^p.*

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Proof. Let $E := \{\theta \in [0, 2\pi] : |\varphi(e^{i\theta})| > c\}$ and $M > 0$ be a constant with $|u(e^{i\theta})| \le M$ m-a.e. on E. From Lemma 2.1, it suffices to show that $\|uC_\varphi f_n\|_p \to 0$, where $\{f_n\}_{n=1}^\infty$ is a bounded sequence in H^p such that $f_n \to 0$ uniformly on compact subsets of D.

Fix any $\varepsilon > 0$. As the set $\{z \in \mathbb{C} : |z| \le c\}$ is compact in D, there is a natural number N_1 such that

$$|f_n(\varphi(e^{i\theta}))|^p < \frac{\varepsilon}{2\|u\|_p^p}$$

whenever $n > N_1$ and $\theta \in [0, 2\pi]\backslash E$. By the compactness of C_φ, we may choose a natural number N_2 such that

$$\int_0^{2\pi} |f_n(\varphi(e^{i\theta}))|^p \, dm = \|C_\varphi f_n\|_p^p < \frac{\varepsilon}{2M^p} \quad \text{for all } n > N_2.$$

If $n > \max\{N_1, N_2\}$, then

$$\begin{aligned}\|uC_\varphi f_n\|_p^p &= \int_E |u(e^{i\theta})|^p |f_n(\varphi(e^{i\theta}))|^p \, dm + \int_{[0,2\pi]\backslash E} |u(e^{i\theta})|^p |f_n(\varphi(e^{i\theta}))|^p \, dm \\ &\le M^p \int_E |f_n(\varphi(e^{i\theta}))|^p \, dm + \frac{\varepsilon}{2\|u\|_p^p} \int_{[0,2\pi]\backslash E} |u(e^{i\theta})|^p \, dm \\ &\le M^p \int_0^{2\pi} |f_n(\varphi(e^{i\theta}))|^p \, dm + \frac{\varepsilon}{2\|u\|_p^p} \int_0^{2\pi} |u(e^{i\theta})|^p \, dm \\ &< \varepsilon.\end{aligned}$$

□

Remark 4.2. (a) Proposition 2.2 is also a direct consequence of Theorem 4.1, for one may take $c = \|\varphi\|_\infty$ and the set in the statement of this theorem is then of zero measure.

(b) In Theorem 4.1, suppose the assumption that C_φ is compact is dropped. Similar norm estimates used to write the proof of this theorem, together with the Littlewood subordination principle, show that uC_φ maps H^p into itself and is therefore bounded (by the closed graph theorem).

Example 4.3. Let $\varphi(z) = 1 - (1 - z)^{1/2}$ and $u(z) = [(z - 1)/(z + 1)]^{1/2p}$. Then, $u \in H^p$ and C_φ is compact on H^p (see Example 3.4). With $|\varphi(-1)| < \frac{1}{2}$, the continuity of φ on T ensures that there exists a constant δ with $0 < \delta < \pi$ such that

$$\{\theta \in [0, 2\pi] : |\varphi(e^{i\theta})| > \tfrac{1}{2}\} \subset [0, 2\pi]\backslash(\pi - \delta, \pi + \delta).$$

Since u is also bounded on the set $\{e^{i\theta} : \theta \in [0, 2\pi]\backslash(\pi - \delta, \pi + \delta)\}$, it follows from Theorem 4.1 that uC_φ is compact on H^p.

The rest of this section is devoted to proving a 'converse' of Theorem 3.3 under extra assumptions on u and φ. We begin with three lemmas. The first one, which appeared in [8, Lemma 2.3 and Proposition 2.4], relates a boundary integral of an H^p-function with an area integral of the function and its derivative.

LEMMA 4.4. *Let $f \in H^p$. Then,*

$$\|f\|_p^p \approx |f(0)|^p + \int_D |f(z)|^{p-2}|f'(z)|^2 \log\frac{1}{|z|}\,dA(z)$$

and

$$\|f\circ\varphi\|_p^p \approx |f(\varphi(0))|^p + \int_D |f(z)|^{p-2}|f'(z)|^2 N_\varphi(z)\,dA(z),$$

where

(a) *A is the normalised Lebesgue area measure on D, that is, $dA = r\,dr\,d\theta/\pi$;*
(b) *N_φ is the Nevanlinna counting function defined in (2.1); and*
(c) *the symbol $\approx$ means that the left-hand side is bounded above and below by positive constant multiples of the right-hand side, and these constants are independent of f.*

An immediate consequence of this lemma is that there are constants $M_1, M_2 > 0$ such that

$$\int_D |f(z)|^{p-2}|f'(z)|^2 \log\frac{1}{|z|}\,dA(z) \le M_1\|f\|_p^p \tag{4.1}$$

and

$$\int_D |f(z)|^{p-2}|f'(z)|^2 N_\varphi(z)\,dA(z) \le M_2\|f\circ\varphi\|_p^p \tag{4.2}$$

for all $f \in H^p$.

