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UNIVALENT AND STARLIKE GENERALIZED HYPERGEOMETRIC FUNCTIONS

SHIGEYOSHI OWA AND H. M. SRIVASTAVA

1. Introduction and definitions. A single-valued function $f(z)$ is said to be *univalent* in a domain $\mathscr{D}$ if it never takes on the same value twice, that is, if $f(z_1) = f(z_2)$ for $z_1, z_2 \in \mathscr{D}$ implies that $z_1 = z_2$. A set $\mathscr{E}$ is said to be *starlike* with respect to $w_0 \in \mathscr{E}$ if the line segment joining w_0 to every other point $w \in \mathscr{E}$ lies entirely in $\mathscr{E}$. If a function $f(z)$ maps $\mathscr{D}$ onto a domain that is starlike with respect to w_0, then $f(z)$ is said to be starlike with respect to w_0. In particular, if w_0 is the origin, then we say that $f(z)$ is a starlike function. Further, a set $\mathscr{E}$ is said to be *convex* if the line segment joining any two points of $\mathscr{E}$ lies entirely in $\mathscr{E}$. If a function $f(z)$ maps $\mathscr{D}$ onto a convex domain, then we say that $f(z)$ is a *convex function* in $\mathscr{D}$.

Let $\mathscr{A}$ denote the class of functions of the form

$$(1.1)\qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $\mathscr{U} = \{z:|z| < 1\}$. Further, let $\mathscr{S}$ denote the class of all functions in $\mathscr{A}$ which are univalent in the unit disk $\mathscr{U}$. Then a function $f(z)$ belonging to $\mathscr{S}$ is said to be *starlike of order* $\alpha (0 \leqq \alpha < 1)$ if and only if (cf. **[1]**, **[3]**, and **[9]**)

$$(1.2)\qquad \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathscr{U})$$

for $0 \leqq \alpha < 1$. We denote by $\mathscr{S}^*(\alpha)$ the class of all functions in $\mathscr{S}$ which are starlike of order α. Throughout this paper, it should be understood that functions such as $zf'(z)/f(z)$, which have removable singularities at $z = 0$, have had these singularities removed in statements like (1.2).

A function $f(z)$ belonging to $\mathscr{S}$ is said to be *convex of order* α $(0 \leqq \alpha < 1)$ if and only if

$$(1.3)\qquad \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathscr{U})$$

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for $0 \leqq \alpha < 1$. We denote by $\mathscr{K}(\alpha)$ the class of all functions in $\mathscr{S}$ which are convex of order α.

We note that $f(z) \in \mathscr{K}(\alpha)$ if and only if $zf'(z) \in \mathscr{S}^*(\alpha)$. We also have

$$(1.4) \qquad \mathscr{S}^*(\alpha) \subseteq \mathscr{S}^*(0) \equiv \mathscr{S}^*, \mathscr{K}(\alpha) \subseteq \mathscr{K}(0) \equiv \mathscr{K}, \text{ and } \mathscr{K}(\alpha) \subset \mathscr{S}^*(\alpha) \subset \mathscr{S}$$

for $0 \leqq \alpha < 1$.

The classes $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$ were first introduced by Robertson [**9**], and were studied subsequently by Schild [**13**], MacGregor [**5**], Pinchuk [**8**], Jack [**3**], and others.

Finally, let a_j ($j = 1, \ldots, p$) and b_j ($j = 1, \ldots, q$) be complex numbers with

$$b_j \neq 0, -1, -2, \ldots; j = 1, \ldots, q.$$

Then the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$$(1.5) \qquad {}_pF_q(z) \equiv {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!} \quad (p \leqq q + 1),$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(1.6) \qquad (\lambda)_n = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda + 1) \ldots (\lambda + n - 1), & \text{if } n \in \mathscr{N} = \{1, 2, 3, \ldots\}. \end{cases}$$

We note that the ${}_pF_q(z)$ series in (1.5) coverges absolutely for $|z| < \infty$ if $p < q + 1$, and for $z \in \mathscr{U}$ if $p = q + 1$. The condition $p \leqq q + 1$ stated with the definition (1.5) will be assumed to hold true throughout this paper.

Merkes and Scott [**6**] proved a result involving starlike hypergeometric functions. More recently, Carlson and Shaffer [**1**] presented a study of certain interesting classes of starlike, convex, and prestarlike hypergeometric functions by applying a linear operator defined by a certain convolution. In the present paper, we prove several interesting results concerning univalent generalized hypergeometric functions, starlike generalized hypergeometric functions of order α, and convex generalized hypergeometric functions of order α. Furthermore, by making use of a certain linear operator involving fractional calculus defined by Equation (5.14) below, we establish several general characterization theorems in terms of fractional calculus of functions $f(z)$ belonging to some of the classes of analytic functions defined above.

2. Univalent generalized hypergeometric functions. A function $f(z)$ belonging to the class $\mathscr{A}$ is said to be *close-to-convex* if there is a convex function $g(z)$ such that

$$(2.1)\qquad \operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0 \quad (z \in \mathscr{U}).$$

We note that $f(z)$ is not required *a priori* to be univalent (cf. Lemma 2 below), and the associated function $g(z)$ need not be a function belonging to the class $\mathscr{A}$. In the sequel we shall require the following lemmas due to Jack [**3**] and Duren [**2**], respectively.

LEMMA 1. *Let $w(z)$ be regular in the unit disk $\mathscr{U}$, with $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r$ $(0 \leqq r < 1)$ at a point z_1, we can write*

$$(2.2)\qquad z_1 w'(z_1) = m w(z_1),$$

where m is real and $m \geqq 1$.

