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SHARP CONSTANTS BETWEEN EQUIVALENT NORMS IN WEIGHTED LORENTZ SPACES

SORINA BARZA and JAVIER SORIA ✉

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Abstract

For an increasing weight w in B_p (or equivalently in A_p), we find the best constants for the inequalities relating the standard norm in the weighted Lorentz space $\Lambda^p(w)$ and the dual norm.

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1. Introduction

Let $1 < p < \infty$. Throughout this paper (R, μ) denotes a σ-finite nonatomic measure space, and w a nonnegative and locally integrable function on $\mathbb{R}_+$. We recall the definition of the weighted Lorentz space $\Lambda^p(w)$ (see [3]):

$$\Lambda^p(w) = \left\{ f : \|f\|_{p,w} = \left(\int_0^\infty f^*(t)^p w(t)\, dt \right)^{1/p} < \infty \right\},$$

where f^* is the nonincreasing rearrangement of f on (R, μ). It is known that $\|\cdot\|_{p,w}$ is a norm if and only if w is a nonincreasing function [12, 13]. Also, it was proved in [15] that $\Lambda^p(w)$ is equivalently normable if and only if $w \in B_p$, which is a condition that also characterizes the boundedness of the Hardy operator on the cone of $L^p(w)$ decreasing functions (see [1]):

$$\int_t^\infty \frac{w(x)}{x^p}\, dx \le \frac{C}{t^p} \int_0^t w(x)\, dx. \tag{1.1}$$

In this case, the usual equivalent norm is given in terms of the maximal function (see [6] for more information):

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(x)\, dx.$$

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If we set

$$\|f\|_{p,w}^* = \left(\int_0^\infty f^{**}(t)^p w(t)\, dt\right)^{1/p},$$

then for $w \in B_p$ (see [4]),

$$\|f\|_{p,w} \le C_1 \|f\|_{p,w}^* \quad \text{and} \quad \|f\|_{p,w}^* \le C_2 \|f\|_{p,w}, \tag{1.2}$$

where, writing $W(r) = \int_0^r w(t)\, dt$,

$$C_1^p = \sup_{t>0} \frac{W(t)}{W(t) + t^p \int_t^\infty x^{-p} w(x)\, dx}$$

and

$$C_2 = \int_0^1 \left(\sup_{r>0} \frac{W(r/x)}{W(r)}\right)^{1/p} dx.$$

The constant C_1 is sharp (for example, if $w(t) = t^{s/p-1}$ and hence $\Lambda^s(w) = L^{p,s}$, then $C_1 = (p')^{-1/s} < 1$, where $1/p + 1/p' = 1$). Also, for power weights $w(t) = t^\alpha$, when $-1 < \alpha < p - 1$, we have that $C_2 = (p/(\alpha + 1))'$ is also sharp (this follows by Hardy's inequalities).

In a recent paper [2], the authors considered, for the case of an increasing power weight (that is, for the classical $L^{p,s}$ spaces with $p < s$), other types of estimates with respect to the so-called dual and decomposition norms, and they also found the best constants for the equivalence of these norms (the case $s \le p$ is trivial since both norms coincide with the Lorentz norm). In this work we will extend this result in the case of the dual norm, for more general increasing weights. As we mentioned above, since we need the space $\Lambda^p(w)$ to be normable, we will necessarily have to assume that $w \in B_p$ (see (1.1)).

So, from now on, we will consider w to be an increasing weight in B_p, $1 < p < \infty$. Hence $\|\cdot\|_{p,w}$ is not a norm but it is equivalent to a norm (see (1.2)).

Denote $\widetilde{w} = w^{1-p'}$ (observe that $\widetilde{w}$ is a decreasing weight). We consider the dual norm, defined in terms of Köthe duality, as follows:

$$\|f\|'_{p,w} = \sup\left\{\int_R fg\, d\mu : \|g\|_{p',\widetilde{w}} = 1\right\}.$$

It is easy to see that $\|\cdot\|'_{p,w}$ is a norm, equivalent to $\|\cdot\|_{p,w}$. Observe that, since w is increasing, we have $W(t)/t \le w(t)$ and hence w is a regular weight (see [5, Definition 2.4.11]). Therefore, using [5, Theorem 2.4.12] (see also [15]), we get

$$[\Lambda^p(w)]' = \Lambda^{p'}(\widetilde{w}).$$

Our main result (Theorem 2.9) shows that the best constant for the equivalence between $\|\cdot\|_{p,w}$ and $\|\cdot\|'_{p,w}$ depends, in fact, on the A_p norm of w. Recall that A_p is the classical Muckenhoupt class associated to the boundedness of the Hardy–Littlewood maximal function (see [14] and Definition 2.7). This condition appears

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naturally when dealing with level functions (see Theorem 2.6), which is one of the main tools used in the proof of Theorem 2.9. The key argument to prove sharpness of the constant is Proposition 2.8.