LEMMA 4.5.

(a) (i) *If $0 < |z| < 1$, then $1 - |z|^2 \le 2\log 1/|z|$.*
(ii) *If $\frac{1}{2} \le |z| < 1$, then $\log 1/|z| \le 2(1 - |z|^2)$.*
(b) *For $0 < r < 1$, there exists a positive constant c (depending only on r) such that*

$$\log\frac{1}{|z|} \le c\Big(\log\frac{r+1}{2} + \log\frac{1}{|z|}\Big) \quad \text{if } 0 < |z| \le r.$$

PROOF. We first prove (a). Fix any $z \in \mathbb{C}$ with $0 < |z| < 1$. By the generalised mean value theorem, there exists some $\zeta \in (|z|, 1)$ such that

$$\frac{\log 1/|z|}{1 - |z|^2} = \frac{1/\zeta}{2\zeta} = \frac{1}{2\zeta^2}.$$

As $1/(2\zeta^2) \ge \frac{1}{2}$, (i) follows. If, in addition, $|z| \ge \frac{1}{2}$, then $1/(2\zeta^2) \le 2$. This gives (ii).

For (b), we fix any r with $0 < r < 1$ and define

$$f(x) := \frac{\log 1/x}{\log\frac{1}{2}(r+1) + \log 1/x} \quad \text{for } 0 < x \le r.$$

The existence of c follows because f is positive and continuous on $(0, r]$, and $\lim_{x\to 0^+} f(x) = 1$. □

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Lemma 4.6. *If $f \in H^p$, then*

$$\int_D |f(z)|^p \, dA(z) \le \|f\|_p^p.$$

Proof. If $f \in H^p$, then

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \le \|f\|_p^p \quad \text{for all } 0 \le r < 1.$$

Thus,

$$\begin{aligned} \int_D |f(z)|^p \, dA(z) &= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 |f(re^{i\theta})|^p r dr d\theta = \frac{1}{\pi} \int_0^1 r \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta dr \\ &\le 2\|f\|_p^p \int_0^1 r \, dr = \|f\|_p^p. \end{aligned}$$

□

Theorem 4.7. *Let $u \in H^p$. Assume that φ is univalent on D and*

$$\lim_{|z| \to 1^-} |u(z)|^{p-2} |u'(z)|^2 (1 - |z|^2) = 0. \tag{4.3}$$

If

$$\lim_{|z| \to 1^-} \frac{|u(z)|^p (1 - |z|^2)}{1 - |\varphi(z)|^2} = 0, \tag{4.4}$$

then uC_φ is compact on H^p.

Proof. Fix any $\varepsilon > 0$. By (4.3) and (4.4), there exists r with $\frac{1}{2} < r < 1$ for which

$$|u(z)|^p (1 - |z|^2) < \varepsilon (1 - |\varphi(z)|^2) \quad \text{and} \quad |u(z)|^{p-2} |u'(z)|^2 (1 - |z|^2) < \varepsilon$$

whenever $r < |z| < 1$. The value of this r will be fixed for the entire proof. Let $\{f_n\}_{n=1}^\infty$ be a sequence in H^p such that $\|f_n\|_p \le 1$ for all n and $f_n \to 0$ uniformly on compact subsets of D. By Lemma 4.4,

$$\|uC_\varphi f_n\|_p^p \le M\Big[|u(0)|^p |f_n(\varphi(0))|^p + \int_D |u(z)(f_n \circ \varphi)(z)|^{p-2} |(u \cdot f_n \circ \varphi)'(z)|^2 \log \frac{1}{|z|} \, dA(z)\Big] \tag{4.5}$$

for a constant $M > 0$. Let $rD := \{z \in \mathbb{C} : |z| \le r\}$. Since

$$\begin{aligned} |(u \cdot f_n \circ \varphi)'(z)|^2 &= |u(z)(f_n \circ \varphi)'(z) + (f_n \circ \varphi)(z) u'(z)|^2 \\ &\le 2|u(z)(f_n \circ \varphi)'(z)|^2 + 2|(f_n \circ \varphi)(z) u'(z)|^2, \end{aligned}$$

