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On perturbed stochastic discrete systems

B.G. Pachpatte

The object of this paper is to study a stochastic discrete system, including an operator T , of the form

$$x_{n+1}(\omega) = A(\omega)x_n(\omega) + f_n\big(\omega, x_n(\omega), (Tx_n)(\omega)\big) , \quad x_0(\omega) = x_0 ,$$

as a perturbation of the linear stochastic discrete system

$$y_{n+1}(\omega) = A(\omega)y_n(\omega) , \quad y_0(\omega) = x_0 ,$$

where $\omega \in \Omega$, the supporting set of probability measure space (Ω, A, P) and $n \in N$, the set of nonnegative integers. We are concerned with the existence, uniqueness, boundedness, and asymptotic behavior of random solutions of the above equation.

1. Introduction

The theory of stochastic or random equations is in a process of continuous development and it has become significant for its various applications in the general areas of the engineering, biological, and physical sciences. Recently, attempts have been made by many scientists and mathematicians to develop and unify the theory of stochastic or random equations using the concepts and methods of probability theory and functional analysis; see [1], [2]-[4], [5], [7], and some of the references given there. Observing a certain physical system with random parameter, in which the independent variable may conveniently be assumed to have only a discrete set of possible values, often lead to mathematical models involving stochastic discrete systems. In this paper we shall study a stochastic

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discrete system, including an operator T, of the form

$$(1)\quad x_{n+1}(\omega) = A(\omega)x_n(\omega) + f_n\big(\omega, x_n(\omega), (Tx_n)(\omega)\big), \quad x_0(\omega) = x_0, \quad n \in N .$$

The system will be studied as a perturbation of the linear stochastic discrete system

$$(2)\qquad y_{n+1}(\omega) = A(\omega)y_n(\omega), \quad y_0(\omega) = x_0, \quad n \in N .$$

Here x_n, y_n are stochastic processes, $A(\omega)$ is an $r \times r$ matrix whose elements are measurable functions, for each n, f_n is a vector valued function defined on $\Omega \times R^r \times R^r \to R^r$, R^r is an euclidean r space and T is an operator which maps R^r into R^r.

Recently, Morozan [2]-[4], and Tsokos and Padgett [7, pp. 121-129], have studied the stability of the random solutions of some special forms of (1). The problem considered in this paper is in the general spirit of the investigations in [2]-[4], [7]. The allowable perturbation terms here include more than just ordinary random function type perturbations. In particular if we impose on T various meanings, it is apparent that equation (1) has a great diversity. For example, the operators we have in mind are of the form

$$g_n\big(\omega, x_n(\omega)\big) \quad \text{or} \quad \sum_{s=0}^{n-1} k_{n,s}(\omega)x_s(\omega) \quad \text{or} \quad \sum_{s=0}^{n-1} k_{n,s}(\omega)g_s\big(\omega, x_s(\omega)\big)$$

and so on. The particular concern of this paper is the existence, uniqueness, boundedness, and asymptotic behavior of a random solution of the stochastic discrete system (1) under some suitable conditions on f_n and on the operator T. The tools that will be employed are the well known fixed point theorem of Banach and the finite difference inequality recently established in [6].

2. Preliminaries

In this section we shall define various notations and terms which will be used in our subsequent discussion. The symbol $|\cdot|$ will denote some convenient norm on R^r as well as a corresponding consistent matrix norm. Let $BC[0, \infty)$ denote the set of bounded functions $x_n(\omega)$ $(n \in N, \omega \in \Omega)$

in R^r, and if $x_n(\omega) \in BC[0, \infty)$, denote its norm by $\|x_n(\omega)\| = \sup_{n \in N} |x_n(\omega)|$. By a *random solution* of a stochastic discrete system (1) we shall mean that for each $n \in N$, $x_n(\omega)$ satisfies the equation almost surely.

Let $k_{n,s}(\omega)$ be a stochastic kernel bounded in the ordinary sense except perhaps on a set with probability measure zero for each n and s satisfying $0 \le s \le n$ such that

$$\left|Y_n(\omega)Y_{s+1}^{-1}(\omega)\right| \le k_{n,s}(\omega) , \tag{3}$$

where $Y_n(\omega)$ is the stochastic fundamental matrix solution of the homogeneous system (2) such that $Y_0(\omega)$ is the unit matrix. Let $y_n(\omega)$ be a bounded random solution of a linear homogeneous system (2) such that $y_0(\omega) = x_0$. It is easy to observe that $y_n(\omega) = Y_n(\omega)x_0$.

To obtain our results in Section 4, we require the following finite difference inequality proved in [*6*].

