Bull. Austral. Math. Soc. 34D05, 34D40
Vol. 53 (1996) [373–389]

GLOBAL ASYMPTOTIC STABILITY OF A PERIODIC SYSTEM OF DELAY LOGISTIC EQUATIONS

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Sufficient conditions are obtained for the existence and global asymptotic stability of a periodic solution in Volterra's population system of integrodifferential equations with periodic coefficients. It is shown that if (i) the intraspecific negative feedbacks are instantaneous and dominate the interspecific effects (ii) the minimum possible growth rates are stronger than the maximum interspecific effects weighted with the respective sizes of all species, when they are near their potential maximum sizes, then the system of integrodifferential equations has a unique componentwise periodic solution which is globally asymptotically stable.

1. Introduction

In this paper we propose to derive a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly positive (componentwise) periodic solution of the integro-differential system

$$\frac{dx_i}{dt} = x_i(t)\left[b_i(t) - a_{ii}(t)x_i(t) - \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)x_j(s)ds\right] \tag{1.1}$$
$$i = 1, 2, \ldots, n, \qquad t > 0$$

with initial conditions

$$x_i(t) = \phi_i(t),\ t \leqslant 0, \quad i = 1, 2, \ldots, n \tag{1.2}$$

where ϕ_i $(i = 1, 2, \ldots, n)$ are bounded nonnegative continuous functions on $(-\infty, 0)$ with possible jump discontinuities at $t = 0$ so that

$$\phi_i(0) > 0, \quad i = 1, 2, \ldots, n. \tag{1.3}$$

The coefficients $a_{ii}(t)$, $b_i(t)$, $c_{ij}(t)$, $(i, j = 1, 2, \ldots, n)$ are given real p-periodic functions defined for all $t \geqslant 0$, and satisfying the conditions

$$a_{ii}(t) > 0,\ b_i(t) > 0,\ c_{ij}(t) > 0, \quad i, j = 1, 2, \ldots, n. \tag{1.4}$$

Received 18 July 1995
We wish to thank Dr. K. Gopalsamy for suggesting this problem for our consideration.

The delay kernels $k_{ij}(t)$ $(i, j = 1, 2, \dots, n)$ are continuous functions defined for all $t \geqslant 0$ and satisfy the conditions:

$$k_{ij}(t) \geqslant 0; \ t \geqslant 0; \quad i, j = 1, 2, \dots, n \tag{1.5}$$

and

$$\int_0^\infty k_{ij}(t)dt = 1; \quad i, j = 1, 2, \dots, n. \tag{1.6}$$

In mathematical ecology, (1.1) denotes a model of the dynamics of an n-species system in which each individual competes with all others of the system for a common pool of resources and the interspecific competition involves a time delay extending over the entire past as typified by the delay kernels in (1.1). The assumption of periodicity of the parameters b_i, a_{ij} $(i, j = 1, 2, \dots, n)$ is a way of incorporating the periodicity of the environment (for example the seasonal effects of weather, food supplies, mating habits et cetera).

A number of authors have considered scalar systems of the form (1.1). Miller [**5**] considered the integro-differential equation

$$\frac{dx}{dt} = x(t)\left[b - ax(t) - c\int_{-\infty}^{t} k(t-s)x(s)ds\right], \quad t > 0,$$

where a, b, c are real positive numbers. He proved the existence of a globally asymptotically stable positive solution under prescribed conditions on the coefficients a, b and c. Cushing [**2**] considered the periodic logistic equation and derived sufficient conditions for the existence of a non-trivial, non-negative, periodic solution. Badii and Schiaffino [**1**] obtained sufficient conditions for the existence of a positive globally asymptotically stable periodic solution of the integrodifferential equation

$$\frac{dx}{dt} = x(t)\left[b(t) - a(t)\,x(t) - c(t)\int_{-\infty}^{t} k(t-s)x(s)dx\right],$$

where the coefficients $a(t)$, $b(t)$ and $c(t)$ are periodic with common period $p > 0$. Gopalsamy [**3**] provided sufficient conditions for the existence of a positive periodic solution and its global asymptotic stability in Volterra's population system incorporating hereditary effects, in a constant environment.