it follows that

$$\begin{aligned} &\int_D |u(z)(f_n \circ \varphi)(z)|^{p-2} |(u \cdot f_n \circ \varphi)'(z)|^2 \log \frac{1}{|z|} \, dA(z) \\ &\le 2 \int_D |u(z)|^p |(f_n \circ \varphi)(z)|^{p-2} |(f_n \circ \varphi)'(z)|^2 \log \frac{1}{|z|} \, dA(z) \\ &\quad + 2 \int_D |u(z)|^{p-2} |u'(z)|^2 |(f_n \circ \varphi)(z)|^p \log \frac{1}{|z|} \, dA(z) \\ &= 2(P_n + Q_n + R_n + S_n), \end{aligned} \tag{4.6}$$

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where

$$P_n := \int_{rD} |u(z)|^p |(f_n \circ \varphi)(z)|^{p-2} |(f_n \circ \varphi)'(z)|^2 \log \frac{1}{|z|}\, dA(z),$$
$$Q_n := \int_{D \setminus rD} |u(z)|^p |f_n(\varphi(z))|^{p-2} |f_n'(\varphi(z))|^2 |\varphi'(z)|^2 \log \frac{1}{|z|}\, dA(z),$$
$$R_n := \int_{rD} |u(z)|^{p-2} |u'(z)|^2 |f_n(\varphi(z))|^p \log \frac{1}{|z|}\, dA(z),$$
$$S_n := \int_{D \setminus rD} |u(z)|^{p-2} |u'(z)|^2 |(f_n \circ \varphi)(z)|^p \log \frac{1}{|z|}\, dA(z).$$

As the sets $\{\varphi(0)\}$ and $\varphi(rD)$ are compact in D, we may choose a natural number N_1 such that if $n > N_1$ and $z \in rD$, then

$$|f_n(\varphi(0))|^p < \varepsilon \quad \text{and} \quad |f_n(\varphi(z))|^p < \varepsilon.$$

From the inequality in (4.1),

$$\begin{aligned} |u(0)|^p |f_n(\varphi(0))|^p + 2R_n &\le |u(0)|^p \varepsilon + 2\varepsilon \int_{rD} |u(z)|^{p-2} |u'(z)|^2 | \log \frac{1}{|z|}\, dA(z) \\ &\le [|u(0)|^p + 2M_1 \|u\|_p^p]\varepsilon. \end{aligned} \tag{4.7}$$

To estimate the value of S_n, we first use Lemma 4.5(a)(ii):

$$\begin{aligned} S_n &\le 2 \int_{D \setminus rD} |u(z)|^{p-2} |u'(z)|^2 |(f_n \circ \varphi)(z)|^p (1 - |z|^2)\, dA(z) \\ &\le 2\varepsilon \int_{D \setminus rD} |(f_n \circ \varphi)(z)|^p\, dA(z). \end{aligned}$$

The fact that C_φ is bounded on H^p, together with Lemma 4.6, gives

$$S_n \le 2\varepsilon \int_D |(f_n \circ \varphi)(z)|^p\, dA(z) \le 2\varepsilon \|f_n \circ \varphi\|_p^p \le 2\|C_\varphi\|^p \|f_n\|_p^p \varepsilon \le 2\|C_\varphi\|^p \varepsilon. \tag{4.8}$$

Using Lemma 4.5(a)(ii) again,

$$\begin{aligned} Q_n &\le 2 \int_{D \setminus rD} |u(z)|^p |f_n(\varphi(z))|^{p-2} |f_n'(\varphi(z))|^2 |\varphi'(z)|^2 (1 - |z|^2)\, dA(z) \\ &\le 2\varepsilon \int_{D \setminus rD} |f_n(\varphi(z))|^{p-2} |f_n'(\varphi(z))|^2 (1 - |\varphi(z)|^2) |\varphi'(z)|^2\, dA(z) \\ &\le 2\varepsilon \int_D |f_n(\varphi(z))|^{p-2} |f_n'(\varphi(z))|^2 (1 - |\varphi(z)|^2) |\varphi'(z)|^2\, dA(z). \end{aligned}$$