LEMMA 2. *Every close-to-convex function is univalent.*

We now prove our first result on univalent generalized hypergeometric functions, contained in

THEOREM 1. *Let the generalized hypergeometric function ${}_pF_q(z)$ defined by* (1.5) *satisfy the condition*

$$(2.3)\qquad \left| {}_pF_q'(a_1, \ldots, a_p; b_1, \ldots, b_q; z) - \frac{\prod_{j=1}^{p} a_j}{\prod_{j=1}^{q} b_j} \right|^{1-\beta}$$

$$\times \left| \frac{z\, {}_pF_q''(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}{{}_pF_q'(a_1, \ldots, a_p; b_1, \ldots, b_q; z)} \right|^{\beta} < \left(\frac{\prod_{j=1}^{p} a_j}{\prod_{j=1}^{q} b_j} \right)^{1-\beta} \left(\frac{1}{2}\right)^{\beta}$$

for some fixed $\beta \geqq 0$ and for all $z \in \mathscr{U}$, where

$$(2.4)\qquad \frac{\prod_{j=1}^{p} a_j}{\prod_{j=1}^{q} b_j} > 0.$$

Then ${}_pF_q(z)$ is univalent in the unit disk $\mathscr{U}$.

Remark 1. In Theorem 1 and elsewhere in this paper, α, β, a_j ($j = 1, \ldots, p$) and b_j ($j = 1, \ldots, q$) are thought of as being fixed and z varies over the unit disk $\mathcal{U}$.

Proof. For the function $H(z)$ defined by

$$(2.5)\qquad H(z) = \frac{\prod_{j=1}^{q} b_j}{\prod_{j=1}^{p} a_j}\{{}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) - 1\}, \quad z \in \mathcal{U},$$

the condition (2.3) implies

$$(2.6)\qquad |H'(z) - 1|^{1-\beta}\left|\frac{zH''(z)}{H'(z)}\right|^{\beta} < \left(\frac{1}{2}\right)^{\beta}.$$

Further, it is clear that $H(z) \in \mathcal{A}$.

Now define the function $w(z)$ by

$$(2.7)\qquad w(z) = H'(z) - 1$$

for $z \in \mathcal{U}$. Then it follows that $w(z)$ is analytic in the unit disk $\mathcal{U}$ with

$$w(0) = 0.$$

Substituting for $H(z)$ into the left-hand side of (2.6) from (2.7), we get

$$(2.8)\qquad |w(z)|^{1-\beta}\left|\frac{zw'(z)}{1 + w(z)}\right|^{\beta} < \left(\frac{1}{2}\right)^{\beta},$$

that is,

$$(2.9)\qquad |w(z)| \cdot \left|\frac{zw'(z)}{w(z)} \cdot \frac{1}{1 + w(z)}\right|^{\beta} < \left(\frac{1}{2}\right)^{\beta},$$

where the comment about removable singularities applies just as in (1.2). Assume that there exists a point $z_1 \in \mathcal{U}$ such that

$$(2.10)\qquad \max_{|z| \leqq |z_1|} |w(z)| = |w(z_1)| = 1.$$

Then we can put

$$z_1\frac{w'(z_1)}{w(z_1)} = m \geqq 1$$

by means of Lemma 1. Therefore, we obtain

$$(2.11)\qquad |w(z_1)| \cdot \left|\frac{z_1w'(z_1)}{w(z_1)} \cdot \frac{1}{1 + w(z_1)}\right|^{\beta} \geqq \left(\frac{m}{2}\right)^{\beta} \geqq \left(\frac{1}{2}\right)^{\beta},$$

which contradicts the condition (2.9), and so also (2.3). This shows that

$$(2.12)\quad |w(z)| = |H'(z) - 1| < 1,$$

which implies that $\mathrm{Re}(H'(z)) > 0$ for $z \in \mathscr{U}$. Note that $g(z) = z$ is convex in the unit disk $\mathscr{U}$. For such $g(z)$, $H(z)$ satisfies

$$(2.13)\quad \mathrm{Re}\left(\frac{H'(z)}{g'(z)}\right) > 0 \quad (z \in \mathscr{U}).$$

Consequently, we get

$$(2.14)\quad \mathrm{Re}\left(\frac{{}_pF_q'(a_1, \dots, a_p; b_1, \dots, b_q; z)}{g'(z)}\right) > 0 \quad (z \in \mathscr{U}),$$

provided that the inequality (2.4) holds true. The inequality (2.14) implies that the generalized hypergeometric function ${}_pF_q(z) - 1$ is close-to-convex in the unit disk $\mathscr{U}$. Thus we have the theorem by virtue of Lemma 2.

COROLLARY 1. *Let the generalized hypergeometric function ${}_pF_q(z)$ defined by* (1.5) *satisfy the condition*

$$(2.15)\quad \left| {}_pF_q'(a_1, \dots, a_p; b_1, \dots, b_q; z) - \frac{\prod_{j=1}^{p} a_j}{\prod_{j=1}^{q} b_j} \right| < \frac{\prod_{j=1}^{p} a_j}{\prod_{j=1}^{q} b_j}$$

for $z \in \mathscr{U}$, where (2.4) *holds true.*
Then ${}_pF_q(z)$ is univalent in the unit disk $\mathscr{U}$.

Proof. Corollary 1 follows immediately from Theorem 1 when we set $\beta = 0$.

COROLLARY 2. *Let the generalized hypergeometric function ${}_pF_q(z)$ defined by* (1.5) *satisfy the condition*

$$(2.16)\quad \left| \frac{z\, {}_pF_q''(a_1, \dots, a_p; b_1, \dots, b_q; z)}{{}_pF_q'(a_1, \dots, a_p; b_1, \dots, b_q; z)} \right| < \frac{1}{2}$$

for $z \in \mathscr{U}$, where (2.4) *holds true.*
Then ${}_pF_q(z)$ is univalent in the unit disk $\mathscr{U}$.

Proof. Taking $\beta = 1$ in Theorem 1, we readily have Corollary 2.

3. Starlike generalized hypergeometric functions of order α. We begin by applying Lemma 1 to prove

LEMMA 3. *Let the function $f(z)$ defined by* (1.1) *satisfy the condition*

$$(3.1)\qquad \left|\frac{zf'(z)}{f(z)} - 1\right|^{1-\beta}\left|\frac{zf''(z)}{f'(z)}\right|^{\beta} < (1-\alpha)^{1-2\beta}\left(1 - \frac{3}{2}\alpha + \alpha^2\right)^{\beta}$$

for some fixed α *and* β ($\beta \geqq 0$; $0 \leqq \alpha \leqq 1/2$), *and for all* $z \in \mathcal{U}$.

Then $f(z)$ *is in the class* $\mathcal{S}^*(\alpha)$.

