A NOTE ON ANNIHILATOR RELATIONS

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In a Frobenius algebra A over a field K, there exists a linear function λ of A into K which does not map any proper ideal of A onto 0.[1] Then the map $\varphi : x \rightarrow x^{\varphi}$, where

$$\lambda(xy) = \lambda(yx^{\varphi}) \qquad \text{for all } y \in A,$$

defines an automorphism φ of A onto itself. This automorphism is called Nakayama's automorphism. Now the following result is well known.

THEOREM 1.[1] *For any two-sided ideal $\mathfrak{z}$ of a Frobenius algebra A, we have*

$$r(\mathfrak{z}) = l(\mathfrak{z})^{\varphi} = l(\mathfrak{z}^{\varphi}),$$

where $r(\mathfrak{z}) = \{x \mid \mathfrak{z}x = 0\}$ *and* $l(\mathfrak{z}) = \{x \mid x\mathfrak{z} = 0\}$.

This result is written as follows:

$$r^2(\mathfrak{z}) = \mathfrak{z}^{\varphi}, \qquad l^2(\mathfrak{z}) = \mathfrak{z}^{\varphi^{-1}}.$$

Therefore we have

COROLLARY. *For any two-sided ideals $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ of a Frobenius algebra A, we have*

$$l^2(\mathfrak{a}_1 \mathfrak{a}_2 \cdots \mathfrak{a}_n) = l^2(\mathfrak{a}_1)\, l^2(\mathfrak{a}_2) \cdots l^2(\mathfrak{a}_n)$$

and

$$r^2(\mathfrak{a}_1 \mathfrak{a}_2 \cdots \mathfrak{a}_n) = r^2(\mathfrak{a}_1)\, r^2(\mathfrak{a}^2) \cdots r_2(\mathfrak{a}_n).$$

Our aim, in this note, is to analyse the above relation of annihilators.

THEOREM 2.[2] *Let A be a ring and X_i, Y_i $(i = 1, \ldots, n)$ the sets of A satisfying the following relations*

$$r(X_i) \subseteq l(Y_i) \qquad (i = 1, \ldots, n).$$

Then we have

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1) T. Nakayama, On Frobeniusean algebras II, Ann. of Math., **42** (1941), pp. 1–21.

2) This formulation of theorem is due to T. Nakayama. The writer's original theorem was more special.

$$r(X_1 \cdots X_n) \subseteq l(Y_1 \cdots Y_n).$$[3]

The proof of this theorem is as follows:

$$\begin{aligned} x \in r(X_1 \cdots X_n) &\Longleftrightarrow (X_1 \cdots X_n)x = 0 \\ \Longrightarrow (X_2 \cdots X_n)x &\subseteq r(X_1) \subseteq l(Y_1) \\ \Longrightarrow (X_2 \cdots X_n)xY_1 &= 0 \Longrightarrow \cdots\cdots\cdots \\ \Longrightarrow x(Y_1 \cdots Y_n) &= 0 \Longleftrightarrow x \in l(Y_1 \cdots Y_n). \end{aligned}$$

From this fundamental theorem, we deduce directly

COROLLARY 1. *If the sets* $X_1, \ldots, X_n$ *of a ring* A *satisfy the following relations*

$$r(l(X_i)) \subseteq l(r(X_i)) \qquad \textit{for } i = 1, \ldots, n,$$

then we have

$$r(l(X_1) \cdots l(X_n)) \subseteq l(r(X_1) \cdots r(X_n)). \tag{1}$$

In particular, if there holds $r(l(X_i)) = l(r(X_i))$ *for each* i, *then we have* $r(l(X_1) \cdots l(X_n)) = l(r(X_1) \cdots r(X_n))$. *Therefore*

$$l(r(l(X_1) \cdots l(X_n))) = l^2(r(X_1) \cdots r(X_n)). \tag{2}$$

COROLLARY 2. *Let* A *be a ring satisfying the annihilator relation* $r(l(\mathfrak{a})) = \mathfrak{a}$ *for all two-sided ideals* $\mathfrak{a}$ *in* A. *Then we have*

$$r(l(\mathfrak{a}_1) \cdots l(\mathfrak{a}_n)) \subseteq l(r(\mathfrak{a}_1) \cdots r(\mathfrak{a}_n)) \tag{3}$$

for any two-sided ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ *of* A. *Further if there hold* $r(l(\mathfrak{a})) = \mathfrak{a} = l(r(\mathfrak{a}))$[4] *for all two-sided ideals* $\mathfrak{a}$ *in* A *we have*

$$l^2(\mathfrak{a}_1) \cdots l^2(\mathfrak{a}_n) = l^2(\mathfrak{a}_1 \cdots \mathfrak{a}_n) \tag{4}$$

and

$$r^2(\mathfrak{a}_1) \cdots r^2(\mathfrak{a}_n) = r^2(\mathfrak{a}_1 \cdots \mathfrak{a}_n). \tag{5}$$

Proof. Since we have $l(r(\mathfrak{a})) \supseteq \mathfrak{a}$, for any ideal $\mathfrak{a}$ of A, we deduce (3) from (1). The relation (4) follows from (2) if we put $\mathfrak{a}_i = r(\mathfrak{b}_i)$ for suitable ideals $\mathfrak{b}_i$. Similarly we have (5).

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3) This theorem is valid if A is a semi-group with zero.

4) It is well known that this relation holds in a quasi-Frobenius ring.