Put $\omega = \varphi(z)$. As φ is univalent, the Cauchy–Riemann equations and the change-of-variables formula yield $dA(\omega) = |\varphi'(z)|^2\, dA(z)$. Then

$$Q_n \le 2\varepsilon \int_D |f_n(\omega)|^{p-2} |f_n'(\omega)|^2 (1 - |\omega|^2)\, dA(\omega).$$

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By Lemma 4.5(a)(i) and the inequality in (4.1),

$$Q_n \le 4\varepsilon \int_D |f_n(\omega)|^{p-2}|f_n'(\omega)|^2 \log\frac{1}{|\omega|}\, dA(\omega) \le 4M_1\|f_n\|_p^p\varepsilon \le 4M_1\varepsilon. \tag{4.9}$$

It remains to estimate the value of P_n. Put $g_n = f_n \circ \varphi$. With the continuity of u on the compact set rD and Lemma 4.5(b), there is a constant $M' > 0$ such that

$$\begin{aligned} P_n &\le M' \int_{rD} |g_n(z)|^{p-2}|g_n'(z)|^2\Big(\log\frac{r+1}{2} + \log\frac{1}{|z|}\Big)\, dA(z) \\ &= M' \int_{rD} |g_n(z)|^{p-2}|g_n'(z)|^2 N_\sigma(z)\, dA(z), \end{aligned}$$

where $\sigma(z) = \frac{1}{2}(r+1)z$. From the inequality in (4.2),

$$\int_{rD} |g_n(z)|^{p-2}|g_n'(z)|^2 N_\sigma(z)\, dA(z) \le M_2\|g_n \circ \sigma\|_p^p.$$

Since $g_n \to 0$ uniformly on the compact set $\sigma(T)$ in D, there is a natural number N_2 such that if $n > N_2$, then

$$\|g_n \circ \sigma\|_p^p = \int_0^{2\pi} |g_n(\sigma(e^{i\theta}))|^p\, dm < \varepsilon.$$

Thus,

$$P_n < M'M_2\varepsilon. \tag{4.10}$$

It now follows from (4.5)–(4.10) that

$$\|uC_\varphi f_n\|_p^p < M[|u(0)|^p + 4\|C_\varphi\|^p + 2M_1(\|u\|_p^p + 4) + 2M'M_2]\varepsilon$$

whenever $n > \max\{N_1, N_2\}$. Hence $\|uC_\varphi f_n\|_p \to 0$, as desired. □

In particular, we obtain the following result, which is essentially due to MacCluer and Shapiro.

Corollary 4.8 [7, page 39]. *If φ is univalent on D and*

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0,$$

then C_φ is compact on H^p.

Example 4.9. Let $\varphi(z) = \frac{1}{2}(z+1)$ and $u(z) = (z-1)^{2/p}$. For $|z| < 1$,

$$\frac{|u(z)|^p}{1-|\varphi(z)|^2} = \frac{4(1-2\,\Re(z)+|z|^2)}{3-2\,\Re(z)-|z|^2} \quad \text{and} \quad 0 < \frac{1-2\,\Re(z)+|z|^2}{3-2\,\Re(z)-|z|^2} < 1$$

and so

$$\lim_{|z|\to 1^-} \frac{|u(z)|^p(1-|z|^2)}{1-|\varphi(z)|^2} = 0.$$

A direct computation gives $\lim_{|z|\to 1^-} |u(z)|^{p-2}|u'(z)|^2(1-|z|^2) = 0$. By Theorem 4.7, uC_φ is compact on H^p. This example also shows that even if C_φ is noncompact (for φ has an angular derivative at $z = 1$), the weighted operator uC_φ can be compact on H^p with an appropriate choice of u.

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The condition '$\lim_{|z|\to 1^-} |u(z)|^{p-2}|u'(z)|^2(1-|z|^2) = 0$' is not necessary for uC_φ to be compact on H^p. For example, let $\varphi(z) = z/2$ and $u(z) = 1/(1-z)^{1/2p}$. The operator C_φ is compact on H^p, since φ is univalent and has no angular derivative at each point of $\mathbb{T}$ (in fact, $|\varphi(e^{i\theta})| = \frac{1}{2} < 1$ for $\theta \in [0, 2\pi]$). As $u \in H^p$, an application of Theorem 4.1 (by taking $c = \frac{1}{2}$) yields the compactness of uC_φ on H^p. However,

$$[u(r)]^{p-2}[u'(r)]^2(1-r^2) = \frac{1+r}{4p^2(1-r)^{3/2}} \to \infty \quad \text{as } r \to 1^-.$$

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