LEMMA 1. *Let* $u(n)$, $p(n)$, *and* $q(n)$ *be real valued non-negative functions defined on* N *for which the inequality*

$$u(n) \le u_0 + \sum_{s=0}^{n-1} p(s)u(s) + \sum_{s=0}^{n-1} p(s)\left(\sum_{\tau=0}^{s-1} q(\tau)u(\tau)\right)$$

holds for all $n \in N$, *where* u_0 *is a non-negative constant. Then*

$$u(n) \le u_0\left[1 + \sum_{s=0}^{n-1} p(s)\left(\prod_{\tau=0}^{s-1} \left(1+p(\tau)+q(\tau)\right)\right)\right] , \quad n \in N .$$

3. Existence of a random solution

In this section we state and prove a theorem that gives conditions under which the stochastic discrete system (1) possesses a unique random solution. A well known Banach fixed point theorem will be used in the proof.

THEOREM 1. *Consider the stochastic discrete system* (1) *subject to the following conditions:*

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(H_1) *suppose that there exists a constant* $M > 0$ *such that*

$$\sum_{s=0}^{n-1} k_{n,s}(\omega) \le M < \infty , \quad n \in N ;$$

(H_2) $f_n(\omega, 0, 0) \equiv 0$ *for any fixed* $n \in N$ *;*

(H_3) *for each* $\alpha > 0$ *, there exists* $\eta > 0$ *such that*

$$|f_n(\omega, x_n(\omega), (Tx_n)(\omega)) - f_n(\omega, \bar{x}_n(\omega), (T\bar{x}_n)(\omega))| \le \alpha[|x_n(\omega) - \bar{x}_n(\omega)| + |(Tx_n)(\omega) - (T\bar{x}_n)(\omega)|]$$

for all $n \in N$ *, whenever* $|x_n(\omega)|, |\bar{x}_n(\omega)| \le \eta$ *;*

(H_4) $|(Tx_n)(\omega) - (T\bar{x}_n)(\omega)| \le \gamma|x_n(\omega) - \bar{x}_n(\omega)|$ *, where* $\gamma > 0$ *is a constant and* $|x_n(\omega)|, |\bar{x}_n(\omega)| \le \eta$.

Then there exists a number $\varepsilon_0 > 0$ *such that to any* $\varepsilon \in (0, \varepsilon_0]$ *, if* $\|y_n(\omega)\| \le \lambda\varepsilon$ *for some fixed* $\lambda \in (0, 1)$ *, then there exists a unique random solution* $x_n(\omega)$ *of the equation* (1) *for all* $n \in N$ *, satisfying* $\|x_n(\omega)\| \le \varepsilon$.

Proof. Fix $\alpha > 0$, $\gamma > 0$, such that $\alpha M(1+\gamma) < 1$. Fix a number $\beta > 0$ such that $\beta M(1+\gamma) \le 1 - \lambda$. Using (H_2), (H_3), and (H_4), pick $\delta > 0$ such that

$$(4) \qquad |f_n(\omega, x_n(\omega), (Tx_n)(\omega))| \le \beta(1+\gamma)|x_n(\omega)| ,$$

for all $n \in N$, whenever $|x_n(\omega)| \le \delta$. Define $\varepsilon_0 = \min(\delta, \eta)$. For any ε , $0 \le \varepsilon \le \varepsilon_0$ we define

$$S(\varepsilon) = \{x_n(\omega) : x_n(\omega) \in BC[0, \infty), \|x_n(\omega)\| \le \varepsilon\} .$$

Define the operator U by the relation

$$(5) \qquad (Ux_n)(\omega) = y_n(\omega) + \sum_{s=0}^{n-1} Y_n(\omega)Y_{s+1}^{-1}(\omega)f_s(\omega, x_s(\omega), (Tx_s)(\omega))$$

for $x_n(\omega) \in S(\varepsilon)$, whose fixed point corresponds to the solution of the equation (1). Using (3), (H_1), (4), and (5), we obtain

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$$|(Ux_n)(\omega)| \le \|y_n(\omega)\| + \sum_{s=0}^{n-1} \left| Y_n(\omega)Y_{s+1}^{-1}(\omega)f_s\left(\omega,\ x_s(\omega),\ (Tx_s)(\omega)\right) \right|$$

$$\le \lambda\varepsilon + \sum_{s=0}^{n-1} k_{n,s}(\omega)\beta(1+\gamma)\|x_s(\omega)\|$$

$$\le \lambda\varepsilon + \beta M(1+\gamma)\varepsilon$$

$$\le \lambda\varepsilon + (1-\lambda)\varepsilon = \varepsilon\ .$$

Hence U maps $S(\varepsilon)$ into itself. On the other hand using (3), (H_1), (H_3), (H_4), and (5) we have

$$\|(Ux_n)(\omega)-(U\bar{x}_n)(\omega)\| \le \sum_{s=0}^{n-1} k_{n,s}(\omega)\alpha(1+\gamma)\|x_s(\omega)-\bar{x}_s(\omega)\|$$

and

$$\|(Ux_n)(\omega)-(U\bar{x}_n)(\omega)\| \le \alpha M(1+\gamma)\|x_n(\omega)-\bar{x}_n(\omega)\|$$

for any $x_n(\omega),\ \bar{x}_n(\omega) \in S(\varepsilon)$. Since $\alpha M(1+\gamma) < 1$, U is a contraction on $S(\varepsilon)$. Hence by the well known contraction mapping principle, the equation (1) has a unique solution $x_n(\omega) \in S(\varepsilon)$ with $\|x_n(\omega)\| \le \varepsilon$. This completes the proof of the theorem.