Proof. It suffices to show that

$$\operatorname{Re}(zf'(z)/f(z)) > \alpha$$

under the condition (3.1). Define a function $w(z)$ by

$$(3.2)\qquad \frac{zf'(z)}{f(z)} = \frac{1-(1-2\alpha)w(z)}{1+w(z)}$$

for $0 \leqq \alpha \leqq 1/2$ and $z \in \mathcal{U}$. Then it is clear that $w(0) = 0$. Differentiating both sides of (3.2) logarithmically, we obtain

$$(3.3)\qquad \frac{zf''(z)}{f'(z)} = -\frac{2(1-\alpha)w(z)}{1+w(z)}\left[1 + \frac{zw'(z)}{\{1-(1-2\alpha)w(z)\}w(z)}\right],$$

whence

$$(3.4)\qquad \left|\frac{zf'(z)}{f(z)} - 1\right|^{1-\beta}\left|\frac{zf''(z)}{f'(z)}\right|^{\beta}$$
$$= \left|\frac{2(1-\alpha)w(z)}{1+w(z)}\right| \cdot \left|1 + \left(\frac{zw'(z)}{w(z)}\right)\left(\frac{1}{1-(1-2\alpha)w(z)}\right)\right|^{\beta}.$$

It should be observed that

$$(3.5)\qquad \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \Leftrightarrow |w(z)| < 1 \quad (z \in \mathcal{U})$$

and, in particular, that the inequality involving α holds true at $z = 0$, since $w(0) = 0$. If

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \alpha \text{ for } z = z_1 \in \mathcal{U}$$

and

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \text{ for } |z| < |z_1|,$$

then we have

$$|w(z)| < |w(z_1)| = 1 \quad \text{for } |z| < |z_1|$$

and, in view of the definition (3.2) above, $w(z_1) \neq -1$.

Now, applying Lemma 1 to $w(z)$ at $z = z_1 \in \mathcal{U}$ and putting

$$z_1 w'(z_1) = m w(z_1) \quad (m \text{ real and } m \geqq 1),$$

we have

$$(3.6) \quad \left|\frac{z_1 f'(z_1)}{f(z_1)} - 1\right|^{1-\beta} \left|\frac{z_1 f''(z_1)}{f'(z_1)}\right|^{\beta} \geqq (1-\alpha)\left\{1 + \frac{m\alpha}{2(1-\alpha)^2}\right\}^{\beta}$$
$$\geqq (1-\alpha)^{1-2\beta}\left(1 - \frac{3}{2}\alpha + \alpha^2\right)^{\beta}$$

from (3.4). The restriction $0 \leqq \alpha \leqq 1/2$ is dictated by the definition (1.2) and by the fact that the condition $1 - 2\alpha \geqq 0$ is required in our transition from (3.4) to (3.6).

The inequality in (3.6) contradicts the condition (3.1). Thus we can conclude that

$$\operatorname{Re}(zf'(z)/f(z)) > \alpha \quad \text{for } z \in \mathscr{U},$$

that is, that $f(z) \in \mathscr{S}^*(\alpha)$, and the proof of Lemma 3 is completed.

By applying Lemma 3, we next prove the following theorems involving starlike generalized hypergeometric functions of order α.

THEOREM 2. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (1.5) *satisfy the condition*

$$(3.7) \quad \left|\frac{z\, {}_pF_q'(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}{{}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}\right| < 1 - \alpha \quad (z \in \mathscr{U})$$

for $0 \leqq \alpha \leqq 1/2$.

Then the function $z\, {}_pF_q(z)$ *is in the class* $\mathscr{S}^*(\alpha)$.

Proof. Define the function $G(z)$ by

$$(3.8) \quad G(z) = z\, {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \quad \text{for } z \in \mathscr{U}.$$

Then the condition (3.7) becomes

$$(3.9) \quad \left|\frac{zG'(z)}{G(z)} - 1\right| < 1 - \alpha \quad (z \in \mathscr{U}),$$

where the comment about removable singularities applies just as in (1.2). By taking $\beta = 0$ in Lemma 3, we thus conclude from (3.9) that

$$G(z) \in \mathscr{S}^*(\alpha),$$

which proves Theorem 2.

Remark 2. Evidently, since $\mathscr{S}^*(\alpha) \subseteq \mathscr{S}^* \subset \mathscr{S}$, the function $G(z)$ is univalent in $\mathscr{U}$ under the hypotheses of Theorem 2.

COROLLARY 3. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined* *by* (1.5) *satisfy the condition* (3.7).

Then

$$(3.10)\quad \frac{1}{(1+|z|)^{2(1-\alpha)}} \leqq |{}_pF_q(a_1,\ldots,a_p;\, b_1,\ldots,b_q;\, z)| \leqq \frac{1}{(1-|z|)^{2(1-\alpha)}}$$

for $0 \leqq \alpha \leqq 1/2$ *and* $z \in \mathscr{U}$.

The result (3.10) *is sharp for the generalized hypergeometric function* ${}_pF_q(z)$ *for which*

$$(3.11)\quad \left|\frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n}\right| = (2-2\alpha)_n \quad (n \geqq 1).$$

Proof. It is well known from the work of Robertson [**9**, p. 385] that

$$(3.12)\quad \frac{|z|}{(1+|z|)^{2(1-\alpha)}} \leqq |z\,{}_pF_q(a_1,\ldots,a_p;\, b_1,\ldots,b_q;\, z)| \leqq \frac{|z|}{(1-|z|)^{2(1-\alpha)}}$$

for

$$z\,{}_pF_q(a_1,\ldots,a_p;\, b_1,\ldots,b_q;\, z) \in \mathscr{S}^*(\alpha),$$

which implies (3.10). Furthermore, again following Robertson [**9**, p. 385], the sharpness condition (3.11) is readily obtained.