2. Main result

We begin with a few lemmas.

LEMMA 2.1. *Let* $1 < p < \infty$*, and let* w *be any weight and* $f \in \Lambda^p(w)$*. Then*

$$\|f\|'_{p,w} = \|f^*\|'_{p,w}.$$

PROOF. For the proof, see [3, pp. 45–49]. □

LEMMA 2.2. *Let* $0 < p \le q < \infty$ *and let* v, w *be general weights. Then the inequality*

$$\left(\int_0^\infty f^q(t)w(t)\,dt\right)^{1/q} \le C\left(\int_0^\infty f^p(t)v(t)\,dt\right)^{1/p} \tag{2.1}$$

holds for all positive decreasing functions f *if and only if*

$$\sup_{t>0}\left(\int_0^t w(s)\,ds\right)^{1/q}\left(\int_0^t v(s)\,ds\right)^{-1/p} < \infty$$

and

$$C = \sup_{t>0}\left(\int_0^t w(s)\,ds\right)^{1/q}\left(\int_0^t v(s)\,ds\right)^{-1/p}$$

is the best constant in (2.1).

PROOF. For the proof see, for example, [4, p. 408]. □

The following corollary will be useful to find the dual norm of the characteristic function of an interval.

COROLLARY 2.3. *Let* u *be a decreasing weight and let* $r > 0$*. Then, the inequality*

$$\int_0^r f(t)\,dt \le C\int_0^r f(t)u(t)\,dt \tag{2.2}$$

holds for all positive and decreasing functions f*, and*

$$C = \frac{r}{\int_0^r u(s)\,ds}$$

is the best constant in (2.2).

PROOF. Take $p = q = 1$ and $w(t) = \chi_{[0,r]}(t)$, $v(t) = \chi_{[0,r]}(t)u(t)$ in (2.1). Observe that $r(\int_0^r u(s)\,ds)^{-1}$ is an increasing function. □

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LEMMA 2.4. *Let $r > 0$, $h(t) = \chi_{[0,r]}(t)$, and $p > 1$. Suppose that w is an increasing weight function. Then*

$$\|h\|'_{p,w} = r\left(\int_0^r \widetilde{w}(t)\,dt\right)^{-1/p'}.$$

PROOF. To evaluate the dual norm of h, we assume that $g \in \Lambda^{p'}(\widetilde{w})$, $g \geq 0$, and $\|g\|_{p',\widetilde{w}} = 1$. Applying Corollary 2.3 (with $u = \widetilde{w}$) and Hölder's inequality we get

$$\begin{aligned}\int_0^r g^*(t)\,dt &\leq r\left(\int_0^r \widetilde{w}(s)\,ds\right)^{-1}\int_0^r g^*(s)\widetilde{w}(s)\,ds \\ &\leq r\left(\int_0^r \widetilde{w}(s)\,ds\right)^{-1}\|g\|_{p',\widetilde{w}}\left(\int_0^r \widetilde{w}(s)\,ds\right)^{1/p} \\ &= r\left(\int_0^r \widetilde{w}(s)\,ds\right)^{-1/p'}.\end{aligned}$$

On the other hand, if

$$g(t) = \frac{\chi_{[0,r]}(t)}{\|\chi_{[0,r]}\|_{p',\widetilde{w}}},$$

then

$$\|\chi_{[0,r]}\|'_{p,w} \geq \int_0^r \frac{\chi_{[0,r]}(t)}{\|\chi_{[0,r]}\|_{p',\widetilde{w}}}\,dt = r\left(\int_0^r \widetilde{w}(t)\,dt\right)^{-1/p'},$$

which completes the proof of the equality. □

LEMMA 2.5. *If f is a monotone function on $[a, b]$ and $[\alpha, \beta] \subset [a, b]$ is an interval for which*

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)\,dt = \frac{1}{b-a}\int_a^b f(t)\,dt,$$

then, for every convex function ϕ,

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(f(t))\,dt \leq \frac{1}{b-a}\int_a^b \phi(f(t))\,dt.$$

PROOF. This result is a corollary of [9, Theorems 249, 250] and can be found in [11, Lemma 6]. □

Given f, the notion of the level function f° was first introduced by Halperin [8], and it is essentially the derivative of the least concave majorant of the primitive of f.