4. Boundedness and asymptotic behavior

In this section we shall study the boundedness and asymptotic behavior of random solutions of equation (1) as a perturbation of the system (2).

Theorem 2 below establishes that to every bounded random solution of (2) there corresponds a bounded random solution of (1) under some suitable conditions on f_n and on the operator T.

THEOREM 2. *Suppose that*

(6) $$\left| Y_n(\omega)Y_{s+1}^{-1}(\omega)f_s\left(\omega,\ x_s(\omega),\ (Tx_s)(\omega)\right) \right| \le p_s(\omega)\left(|x_s(\omega)|+|(Tx_s)(\omega)|\right)$$

where $p_s(\omega)$ *is a non-negative random function defined for* $s \in N$, $\omega \in \Omega$, *and* $\sum_{s=0}^{\infty} p_s(\omega) < \infty$. *Further, suppose that the operator* T *satisfies the inequality*

$$(7) \qquad |(Tx_n)(\omega)| \le \sum_{s=0}^{n-1} q_s(\omega)|x_s(\omega)| ,$$

where $q_n(\omega)$ *is a non-negative random function defined for* $n \in N$, $\omega \in \Omega$, *and* $\prod_{s=0}^{n-1} \big(1+p_s(\omega)+q_s(\omega)\big) < \infty$. *Then to every bounded random solution* $y_n(\omega)$ *of* (2) *on* N, *the corresponding random solution* $x_n(\omega)$ *of* (1) *is bounded on* N.

Proof. The random solutions of (1) and (2) with the same initial values are related by

$$(8) \qquad x_n(\omega) = y_n(\omega) + \sum_{s=0}^{n-1} Y_n(\omega)Y_{s+1}^{-1}(\omega)f_s\big(\omega, x_s(\omega), (Tx_s)(\omega)\big) .$$

From (6), (7), and (8), we obtain

$$|x_n(\omega)| \le c + \sum_{s=0}^{n-1} p_s(\omega)|x_s(\omega)| + \sum_{s=0}^{n-1} p_s(\omega)\left(\sum_{\tau=0}^{s-1} q_\tau(\omega)|x_\tau(\omega)|\right) ,$$

where c is the upper bound for $|y_n(\omega)|$. Now an application of Lemma 1 with $u(n) = |x_n(\omega)|$ yields

$$|x_n(\omega)| \le c\left[1 + \sum_{s=0}^{n-1} p_s(\omega)\left(\prod_{\tau=0}^{s-1} \big(1+p_\tau(\omega)+q_\tau(\omega)\big)\right)\right] .$$

The above estimation implies the boundedness of $|x_n(\omega)|$ on N, and the theorem is proved.

Our next theorem shows that under some suitable conditions on the fundamental matrix of (2) and the perturbation term in (1), all the random solutions of (1) approach zero as $n \to \infty$.

THEOREM 3. *Let the fundamental matrix* $Y_n(\omega)$ *of* (2) *satisfy the inequalities*

$$(9) \qquad \left|Y_n(\omega)Y_{s+1}^{-1}(\omega)\right| \le Me^{-\alpha(n-s)} , \quad |Y_n(\omega)| \le Me^{-\alpha n} ,$$

where $M > 0$, $\alpha > 0$ *are constants. Suppose that the perturbation term* $f_n\big(\omega, x_n(\omega), (Tx_n)(\omega)\big)$ *in* (1) *satisfies*

(10) $$\left|f_n\left(\omega,\ x_n(\omega),\ \left(Tx_n\right)(\omega)\right)\right| \le p_n(\omega)\left(\left|x_n(\omega)\right|+\left|\left(Tx_n\right)(\omega)\right|\right) ,$$

where $p_n(\omega)$ *is a non-negative random function defined for* $n \in N$ *,* $\omega \in \Omega$ *, and* $\sum_{s=0}^{\infty} p_s(\omega) < \infty$ *. Further, suppose that the operator* T *satisfies the inequality*