THEOREM 3. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (1.5) *satisfy the condition*

$$(3.13)\quad \left|\frac{z\,{}_pF_q''(a_1,\ldots,a_p;\, b_1,\ldots,b_q;\, z)}{{}_pF_q'(a_1,\ldots,a_p;\, b_1,\ldots,b_q;\, z)}\right| < (1-\alpha)^{-1}\left(1-\frac{3}{2}\alpha+\alpha^2\right) \quad (z \in \mathscr{U})$$

for $0 \leqq \alpha \leqq 1/2$ *and*

$$(3.14)\quad \prod_{j=1}^{p} a_j \neq 0.$$

Then ${}_pF_q(z)$ *is starlike of order* α *with respect to* 1.

Proof. The function $H(z)$ defined by (2.5) is in the class $\mathscr{A}$ and satisfies

$$(3.15) \quad \left|\frac{zH''(z)}{H'(z)}\right| = \left|\frac{z\,{}_pF_q''(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}{{}_pF_q'(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}\right| < (1 - \alpha)^{-1}\left(1 - \frac{3}{2}\alpha + \alpha^2\right)$$

for $z \in \mathscr{U}$. Consequently, we can see (upon setting $\beta = 1$ in Lemma 3) that

$$H(z) \in \mathscr{S}^*(\alpha),$$

that is, that $H(z)$ is starlike of order α with respect to the origin for $0 \leqq \alpha \leqq 1/2$.

Now Theorem 3 follows from the definition (2.5).

Remark 3. Clearly, since $\mathscr{S}^*(\alpha) \subseteq \mathscr{S}^* \subset \mathscr{S}$, the function $H(z)$ defined by (2.16) is univalent in $\mathscr{U}$ under the hypotheses of Theorem 3.

Merkes and Scott [**6**] established a theorem on starlike hypergeometric functions. By using their technique *mutatis mutandis*, we can easily prove

THEOREM 4. *Let* $0 < b \leqq 2$ *and* $b \leqq a < c$. *Also let the hypergeometric function* ${}_2F_1(a, b; c; z)$ *be defined by* (1.5) *with* $p = 2$ *and* $q = 1$.
Then the function $z\,{}_2F_1(a, b; c; z)$ *is in the class* $\mathscr{S}^*(1 - (1/2)b)$.

Remark 4. Since $\mathscr{S}^*(1 - (1/2)b) \subseteq \mathscr{S}^* \subset \mathscr{S}$, $0 < b \leqq 2$, the function $\Lambda(z)$ defined by

$$\Lambda(z) = z\,{}_2F_1(a, b; c; z)$$

is univalent in $\mathscr{U}$ under the hypotheses of Theorem 4.

4. Convex generalized hypergeometric functions of order α. Corresponding to Theorem 2 and Theorem 4, we have the following results (Theorem 5 and Theorem 6) on convex generalized hypergeometric functions of order α.

THEOREM 5. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (1.5) *satisfy the condition* (3.7) *for* $0 \leqq \alpha \leqq 1/2$.

Then the function

$$z\,{}_{p+1}F_{q+1}(a_1, \ldots, a_p, 1; b_1, \ldots, b_q, 2; z)$$

is in the class $\mathscr{K}(\alpha)$.

Proof. First observe that $zf'(z) \in \mathscr{S}^*(\alpha)$ is equivalent to $f(z) \in \mathscr{K}(\alpha)$, and that

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$$z\ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \in \mathcal{S}^*(\alpha)$$

under the condition (3.7), in view of Theorem 2. Therefore, we have

$$\int_0^z {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; t)dt \in \mathcal{K}(\alpha).$$

Further, by a simple computation using (1.5), we obtain

$$\begin{aligned}(4.1)\quad &\int_0^z {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; t)dt \\ &= z\ {}_{p+1}F_{q+1}(a_1, \ldots, a_p, 1; b_1, \ldots, b_q, 2; z),\end{aligned}$$

which completes the proof of Theorem 5.

Remark 5. Since $\mathcal{K}(\alpha) \subseteq \mathcal{K} \subset \mathcal{S}$, the function

$$z\ {}_{p+1}F_{q+1}(a_1, \ldots, a_p, 1; b_1, \ldots, b_q, 2; z)$$

is univalent in $\mathcal{U}$ under the hypotheses of Theorem 5.

THEOREM 6. *Let* $0 < b \leqq 2$ *and* $b \leqq a < c$. *Also let the hypergeometric function* ${}_2F_1(a, b; c; z)$ *be defined by* (1.5) *with* $p = 2$ *and* $q = 1$.
Then the function

$$z\ {}_3F_2(a, b, 1; c, 2; z)$$

is in the class $\mathcal{K}(1 - (1/2)b)$ *for* $z \in \mathcal{U}$.

Proof. The proof of Theorem 6 is much akin to that of Theorem 5. Indeed, instead of Theorem 2, it uses the assertion of Theorem 4.

5. Applications of fractional calculus. Let the functions $f_j(z)$ be defined by

$$(5.1)\quad f_j(z) = \sum_{n=0}^{\infty} a_{j,n+1}z^{n+1}$$

for $j = 1, 2$. Denote by $f_1 * f_2(z)$ the *Hadamard product* or *convolution* of two functions $f_1(z)$ and $f_2(z)$, that is,

$$(5.2)\quad f_1 * f_2(z) = \sum_{n=0}^{\infty} a_{1,n+1}a_{2,n+1}z^{n+1}.$$

Also let the function $\phi(a, c)$ be defined by

$$(5.3)\quad \phi(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (z \in \mathcal{U}),$$

where (as usual) $c \neq 0, -1, -2, \ldots$. The function $\phi(a, c)$ is an incomplete beta function with

$$\phi(a, c; z) = z\, {}_2F_1(1, a; c; z).$$

Corresponding to the function $\phi(a, c)$, define a linear operator on $\mathscr{A}$ by [**1**, p. 738, Equation (2.2)]

(5.4) $\mathscr{L}(a, c)f = \phi(a, c) * f$

for $f(z) \in \mathscr{A}$. Then $\mathscr{L}(a, c)$ maps $\mathscr{A}$ onto itself. Moreover, if $a \neq 0, -1, -2, \ldots$, then $\mathscr{L}(c, a)$ is an inverse of $\mathscr{L}(a, c)$. Note also that [**1**, p. 739]

(5.5) $\mathscr{K}(\alpha) = \mathscr{L}(1, 2)\mathscr{S}^*(\alpha)$,

and that

(5.6)
$$\begin{aligned} z\, {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) &= \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_{q+1})_n}{(b_1)_n \ldots (b_q)_n} \frac{z^{n+1}}{n!} \\ &= \mathscr{L}(a_1, b_1) \ldots \mathscr{L}(a_q, b_q)\mathscr{L}(a_{q+1}, 1)\phi(1, 1; z). \end{aligned}$$

By using the linear operator $\mathscr{L}(a, c)$, Carlson and Shaffer [**1**] have represented a result of Suffridge [**17**] as follows:

LEMMA 4. *If* $\alpha \leqq \beta \leqq 1$ *and* $\alpha < 1$, *then*

(5.7) $\mathscr{L}(2 - 2\beta, 2 - 2\alpha)\mathscr{S}^*(\alpha) \subset \mathscr{S}^*(\beta) \subset \mathscr{S}^*(\alpha)$.