2. Notation and preliminary results

Let $C_p = C_p(\mathbb{R}, \mathbb{R})$ denote the Banach space of all real p-periodic continuous scalar functions with the supremum norm

$$\|x\|_0 = \operatorname{Sup}_{0 \leqslant t \leqslant p} |x(t)|.$$

$C_p^n = C_p^n(\mathbb{R}, \mathbb{R})$ denotes the Banach space of all real n-dimensional, p-periodic continuous functions endowed with the norm

$$\|(x_1, x_2, \ldots, x_n)\| = \max_{1\leqslant i\leqslant n} \|x_i\|_0 .$$

The natural ordering in C_p^n, that is,

$$x \geqslant y \text{ if } x_i(t) \geqslant y_i(t) \text{ for } i = 1, 2, \ldots, n \text{ and } t \in \mathbb{R}, \text{ will be used.}$$

For x defined on $[0, p]$, the average of x is defined to be

$$[x] = p^{-1}\int_0^p x(s)ds.$$

Let the convolution of k with the bounded function x be denoted by

$$(k * x)(t) = \int_{-\infty}^t k(t-s)x(s)ds.$$

DEFINITION: A solution of the problem (1.1) – (1.2) is a continuous function

$$x(t) = (x_1(t), x_2(t), \ldots, x_n(t)),$$

with

$$x_i\colon (-\infty, \infty) \to [0, \infty) \quad (i = 1, 2, \ldots, n)$$

such that $x_i(t) = \phi_i(t)$ on $(-\infty, 0]$ with $x_i(t)$, $(i = 1, 2, \ldots, n)$ continuously differentiable on $(0, \infty)$ and satisfying (1.1).

THEOREM 2.1. Cushing [2]. *Let $b_i(t)$, $a_{ii}(t)$ $(i = 1, 2, \ldots, n)$ be given real p-periodic continuous functions such that $[b_i] > 0$ and $a_{ii}(t) > 0$ $(i = 1, 2, \ldots, n)$ and $c_{ij}(t) = 0$ $(i, j = 1, 2, \ldots, n)$, $t \in \mathbb{R}$, then (1.1) has a unique positive (componentwise) periodic solution.*

THEOREM 2.2. *Let the coefficients $b_i(t)$, $a_{ii}(t)$, $c_{ij}(t)$ and k_{ij} $(i, j = 1, 2, \ldots, n)$ of the integrodifferential system (1.1) satisfy the conditions (1.4) and (1.5), and let the initial conditions $\phi_i(t)$, $(i = 1, 2, \ldots, n)$, $t \leqslant 0$ be non-negative bounded continuous functions on $(-\infty, 0)$. Then the problem (1.1) – (1.2) has a unique non-negative (componentwise) bounded solution.*

PROOF: The existence and uniqueness of a local solution are proved by standard techniques applicable to integrodifferential equations (see, for instance, Cushing [2]). The non-negativity and boundedness of the solution follow from the form of (1.1) and $a_{ii}(t) > 0$. $\square$

Let G be the set defined by

$$G = \left\{x \in C_p^n\colon \left[b_i(t) - \sum_{j=1}^{n} c_{ij}(t)(k_{ij} * x_j)(t)\right] > 0\right\}, \quad i = 1, 2, \ldots, n.$$

Let the operator $B\colon G \to C_p^n$ be defined as follows:

$$(Bx)(t) = u(t), \quad t \in \mathbb{R},$$

where $u(t) \in C_p^n$ is the unique positive (componentwise) periodic solution of the system

$$\text{(2.1)} \qquad \frac{du_i}{dt} = \left[b_i(t) - \sum_{j=1}^{n} c_{ij}(t)(k_{ij} * x_j)(t)\right]u_i(t) - a_{ii}(t)u_{ii}^2(t), \quad i = 1, 2, \ldots, n$$

which exists as a consequence of Theorem (2.1). It is evident that p-periodic solutions of (1.1) are in one-to-one correspondence with the fixed point of the operator $B\colon G \to C_p^n$, and it immediately follows that G is not empty since $x(t) \equiv 0$ belongs to G.