THEOREM 2.6 [8]. *Assume that f is a nonnegative measurable function on $\mathbb{R}_+$ and that*

$$\int_0^t f(u)\,du = o(t) \quad \text{as } t \to \infty.$$

Then there exists a nonnegative function f° on $\mathbb{R}_+$ satisfying the following conditions:

(a) *the function f° decreases on $\mathbb{R}_+$;*
(b) *$\int_0^t f \leq \int_0^t f^\circ$ for all $t > 0$;*

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(c) *up to a set of measure zero, the set* $\{t \in \mathbb{R}_+ : f(t) \neq f^\circ(t)\}$ *is the union of bounded disjoint intervals* I_k *such that*

$$\int_{I_k} f(u)\,du = \int_{I_k} f^\circ(u)\,du,$$

and $f^\circ(t)$ *is constant on* I_k.

DEFINITION 2.7 [14]. Let $p > 1$. A weight w on $\mathbb{R}_+$ is said to belong to the A_p class if

$$\frac{1}{b-a}\int_a^b w(t)\,dt \le C\left(\frac{1}{b-a}\int_a^b w^{1-p'}(t)\,dt\right)^{1-p}$$

for some constant C and any $0 < a < b < \infty$. We also denote by

$$\|w\|_{A_p} = \sup_{0<a<b}\left(\frac{1}{b-a}\int_a^b w(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b w^{1-p'}(t)\,dt\right)^{p-1}$$

the optimal constant in the above inequality.

PROPOSITION 2.8. *Let* w *be a monotone weight in* A_p*, for some* $p > 1$*. Then,*

$$\|w\|_{A_p} = \sup_{0<r}\left(\frac{1}{r}\int_0^r w(t)\,dt\right)\left(\frac{1}{r}\int_0^r w^{1-p'}(t)\,dt\right)^{p-1}.$$

PROOF. It suffices to prove the case when w is decreasing (if w is increasing, we consider $v = \widetilde{w} \in A_{p'}$). Let us show that we can assume, without loss of generality, that w is a strictly decreasing weight: let $w_\varepsilon(x) = (1 + \varepsilon/(1+x))w(x)$. Then $w_\varepsilon(x) > w_\varepsilon(y)$ if $x < y$ (observe that necessarily $w(x) > 0$, since A_p weights satisfy the doubling property and by monotonicity). Also, if we call

$$\|w\|_{A_{p,0}} = \sup_{0<r}\left(\frac{1}{r}\int_0^r w(t)\,dt\right)\left(\frac{1}{r}\int_0^r w^{1-p'}(t)\,dt\right)^{p-1},$$

then it is easy to prove that $\|w_\varepsilon\|_{A_{p,0}} \to \|w\|_{A_{p,0}}$ and $\|w_\varepsilon\|_{A_p} \to \|w\|_{A_p}$, as $\varepsilon \to 0$. Thus, assume that $w \in A_p$ is strictly decreasing. It is clear that $\|w\|_{A_{p,0}} \le \|w\|_{A_p}$, so let us prove the converse inequality. Fix $0 < a < b < \infty$. Set

$$w_0 = \lim_{r\to 0}\frac{1}{r}\int_0^r w(t)\,dt \quad \text{and} \quad w_\infty = \lim_{r\to\infty}\frac{1}{r}\int_0^r w(t)\,dt.$$