Remark 6. It should be noted that $\mathscr{S}^*(\alpha) \not\subset \mathscr{S}$ when $\alpha < 0$ in Lemma 4. Our definitions for $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$, involving the inequalities (1.2) and (1.3), require that $0 \leqq \alpha < 1$.

A function $f(z)$ defined by (1.1) and belonging to the class $\mathscr{A}$ is said to be *prestarlike of order* α ($\alpha \leqq 1$) if and only if

(5.8)
$$\begin{cases} f * \dfrac{z}{(1 - z)^{2(1-\alpha)}} \in \mathscr{S}^*(\alpha) & (\alpha < 1) \\ \operatorname{Re}\left(\dfrac{f(z)}{z}\right) > \dfrac{1}{2} & (z \in \mathscr{U}; \alpha = 1). \end{cases}$$

We denote by $\mathscr{Q}(\alpha)$ the class consisting of all functions in $\mathscr{A}$ which are prestarlike of order α. The class $\mathscr{Q}(\alpha)$, introduced by Ruscheweyh [**11**], may be represented by

(5.9) $\mathscr{Q}(\alpha) = \mathscr{L}(1, 2 - 2\alpha)\mathscr{S}^*(\alpha) \quad (\alpha < 1)$

and

(5.10) $\mathscr{Q}(1) = \left\{f \in \mathscr{A}: \operatorname{Re}(f(z)/z) > \dfrac{1}{2}, z \in \mathscr{U}\right\}$

by using the linear operator $\mathscr{L}(a, c)$.

Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g., [**10**] and [**16**]). We find it to be convenient to recall here the following definitions which were used recently by Owa [7] (and by Srivastava and Owa [**15**]).

Definition 1. The *fractional integral of order* λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}d\zeta, \tag{5.11}$$

where $\lambda > 0$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z - \zeta)^{\lambda-1}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

Definition 2. The *fractional derivative of order* λ is defined, for a function $f(z)$, by

$$D_z^{\lambda}f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta, \tag{5.12}$$

where $0 \leqq \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z - \zeta)^{-\lambda}$ is removed as in Definition 1 above.

Definition 3. Under the hypotheses of Definition 2, the *fractional derivative of order* $n + \lambda$ is defined by

$$D_z^{n+\lambda}f(z) = \frac{d^n}{dz^n}D_z^{\lambda}f(z), \tag{5.13}$$

where $0 \leqq \lambda < 1$, and $n \in \mathscr{N} \cup \{0\}$.

By using these definitions of fractional calculus we introduce the linear operator Ω^{λ} defined by

$$\Omega^{\lambda}f = \Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z), \quad \lambda \neq 2, 3, 4, \dots \tag{5.14}$$

for functions (1.1) belonging to the class $\mathscr{A}$. Then we observe that

$$\begin{aligned} \Omega^{\lambda}f &= \Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}\left(\sum_{n=0}^{\infty} a_{n+1}z^{n+1}\right) \\ &= \sum_{n=0}^{\infty}\frac{\Gamma(n+2)\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}a_{n+1}z^{n+1} \\ &= \mathscr{L}(2, 2-\lambda)f, \end{aligned} \tag{5.15}$$

where $a_1 = 1$ and, as already pointed out, $\lambda \neq 2, 3, 4, \ldots$.
We now prove

THEOREM 7. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $\mathscr{K}(1/2)$, *then* $\Omega^\lambda f \in \mathscr{S}^*(1/2)$ *for* $0 \leqq \lambda < 1$, *that is,*

$$(5.16)\quad \Omega^\lambda \mathscr{K}\left(\frac{1}{2}\right) \subset \mathscr{S}^*\left(\frac{1}{2}\right) \quad (0 \leqq \lambda < 1).$$

Proof. With the aid of (5.5) and (5.15), we have

$$(5.17)\quad \Omega^\lambda \mathscr{K}\left(\frac{1}{2}\right) = \mathscr{L}(2, 2 - \lambda)\mathscr{L}(1, 2)\mathscr{S}^*\left(\frac{1}{2}\right)$$
$$= \mathscr{L}(1, 2 - \lambda)\mathscr{S}^*\left(\frac{1}{2}\right).$$

Since $\mathscr{S}^*(1/2) \subset \mathscr{S}^*((1/2)\lambda)$ for $0 \leqq (1/2)\lambda < 1/2$,

$$(5.18)\quad \Omega^\lambda \mathscr{K}\left(\frac{1}{2}\right) \subset \mathscr{L}(1, 2 - \lambda)\mathscr{S}^*\left(\frac{1}{2}\lambda\right).$$

Further, putting $\alpha = (1/2)\lambda$ and $\beta = 1/2$ in Lemma 4, we get

$$(5.19)\quad \mathscr{L}(1, 2 - \lambda)\mathscr{S}^*\left(\frac{1}{2}\lambda\right) \subset \mathscr{S}^*\left(\frac{1}{2}\right) \subset \mathscr{S}^*\left(\frac{1}{2}\lambda\right),$$

which implies (5.16).

COROLLARY 4. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (1.5) *satisfy the condition*

$$(5.20)\quad \left|\frac{z\,{}_pF_q'(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}{{}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}\right| < \frac{1}{2} \quad (z \in \mathscr{U}).$$

Then

$$\Omega^\lambda\{z\,{}_{p+1}F_{q+1}(a_1, \ldots, a_p, 1; b_1, \ldots, b_q, 2; z)\} \in \mathscr{S}^*\left(\frac{1}{2}\right),$$

where $0 \leqq \lambda < 1$.