Define

$$u^0(t) = (BO)(t);\ u^0 \in C_p^n.$$

Lemma 2.1. *If $x \in C_o^n(\mathbb{R}, \mathbb{R})$ and $c_{ij}(t)$ are real p-periodic continuous functions $(i, j = 1, 2, \ldots, n)$, then*

$$y = (y_1, y_2, \ldots, y_n) \in C_p^n(\mathbb{R}, \mathbb{R}),$$

where

$$y_i(t) = \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{t} k_{ij}(t - s)x_j(s)\,ds \quad i, j = 1, 2, \ldots, n$$

and

$$\begin{aligned} y_i(t + p) &= \sum_{j=1}^{n} c_{ij}(t + p) \int_{-\infty}^{t+p} k_{ij}(t + p - s)x_j(s)\,ds \\ &= \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{t} k_{ij}(t - s)x_j(s - p)\,ds \\ &= \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{t} k_{ij}(t - s)x_j(s)\,ds \\ &= y_i(t). \end{aligned}$$

Lemma 2.2. *Let $x^{(1)}$ and $x^{(2)}$ belong to G and $x^{(1)} \leqslant x^{(2)}$, then $Bx^{(2)} \leqslant B_x^{(1)}$.*

Proof: Let

$$w_i^{(1)} = b_i(t) - \sum_{j=1}^{n} c_{ij}(t)\Big(k_{ij} * x_j^{(1)}\Big)(t),$$

and

$$w_i^{(2)} = b_i(t) - \sum_{j=1}^{n} c_{ij}(t)\Big(k_{ij} * x_j^{(2)}(t)\Big), \quad i = 1, 2, \ldots, n, \quad t > 0$$

and let $u^{(j)}(t) = \big(Bx^{(j)}\big)(t)$ for $t \in \mathbb{R}$ with $j = 1, 2$. Then we have

$$w^{(1)}(t) \geqslant w^{(2)}(t), \text{ where } w^{(j)} = \Big(w_1^{(j)}, w_2^{(j)}, \ldots, w_n^{(j)}\Big), \quad j = 1, 2.$$

Further, $[w_i^{(j)}] = [a_{ii}u_i^{(j)}]$ since $u_i^{(j)}(t)$ $(i = 1, 2, \ldots, n;\ j = 1, 2)$ are periodic by virtue of Lemma 2.1. Hence,

$$[a_{ii}u_i^{(1)}] \geqslant [a_{ii}u_i^{(2)}], \quad (i = 1, 2, \ldots, n)$$

for some

$$t_0 \in \mathbb{R}, \quad u_i^{(2)}(t_0) \geqslant u_i^{(1)}(t_0), \quad (i = 1, 2, \ldots, n).$$

Now let $v_i(t) = u_i^{(1)}(t) - u_i^{(2)}(t)$; hence it follows that:

$$\begin{aligned}
\frac{dv_i}{dt} &= \frac{du_i^{(1)}}{dt} - \frac{du_i^{(2)}}{dt} \\
&= w_i^{(1)}(t)u_i^{(1)}(t) - b_i(t)\Big(u_i^{(1)}(t)\Big)^2 - w_i^{(2)}(t)u_i^{(2)}(t) - b_i(t)\Big(u_i^{(2)}(t)\Big)^2 \\
&= w_i^{(1)}(t)u_i^{(1)}(t) - w_i^{(2)}(t)u_i^{(2)}(t) - b_i(t)\Big(u_i^{(1)}(t) - u_i^{(2)}(t)\Big)\Big(u_i^{(1)}(t) + u_i^{(2)}(t)\Big) \\
&\geqslant [w_i^1(t) - b_i(t)\Big(u_i^{(1)}(t) + u_i^{(2)}(t)\Big)]v_i(t).
\end{aligned}$$