(11) $$\left|\left(Tx_n\right)(\omega)\right| \le e^{-\alpha n} \sum_{s=0}^{n-1} q_s(\omega)\left|x_s(\omega)\right| ,$$

where $q_n(\omega)$ *is a non-negative random function defined for* $n \in N$ *,* $\omega \in \Omega$ *, and* $\prod_{s=0}^{n-1}\left(1+Mp_s(\omega)+q_s(\omega)e^{-\alpha s}\right) < \infty$ *. Then all random solutions of* (1) *approach zero as* $n \to \infty$ *.*

Proof. The random solutions of (1) and (2) with the same initial values are related by

(12) $$x_n(\omega) = Y_n(\omega)x_0 + \sum_{s=0}^{n-1} Y_n(\omega)Y_{s+1}^{-1}(\omega)f_s\left(\omega,\ x_s(\omega),\ \left(Tx_s\right)(\omega)\right) .$$

Using (9), (10), (11), and (12), we obtain

$$\left|x_n(\omega)\right| \le M\left|x_0\right|e^{-\alpha n} + e^{-\alpha n}\sum_{s=0}^{n-1} Me^{\alpha s} \times \left[p_s(\omega)\left(\left|x_s(\omega)\right| + e^{-\alpha s}\sum_{\tau=0}^{s-1} q_\tau(\omega)\left|x_\tau(\omega)\right|\right)\right]$$

Multiplying both sides of the above inequality by $e^{\alpha n}$, applying Lemma 1 with $u(n) = \left|x_n(\omega)\right|e^{\alpha n}$, then multiplying by $e^{-\alpha n}$, we obtain

$$\left|x_n(\omega)\right| \le M\left|x_0\right|e^{-\alpha n}\left[1 + \sum_{s=0}^{n-1} Mp_s(\omega)\left(\prod_{\tau=0}^{s-1}\left(1+Mp_\tau(\omega)+q_\tau(\omega)e^{-\alpha\tau}\right)\right)\right] .$$

The above estimate yields the desired result if we choose M and $\left|x_0\right|$ small enough, and the proof of the theorem is complete.

The random function $z_n(\omega)$ will be called *slowly growing* if and only

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if for every $\varepsilon > 0$ there exists a constant M, which may depend on ε, such that

$$|z_n(\omega)| \le Me^{\varepsilon n} , \quad n \in N .$$

Theorem 4 below demonstrates that the random solution of (1) grows more slowly than any positive exponential.

THEOREM 4. *Let the fundamental matrix* $Y_n(\omega)$ *of* (2) *satisfy the inequalities*

$$\left|Y_n(\omega)Y_{s+1}^{-1}(\omega)\right| \le Ke^{\varepsilon(n-s)} , \quad |Y_n(\omega)| \le Ke^{\varepsilon n} ,$$

where $K > 0$, $\alpha > 0$ *are constants. Suppose that the perturbation term in* (1) *satisfies the condition* (10) *of Theorem* 3, *and the operator* T *satisfies the inequality*

$$|(Tx_n)(\omega)| \le e^{\varepsilon n} \sum_{s=0}^{n-1} q_s(\omega)|x_s(\omega)| ,$$

where $q_n(\omega)$ *is a non-negative random function defined for* $n \in N$, $\omega \in \Omega$, *and*

$$1 + \sum_{s=0}^{n-1} Kp_s(\omega)\left(\prod_{\tau=0}^{s-1} \left[1+Kp_\tau(\omega)+q_\tau(\omega)e^{\varepsilon\tau}\right]\right) \le c ,$$

where $c > 0$ *is a constant; then all solutions of* (1) *are slowly growing.*

The proof of this theorem follows by a similar argument as in the proof of Theorem 3, and hence we omit the details.

We observe that the stochastic discrete system (1) may be written in the equivalent form

$$(13) \qquad x_n(\omega) = y_n(\omega) + \sum_{s=0}^{n-1} Y_n(\omega)Y_{s+1}^{-1}(\omega)f_s\big(\omega, x_s(\omega), (Tx_s)(\omega)\big) ,$$

where $y_n(\omega)$ is the random solution of (2) and $Y_n(\omega)$ is the stochastic fundamental matrix solution of (2). If we let

$$Y_n(\omega)Y_{s+1}^{-1}(\omega) = k_{n,s}(\omega) ,$$

then the equation (13) reduces to

$$(14)\qquad x_n(\omega) = y_n(\omega) + \sum_{s=0}^{n-1} k_{n,s}(\omega) f_s\big(\omega,\ x_s(\omega),\ (Tx_s)(\omega)\big)\ .$$

The stochastic equation (14) is a generalization of the recent study of Tsokos and Padgett [7, Chapter V] in which the equation (14) with $f_s\big(\omega,\ x_s(\omega),\ (Tx_s)(\omega)\big) = f_s\big(x_s(\omega)\big)$ is studied with respect to the existence of a unique random solution.

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Department of Mathematics,
Deogiri College,
Aurangabad (Maharashtra), India.

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