The proof follows from Theorem 5 and Theorem 7. Furthermore, from Theorem 6 and Theorem 7, we have

COROLLARY 5. *Let* $1 \leqq a < c$. *Then*

$$\Omega^\lambda\{z\,{}_3F_2(a, 1, 1; c, 2; z)\} \in \mathscr{S}^*\left(\frac{1}{1}\right),$$

where $0 \leqq \lambda < 1$.

THEOREM 8. *Let* $0 \leqq \lambda < 2$. *Then*

$$(5.21)\quad \Omega^\lambda \mathscr{K}\left(\frac{1}{2}\lambda\right) = \mathscr{Q}\left(\frac{1}{2}\lambda\right).$$

Proof. By using (5.5) and (5.15), and then (5.9), we can see that

$$(5.22)\quad \Omega^\lambda \mathscr{K}\left(\frac{1}{2}\lambda\right) = \mathscr{L}(2, 2 - \lambda)\mathscr{L}(1, 2)\mathscr{S}^*\left(\frac{1}{2}\lambda\right)$$

$$= \mathscr{L}(1, 2 - \lambda)\mathscr{S}^*\left(\frac{1}{2}\lambda\right)$$

$$= \mathscr{Q}\left(\frac{1}{2}\lambda\right),$$

provided, by virtue of the definition (1.3), that $0 \leqq (1/2)\lambda < 1$.

In view of Theorem 5 and Theorem 6, Theorem 8 yields the following corollaries.

COROLLARY 6. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (1.5) *satisfy the condition*

$$(5.23)\quad \left|\frac{z\,{}_pF_q'(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}{{}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}\right| < 1 - \frac{1}{2}\lambda \quad (z \in \mathscr{U})$$

for $0 \leqq \lambda \leqq 1$.

Then

$$(5.24)\quad \Omega^\lambda\{z\,{}_{p+1}F_{q+1}(a_1, \ldots, a_p, 1; b_1, \ldots, b_q, 2; z)\} \in \mathscr{Q}\left(\frac{1}{2}\lambda\right).$$

COROLLARY 7. *Let* $0 \leqq \lambda < 2$ *and* $2 - \lambda \leqq a < c$. *Then*

$$(5.25)\quad \Omega^\lambda\{z\,{}_3F_2(a, 2 - \lambda, 1; c, 2; z)\} \in \mathscr{Q}\left(\frac{1}{2}\lambda\right).$$

Finally, we prove the following characterization theorem for the generalized hypergeometric function ${}_pF_q(z)$ by using the linear operator $\mathscr{L}(a, c)$.

THEOREM 9. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (1.5) *satisfy the condition* (3.13) *for* $0 \leqq \alpha \leqq 1/2$, *and let the constraint* (3.14) *hold true.*

Then

$$z\,{}_{p+1}F_{q+1}(a_1 + 1, \ldots, a_p + 1, 1; b_1 + 1, \ldots, b_q + 1, 2; z)$$

is in the class $\mathscr{S}^*(\alpha)$.

Proof. As proved in Theorem 3, we note that the function $H(z)$ defined by (2.5) is in the class $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^* \subset \mathcal{S}$. Hence

$$(5.26) \quad \int_0^z \frac{H(t)}{t} dt \in \mathcal{K}(\alpha),$$

by means of the definition of the class $\mathcal{K}(\alpha)$. In fact,

$$(5.27) \quad \int_0^z \frac{H(t)}{t} dt$$

$$= \frac{\prod_{j=1}^{q} b_j}{\prod_{j=1}^{p} a_j} \int_0^z t^{-1}[{}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; t) - 1]dt$$

$$= \mathcal{L}(1, 2)[z\ {}_{p+1}F_{q+1}(a_1 + 1, \dots, a_p + 1, 1; b_1 + 1, \dots, b_q + 1, 2; z)],$$

so that

$$(5.28) \quad \mathcal{L}(1, 2)[z\ {}_{p+1}F_{q+1}(a_1 + 1, \dots, a_p + 1, 1; b_1 + 1, \dots, b_q + 1, 2; z)] \in \mathcal{K}(\alpha).$$

Applying the linear operator $\mathcal{L}(2, 1)$ in (5.28), and using the relationship (5.5), we complete the proof of Theorem 9.

6. Further applications of the linear operator Ω^λ. We now recall the following lemma due to Ruscheweyh and Sheil-Small [**12**].

LEMMA 5. *Let $h(z)$ and $g(z)$ be analytic in the unit disk $\mathcal{U}$ and satisfy*

$$h(0) = g(0) = 0, \quad h'(0) \neq 0, \quad g'(0) \neq 0.$$

Suppose that, for each σ $(|\sigma| = 1)$ and ρ $(|\rho| = 1)$, we have

$$(6.1) \quad h(z) * \left(\frac{1 + \rho\sigma z}{1 - \sigma z}\right) g(z) \neq 0 \quad (z \in \mathcal{U} - \{0\}).$$

Then, for each function $F(z)$ analytic in the unit disk $\mathcal{U}$ and satisfying the inequality

$$(6.2) \quad \operatorname{Re}\{F(z)\} > 0 \quad (z \in \mathcal{U}),$$

$$(6.3) \quad \operatorname{Re}\left\{\frac{h * k(z)}{h * g(z)}\right\} > 0 \quad (z \in \mathcal{U}),$$

where $k(z) = F(z)g(z)$.

By using Lemma 5, we prove

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THEOREM 10. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}^*$ *and let, for each* σ $(|\sigma| = 1)$ *and* ρ $(|\rho| = 1)$,

$$\text{(6.4)} \qquad \mathcal{L}(2, 2 - \lambda)\left(\frac{1 + \rho\sigma z}{1 - \sigma z} f(z)\right) \neq 0, \quad \forall z \in \mathcal{U} - \{0\}.$$

Then $\Omega^\lambda f(z)$ *is also in the class* $\mathcal{S}^*$.