Then

$$w_0 = \lim_{t\to 0} w(t) \quad \text{and} \quad w_\infty = \lim_{t\to\infty} w(t),$$

and, since

$$w_0 > \frac{1}{b-a}\int_a^b w(t)\,dt > w_\infty,$$

there exists a unique $r > 0$ such that

$$\frac{1}{r}\int_0^r w(t)\,dt = \frac{1}{b-a}\int_a^b w(t)\,dt.$$

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Let us show that $r \geq b$. Assume, for contradiction, that $r < b$. Then, since w is decreasing,

$$\frac{1}{b-a}\int_a^b w(t)\,dt = \frac{1}{r}\int_0^r w(t)\,dt > \frac{1}{b}\int_0^b w(t)\,dt = \frac{1}{b}\biggl(\int_0^a w(t)\,dt + \int_a^b w(t)\,dt\biggr),$$

and hence we reach the contradiction:

$$w(a) > \frac{1}{b-a}\int_a^b w(t)\,dt > \frac{1}{a}\int_0^a w(t)\,dt > w(a).$$

Therefore, since $[a, b] \subset [0, r]$, we can use Lemma 2.5 with $\phi(t) = t^{1-p'}$ to obtain

$$\frac{1}{b-a}\int_a^b w^{1-p'}(t)\,dt \leq \frac{1}{r}\int_0^r w^{1-p'}(t)\,dt,$$

which finally gives us the estimate:

$$\biggl(\frac{1}{b-a}\int_a^b w(t)\,dt\biggr)\biggl(\frac{1}{b-a}\int_a^b w^{1-p'}(t)\,dt\biggr)^{p-1} \leq \biggl(\frac{1}{r}\int_0^r w(t)\,dt\biggr)\biggl(\frac{1}{r}\int_0^r w^{1-p'}(t)\,dt\biggr)^{p-1} \leq \|w\|_{A_{p,0}},$$

that is, $\|w\|_{A_p} \leq \|w\|_{A_{p,0}}$. □

We now find the optimal constants for the inequalities relating both the Lorentz and the dual norms. We observe that the A_p condition, for monotone weights, is equivalent to $w \in B_p$ (see [7]), and hence there is no loss of generality in assuming that $w \in A_p$. Observe also that the only increasing $w \in B_p$ for which $\|\cdot\|_{p,w}$ is a norm is the constant weight $w \equiv C$.

THEOREM 2.9. *Let $p > 1$ and $w \in A_p$ be an increasing weight. Suppose that $f \in \Lambda^p(w)$. Then*

$$\|f\|'_{p,w} \leq \|f\|_{p,w} \leq C_{p,w}\|f\|'_{p,w} \tag{2.3}$$

where

$$C_{p,w} = \|w\|_{A_p}^{1/p}.$$

The constants in the inequalities (2.3) are optimal.

PROOF. The left-hand-side inequality in (2.3) is just a consequence of Hölder's inequality and the Hardy–Littlewood rearrangement inequality. We have equality for functions f such that $\psi(t) = f^*(t)^{p-1}w(t)$ is nonincreasing (for example, $f^*(t) = w^{1-p'}(t)\chi_{[0,1]}(t)$).

By Lemma 2.1, it is enough to prove the right-hand-side inequality in (2.3) for positive nonincreasing functions f. Denote

$$\psi(t) = f(t)^{p-1}w(t)$$

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and take $\psi^\circ(t)$ to be the level function of ψ. Applying Theorem 2.6(b) and the Hardy–Littlewood–Pólya relation (see Hardy's lemma in [3, Proposition 3.6]), we obtain

$$\begin{aligned}\|f\|_{p,w}^p &= \int_0^\infty f(t)\psi(t)\,dt \le \int_0^\infty f(t)\psi^\circ(t)\,dt \\ &\le \|\psi^\circ\|_{p',\widetilde{w}}\|f\|'_{p,w}.\end{aligned}$$

Hence, to show the right-hand-side inequality it suffices to prove that

$$\|\psi^\circ\|_{p',\widetilde{w}} \le C_{p,w}\|f\|_{p,w}^{p-1}.$$

Let $E = \{t \in \mathbb{R}_+ \mid \psi^\circ(t) = \psi(t)\}$. Then, up to a set of measure zero,

$$\mathbb{R}_+ \backslash E = \bigcup_k (a_k, b_k),$$

where (a_k, b_k) are bounded disjoint intervals such that

$$\psi^\circ(t) = \frac{1}{b_k - a_k}\int_{a_k}^{b_k} \psi(u)\,du \quad \forall t \in (a_k, b_k).$$