Thus,

$$\frac{dv_i}{dt} \geqslant q_i(t)v_i(t) \quad \text{for } t \geqslant t_0,$$

where $q_i(t) = w_i^{(1)}(t) - b_i(t)\Big(u_i^{(1)}(t) + u_i^{(2)}(t)\Big)$, $i = 1, 2, \ldots, n$. This implies that $v_i(t) \geqslant 0$ for all $t \geqslant t_0$, $i = 1, 2, \ldots, n$. ∎

Lemma 2.3. *Let k_{ij} $(i, j = 1, 2, \ldots, n)$ satisfy (1.5) and define $k_{ij} = 0$ for $t < 0$, and let $c_{ij}(t)$ and $v_{ij}(t)$ belong to C_p. Then*

$$\left[\sum_{j=1}^{n} c_{ij}(t)(k_{ij} * v_{ij})(t)\right] = \left[\sum_{j=1}^{n} v_{ij}(t)(k_{ij} * c_{ij})(t)\right]$$

PROOF:

$$\left[\sum_{j=1}^{n} c_{ij}(k_{ij} * v_{ij})(t)\right] \stackrel{d}{=} \sum_{j=1}^{n} \sum_{i=-\infty}^{\infty} \int_0^p c_{ij}(t) \int_{ip}^{p(i+1)} k_{ij}(t-s)v_{ij}(s)\,ds\,dt$$
$$= \sum_{j=1}^{n} c_{ij}(t) \sum_{i=-\infty}^{\infty} \int_0^p c_{ij}(t) \int_0^p k_{ij}(t-s-ip)\,ds\,dt$$
$$= \sum_{j=1}^{n} \sum_{i=-\infty}^{\infty} \int_0^p v_{ij}(s) \int_{-ip}^{p(i-1)} k_{ij}(t-s)c_{ij}(t)\,ds\,dt$$
$$= \left[\sum_{j=1}^{n} v_{ij}(k_{ij} * c_{ij})\right].$$

□

3. THE MAIN RESULT

DEFINITION 3.1: The solution $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))$ of (1.1) is called globally asymptotically stable if and only if all solutions of the system (1.1) with initial conditions $x_i(0) > 0$ $(i = 1, 2, \ldots, n)$ defined for all $t > 0$ satisfy:

$$\lim_{t\to\infty} |x_i(t) - x_i^*(t)| = 0; \quad i = 1, 2, \ldots, n. \tag{3.1}$$

By definition of global asymptotic stability it follows that $x^*(t)$ is unique.

THEOREM 3.1. *Consider the set of integrodifferential equations (1.1) along with (1.2) – (1.3). Let the coefficients $a_{ij}(t)$, $b_i(t)$, $c_{ij}(t)$ $(i, j = 1, 2, \ldots, n)$ satisfy conditions (1.4), and let k_{ij} $(i, j = 1, 2, \ldots, n)$ satisfy (1.5) and (1.6).*

Further, if

$$\min_{t\in[0,p]} a_{ii}(t) > \max_{t\in[0,p]} \sum_{j=1}^{n} c_{ij}(t) + \alpha_1 \tag{3.2}$$

for some positive constant α_1, and

$$\min_{t\in[0,p]} b_i(t) > \sum_{j=1}^{n} \max_{t\in[0,p]} c_{ij}(t) \left[\frac{\max_{t\in[0,p]} b_j(t)}{\min_{t\in[0,p]} a_{jj}(t)}\right] + \varepsilon_2 \tag{3.3}$$

for some positive constant ε_2, then the system (1.1) with (1.2) – (1.3) has a strictly positive (componentwise) globally asymptotically stable p-periodic solution.