Proof. In view of (5.15), we have

$$\text{(6.5)} \qquad \operatorname{Re}\left\{\frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)}\right\} = \operatorname{Re}\left\{\frac{\Omega^\lambda(zf'(z))}{\Omega^\lambda f(z)}\right\}$$

$$= \operatorname{Re}\left\{\frac{\mathcal{L}(2, 2 - \lambda)(zf'(z))}{\mathcal{L}(2, 2 - \lambda)f(z)}\right\}$$

$$= \left\{\frac{\left(\sum_{n=0}^{\infty} \frac{(2)_n}{(2 - \lambda)_n} z^{n+1}\right) * (zf'(z))}{\left(\sum_{n=0}^{\infty} \frac{(2)_n}{(2 - \lambda)_n} z^{n+1}\right) * f(z)}\right\}.$$

Putting

$$h(z) = \sum_{n=0}^{\infty} \frac{(2)_n}{(2 - \lambda)_n} z^{n+1},$$

$$F(z) = \frac{zf'(z)}{f(z)},$$

and $g(z) = f(z)$ in Lemma 5, we conclude from (6.5) that

$$\text{(6.6)} \qquad \operatorname{Re}\left\{\frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)}\right\} > 0 \quad (z \in \mathcal{U}),$$

which evidently implies that $\Omega^\lambda f(z) \in \mathcal{S}^*$, and we thus complete the proof of Theorem 10.

Next we recall the following result due to Twomey [**18**].

LEMMA 6. *Let the function* $f(z)$ *be in the class* $\mathcal{S}^*$. *Then*

$$\text{(6.7)} \qquad \left|\frac{zf'(z)}{f(z)}\right| \leqq 1 + \frac{|z|\log\left(\frac{(1 + |z|)^2|f(z)|}{|z|}\right)}{(1 - |z|)\log\left(\frac{1 + |z|}{1 - |z|}\right)}$$

for $z \in \mathcal{U}$. *Equality in* (6.7) *holds true for the Koebe function*

$$(6.8)\quad f(z) = \frac{z}{(1-z)^2}.$$

Applying Lemma 6 to Theorem 10, we immediately have

COROLLARY 8. *Under the hypotheses of Theorem* 10,

$$(6.9)\quad \left|\frac{\Omega^\lambda(zf'(z))}{\Omega^\lambda f(z)}\right| \leqq 1 + \frac{|z|\log\left(\dfrac{(1+|z|)^2|\Omega^\lambda f(z)|}{|z|}\right)}{(1-|z|)\log\left(\dfrac{1+|z|}{1-|z|}\right)}$$

for $z \in \mathcal{U}$. *Equality in* (6.9) *holds true for the function* $f(z)$ *given by*

$$(6.10)\quad f(z) = \mathcal{L}(2-\lambda, 2)\left(\frac{z}{(1-z)^2}\right).$$

Now we recall the following lemma due to Singh [**14**].

LEMMA 7. *Let the function* $f(z)$ *be in the class* $\mathcal{S}^*$. *Then*

$$(6.11)\quad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \frac{1-|z|^2}{|z|}|f(z)|$$

and

$$(6.12)\quad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \leqq \frac{1+|z|}{1-|z|} + \frac{2|z|\log\left(\dfrac{(1-|z|)^2|f(z)|}{|z|}\right)}{(1-|z|^2)\log\left(\dfrac{1+|z|}{1-|z|}\right)}$$

for $z \in \mathcal{U}$.

Equality in (6.11) *is attained for a function of the form*

$$(6.13)\quad f(z) = \frac{z}{(1-ze^{i\gamma})^{2\delta}(1-ze^{-i\gamma})^{2(1-\delta)}} \quad (0 \leqq \delta \leqq 1;\ \gamma\ \textit{real}),$$

and equality in (6.12) *is attained for a function of the form*

$$(6.14)\quad f(z) = \frac{z}{(1-z)^{2\delta}(1+z)^{2(1-\delta)}} \quad (0 \leqq \delta \leqq 1),$$

where δ *satisfies*

$$(6.15)\quad 2\delta\log\left(\frac{1+|z|}{1-|z|}\right) = \log\left(\frac{(1+|z|)^2|f(z)|}{|z|}\right).$$

From Lemma 7 and Theorem 10 we readily have

COROLLARY 9. *Under the hypotheses of Theorem* 10,

$$\text{(6.16)}\quad \operatorname{Re}\left\{\frac{\Omega^\lambda(zf'(z))}{\Omega^\lambda f(z)}\right\} \geqq \frac{1-|z|^2}{|z|}|\Omega^\lambda f(z)|$$

and

$$\text{(6.17)}\quad \operatorname{Re}\left\{\frac{\Omega^\lambda(zf'(z))}{\Omega^\lambda f(z)}\right\} \leqq \frac{1+|z|}{1-|z|} + \frac{2|z|\log\left(\frac{(1-|z|)^2|\Omega^\lambda f(z)|}{|z|}\right)}{(1-|z|^2)\log\left(\frac{1+|z|}{1-|z|}\right)}$$

for $z \in \mathscr{U}$. *Equality in* (6.16) *is attained for the function*

$$\text{(6.18)}\quad f(z) = \mathscr{L}(2-\lambda, 2)\left\{\frac{z}{(1-ze^{i\gamma})^{2\delta}(1-ze^{-i\gamma})^{2(1-\delta)}}\right\},$$

and equality in (6.17) *is attained for the function*

$$\text{(6.19)}\quad f(z) = \mathscr{L}(2-\lambda, 2)\left\{\frac{z}{(1-z)^{2\delta}(1+z)^{2(1-\delta)}}\right\},$$

where γ *is real,* $0 \leqq \delta \leqq 1$, *and* δ *satisfies* (6.15).

For functions belonging to the class $\mathscr{K}$, we prove

THEOREM 11. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathscr{K}$ *and let, for each* σ $(|\sigma| = 1)$ *and* ρ $(|\rho| = 1)$,

$$\text{(6.20)}\quad \mathscr{L}(2, 1)\mathscr{L}(2, 2-\lambda)\left(\frac{1+\rho\sigma z}{1-\sigma z}f(z)\right) \neq 0, \quad \forall z \in \mathscr{U} - \{0\}.$$

Then $\Omega^\lambda f(z)$ *is also in the class* $\mathscr{K}$.