By Hölder's inequality, the definition of ψ and the hypothesis, we obtain

$$\begin{aligned}\psi^\circ(t) &\le \frac{1}{b_k - a_k}\left(\int_{a_k}^{b_k} w(u)\,du\right)^{1/p}\left(\int_{a_k}^{b_k} f^p(u)w(u)\,du\right)^{1/p'} \\ &\le \|w\|_{A_p}^{1/p}\left(\int_{a_k}^{b_k} \widetilde{w}(u)\,du\right)^{-1/p'}\left(\int_{a_k}^{b_k} f^p(u)w(u)\,du\right)^{1/p'}\end{aligned}$$

for all $t \in (a_k, b_k)$. Thus

$$\int_{a_k}^{b_k} \psi^\circ(t)^{p'}\widetilde{w}(t)\,dt \le \|w\|_{A_p}^{p'/p}\int_{a_k}^{b_k} f^p(t)w(t)\,dt.$$

On the set E, we also have

$$\begin{aligned}\int_E \psi^\circ(t)^{p'}\widetilde{w}(t)\,dt &= \int_E \psi(t)^{p'}\widetilde{w}(t)\,dt \\ &= \int_E f^p(t)w(t)\,dt.\end{aligned}$$

Since $\|w\|_{A_p} \ge 1$,

$$\begin{aligned}\|f\|_{p,w}^p &\le \|\psi^\circ\|_{p',\widetilde{w}}\|f\|'_{p,w} \\ &\le \|w\|_{A_p}^{1/p}\left(\sum_k \int_{a_k}^{b_k} f^p(t)w(t)\,dt + \int_E f^p(t)w(t)\,dt\right)^{1/p'}\|f\|'_{p,w} \\ &= \|w\|_{A_p}^{1/p}\|f\|_{p,w}^{p-1}\|f\|'_{p,w}\end{aligned}$$

and we obtain the right-hand side of (2.3), when $1 < p < \infty$.

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Taking $f = \chi_{[0,r]}$ and using Lemma 2.4, we see that

$$C_{p,w} \geq \frac{\|\chi_{[0,r]}\|_{p,w}}{\|\chi_{[0,r]}\|'_{p,w}} = \frac{1}{r}\left(\int_0^r w(t)\,dt\right)^{1/p}\left(\int_0^r \widetilde{w}(t)\,dt\right)^{1/p'}$$

and hence, using Proposition 2.8,

$$C_{p,w} \geq \|w\|_{A_{p,0}}^{1/p} = \|w\|_{A_p}^{1/p},$$

which shows that $\|w\|_{A_p}^{1/p}$ is sharp. □

REMARK 2.10. If $w(t) = t^{s/p-1}$, then $\Lambda^s(w) = L^{p,s}$. Hence, if $1 < p < s < \infty$, then w is an increasing weight and

$$\|w\|_{A_s} = \frac{p}{s}\left(\frac{p'}{s'}\right)^{s-1}.$$

Therefore,

$$\|f\|_{p,s} \leq \left(\frac{p}{s}\right)^{1/s}\left(\frac{p'}{s'}\right)^{1/s'}\|f\|'_{p,s},$$

which is [2, Theorem 4.4].

REMARK 2.11. An interesting application of Theorem 2.9 is the following uniform estimate in the triangle inequality: if $p > 1$ and $w \in A_p$ is an increasing weight, then

$$\left\|\sum_{j=1}^N f_j\right\|_{p,w} \leq \|w\|_{A_p}^{1/p}\sum_{j=1}^N \|f_j\|_{p,w}.$$

In fact, if $f = \sum_{j=1}^N f_j$, then

$$\begin{aligned}\|f\|_{p,w} &\leq \|w\|_{A_p}^{1/p}\|f\|'_{p,w} \leq \|w\|_{A_p}^{1/p}\sum_{j=1}^N \|f_j\|'_{p,w}\\ &\leq \|w\|_{A_p}^{1/p}\sum_{j=1}^N \|f_j\|_{p,w}.\end{aligned}$$

Using Proposition 2.8 and [10, Theorem 3], one can prove that this estimate is optimal.

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SORINA BARZA, Department of Mathematics, Karlstad University,
SE-65188 Karlstad, Sweden
e-mail: sorina.barza@kau.se

JAVIER SORIA, Department of Applied Mathematics and Analysis,
University of Barcelona, E-08007 Barcelona, Spain
e-mail: soria@ub.edu

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