The proof of the Theorem is presented in two parts. In Part 1 the existence of a positive (componentwise) p-periodic solution is demonstrated, and in Part 2 the global asymptotic stability of the solution is shown.

PART 1.

PROOF OF THE EXISTENCE OF A POSITIVE p-PERIODIC SOLUTION..

Consider the set of differential equations:

$$\frac{1}{u_i^0(t)}\frac{du_i^0}{dt} = b_i(t) - a_{ii}(t)u_i^0(t); \quad i = 1, 2, \ldots, n. \tag{3.4}$$

As a consequence of the periodicity of the equations (3.4) it follows that

$$[b_i] = [a_{ii}u_i^0], \quad i = 1, 2, \ldots, n. \tag{3.5}$$

Lemmas (2.3) and (2.3) imply

(3.6)

$$\begin{aligned}[a_{ii}u_i^0] = [b_i] &> \left[\sum_{j=1}^{n} \max_{t\in[0,p]} c_{ij}(t)\left[\frac{\max_{t\in[0,p]} b_j(t)}{\min_{t\in[0,p]} a_{jj}(t)}\right]\right] > \left[\sum_{j=1}^{n} u_j^0(k_{ij} * c_{ij})\right] \\ &= \left[\sum_{j=1}^{n} c_{ij}(t)\big(k_{ij} * u_j^0\big)(t)\right]\end{aligned}$$

and consequently that $u^0 = \left(u_1^0, u_2^0, \ldots, u_n^0\right) \in G$. Since $0 < u_i^0$ $(i = 1, 2, \ldots, n)$, then $Bu^0 \leqslant u^0$.

Now if the function $v \in C_p^n$ satisfying the condition $0 < v < u^0$ is considered, it is clear that $0 < Bu^0 \leqslant Bv \leqslant u^0$.

Therefore the set

$$G_0 = \left\{u \in C_p^n : 0 < v \leqslant u^0\right\} \subset G$$

is left invariant by the operator B. Further

$$Bu^0 \leqslant Bv \leqslant u^0 \Rightarrow Bu^0 \leqslant B^2v \leqslant B^2u^0 \Rightarrow B^3u^0 \leqslant B^3v \leqslant B^2u^0.$$

By induction,

$$B^{2n+1}u^0 \leqslant B^{2n+1}v \leqslant B^{2n}u^0$$

and

$$B^{2n+1}u^0 \leqslant B^{2n+2}v \leqslant B^{2n+2}u^0 \text{ for } n = 0, 1, 2, \ldots. \tag{3.7}$$

It is now to be shown that $\left(B^{2n+1}u^0\right)$ *and* $\left(B^{2n}u^0\right)$ are, respectively, increasing and decreasing sequences.

https://doi.org/10.1017/S0004972700017135 Published online by Cambridge University Press

The method of finite induction will be used to prove the last assertion. Now

$$0 < u^0 \Rightarrow Bu^0 < u^0 \Rightarrow 0 < Bu^0 < B^2u^0 < u^0 \Rightarrow 0 < Bu^0 < B^3u^0 < B^2u^0 < u^0$$
$$\Rightarrow 0 < Bu^0 < B^3u^0 < B^2u^0 < u^0.$$

Let it be assummed that for $n = k$, the following relationship holds:

(3.8)
$$0 < Bu^0 < B^3u^0 < \ldots < B^{2k-3}u^0 < B^{2k-1}u^0 < B^{2k}u^0 < \ldots < B^4u^0 < B^2u^0 < u^0.$$

We wish to show that

(3.9)
$$0 < Bu^0 < B^3u^0 < \ldots < B^{2k-3}u^0 < B^{2k-1}u^0$$
$$< B^{2k+2}u^0 < B^{2k-2}u^0 < \ldots < B^4u^0 < B^2u^0.$$

Using the operator B on all terms in (3.7), and the monotonic properties of B discussed above together gives

$$0 < Bu^0 < B^3u^0 < \ldots < B^{2k-3}u^0 < B^{2k-1}u^0$$
$$< B^{2k+2}u^0 < B^{2k}u^0 < B^{2k-2}u^0 < \ldots < B^4u^0 < B^2u^0 < u^0$$

and hence (3.9) holds. Therefore, from the principle of finite induction $\{B^{2n+1}u^0\}$ and $\{B^{2n}u^0\}$ are, respectively, increasing, and decreasing sequences.