Proof. With the aid of Theorem 10, we observe that

$$\begin{aligned} f(z) \in \mathscr{K} &\Leftrightarrow zf'(z) \in \mathscr{S}^* \\ &\Rightarrow \Omega^\lambda(zf'(z)) \in \mathscr{S}^* \\ &\Rightarrow z(\Omega^\lambda f(z))' \in \mathscr{S}^* \\ &\Leftrightarrow \Omega^\lambda f(z) \in \mathscr{K}, \end{aligned}$$

which evidently proves Theorem 11.

Finally, in order to prove some interesting generalizations of Theorem 10 and Theorem 11 (contained in Theorem 12 and Theorem 13 below), we recall the following lemma due to Lewis [**4**].

LEMMA 8. *Given* μ, *with* $-\infty < \mu < \infty$, *let*

$$\text{(6.21)}\quad f_\mu(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^\mu} z^{n+1}$$

for $z \in \mathscr{U}$.

Then $f_\mu(z)$ *is in the class* $\mathscr{K}$ *whenever* $\mu \geqq 0$.

THEOREM 12. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathscr{A}$ *and satisfy, for each* σ $(|\sigma| = 1)$ *and* ρ $(|\rho| = 1)$,

$$(6.22)\quad \mathscr{L}(2, 2 - \lambda)\left(\frac{1 + \rho\sigma z}{1 - \sigma z}(f_\mu * f(z))\right)$$

for $\mu \geqq 0$, *the function* $f_\mu(z)$ *being given by* (6.21).

Then $\Omega^\lambda(f_\mu * f(z))$ *is in the class* $\mathscr{S}^*$.

Proof. We note that

$$(6.23)\quad \mathrm{Re}\left\{\frac{z(\Omega^\lambda(f_\mu * f(z)))'}{\Omega^\lambda(f_\mu * f(z))}\right\} = \mathrm{Re}\left\{\frac{\mathscr{L}(2, 2 - \lambda)(f * zf_\mu'(z))}{\mathscr{L}(2, 2 - \lambda)(f * f_\mu(z))}\right\}$$

$$= \mathrm{Re}\left\{\frac{\left(\sum_{n=0}^{\infty}\frac{(2)_n}{(2 - \lambda)_n}a_{n+1}z^{n+1}\right) * (zf_\mu'(z))}{\left(\sum_{n=0}^{\infty}\frac{(2)_n}{(2 - \lambda)_n}a_{n+1}z^{n+1}\right) * f_\mu(z)}\right\}, \quad a_1 = 1$$

and, by Lemma 8, $f_\mu(z) \in \mathscr{K} \subset \mathscr{S}^*$ for $\mu \geqq 0$. Setting

$$h(z) = \sum_{n=0}^{\infty}\frac{(2)_n}{(2 - \lambda)_n}a_{n+1}z^{n+1}, \quad a_1 = 1$$

$$F(z) = \frac{zf_\mu'(z)}{f_\mu(z)},$$

and $g(z) = f_\mu(z)$ in Lemma 5, we find that

$$(6.24)\quad \mathrm{Re}\left\{\frac{z(\Omega^\lambda(f_\mu * f(z)))'}{\Omega^\lambda(f_\mu * f(z))}\right\} > 0 \quad (z \in \mathscr{U}),$$

that is, that $\Omega^\lambda(f_\mu * f(z)) \in \mathscr{S}^*$.

COROLLARY 10. *If* $f(z)$ *is in the class* $\mathscr{S}^*$ *and satisfies the condition* (6.22) *for* $\mu \geqq 0$, *then* $\Omega^\lambda(f_\mu * f(z))$ *is also in the class* $\mathscr{S}^*$, $f_\mu(z)$ *being given by* (6.21).

Remark 7. Taking $f(z) \in \mathscr{S}^*$ and $\mu = 0$ in Theorem 12, we have Theorem 10.

Ruscheweyh and Sheil-Small [**12**] (see also [**2**, p. 248, Theorem 8.6′]) proved the following lemma.

LEMMA 9. *If* $f(z) \in \mathscr{S}^*$ *and* $g(z) \in \mathscr{K}$, *then* $f * g(z) \in \mathscr{S}^*$.

We shall make use of Lemma 8 and Lemma 9 in order to prove

THEOREM 13. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathscr{K}$ *and satisfy, for each* σ $(|\sigma| = 1)$ *and* ρ $(|\rho| = 1)$,

$$(6.25)\quad \mathscr{L}(2, 1)\mathscr{L}(2, 2 - \lambda)\left(\frac{1 + \rho\sigma z}{1 - \sigma z}(f_\mu * f(z))\right) \neq 0 \quad (z \in \mathscr{U} - \{0\})$$

for $\mu \geqq 0$, *the function* $f_\mu(z)$ *being given by* (6.21).
Then $\Omega^\lambda(f_\mu * f(z))$ *is also in the class* $\mathscr{K}$.

Proof. Note that

$$f(z) \in \mathscr{K} \Leftrightarrow zf'(z) \in \mathscr{S}^*.$$

Thus it follows from Lemma 8 and Lemma 9 that

$$f_\mu * (zf'(z)) \in \mathscr{S}^*$$

for $\mu \geqq 0$. Applying Corollary 10, we observe that

$$\begin{aligned} f(z) \in \mathscr{K} &\Leftrightarrow zf'(z) \in \mathscr{S}^* \\ &\Rightarrow f_\mu * (zf'(z)) \in \mathscr{S}^* \\ &\Rightarrow \Omega^\lambda(f_\mu * (zf'(z))) \in \mathscr{S}^* \\ &\Rightarrow z(\Omega^\lambda(f_\mu * f(z)))' \in \mathscr{S}^* \\ &\Leftrightarrow \Omega^\lambda(f_\mu * f(z)) \in \mathscr{K}, \end{aligned}$$

which obviously completes the proof of Theorem 13.

Remark 8. Taking $\mu = 0$ in Theorem 13, we have Theorem 11.

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Kinki University,
Osaka, Japan;
University of Victoria,
Victoria, British Columbia