Now define

$$u^n(t) = \left(B^n u^0\right)(t) = \left(Bu^{n-1}\right)(t).$$

Since the sequence $\left(B^{2n+1}u^0\right)$ is monotonic increasing, and bounded above by every term in the sequence $\left(B^{2n}u^0\right)$, then $u^-(t) = \lim_n u^{2n+1}(t)$ exists.

Similarly, since the sequence $\left(B^{2n}u^0\right)$ is monotonic decreasing and bounded below by every term in the sequence $\left(B^{2n+1}u^0\right)$, then $u^+(t) = \lim_n u^{2n}(t)$ exists, and so

$$0 < u^-(t) \leqslant u^+(t)$$

(that is, $0 < u_i^-(t) \leqslant u_i^+(t)$; $i = 1, 2, \ldots, n$).

To prove the existence of a unique fixed point $u^*(t)$ of the operator B it is sufficient to show that

$$u^-(t) = u^+(t) = u^*(t)$$

(that is, $u_i^-(t) = u_i^+(t) = u_i^*(t)$; $i = 1, 2, \ldots, $).

As a result of the montonocity and uniform boundedness of $\{u^n\}$, the L^2-convergences of the sequences $\{u^{2n+1}\}$ and $\{u^{2n}\}$ follow, and also that of their derivatives, since, by definition

$$\frac{du_i^n(t)}{dt} = \Big(b_i(t) - \sum_{j=1}^{n} c_{ij}(t)\big(k_{ij} * u^{n-1}\big)(t)\Big)u_i^n(t) - a_{ii}(t)(u_i^n(t))^2 \tag{3.10}$$
$$i = 1, 2, \dots, n;\ t > 0.$$

Taking the limit for n gives, for odd and even cases, respectively,

$$\dot{u}_i^- = \Big(b_i(t) - \sum_{j=1}^{n} c_{ij}(t)\big(k_{ij} * u_j^+\big)(t)\Big)u_i^-(t) - a_{ii}(t)\big(u_i^-(t)\big)^2, \quad i = 1, 2, \dots, n \tag{3.11}$$

$$\dot{u}_i^+ = \Big(b_i(t) - \sum_{j=1}^{n} c_{ij}(t)\big(k_{ij} * u_j^-\big)(t)\Big)u_i^+(t) - a_{ii}(t)\big(u_i^+(t)\big)^2, \quad i = 1, 2, \dots, n. \tag{3.12}$$

From dividing (3.11), (3.12) by $u_i^-(t)$, $u_i^+(t)$, respectively, $i = 1, 2, \dots, n$, the result is

$$\frac{\dot{u}_i^-(t)}{u_i^-(t)} = \Big(b_i(t) - \sum_{j=1}^{n} c_{ij}(t)\big(k_{ij} * u_j^+\big)(t)\Big) - a_{ii}(t)\big(u_i^-(t)\big), \quad i = 1, 2, \dots, n; \tag{3.13}$$

$$\frac{\dot{u}_i^+}{u_i^+(t)} = \Big(b_i(t) - \sum_{j=1}^{n} c_{ij}(t)\big(k_{ij} * u_j^-\big)(t)\Big) - a_{ii}(t)\big(u_i^+(t)\big), \quad i = 1, 2, \dots, n. \tag{3.14}$$

Integrating (3.13) and (3.14) with respect to t from 0 to ∞, and using the periodicity of $\log u_i^-(t)$ and $u_i^+(t)$ $(i = 1, 2, \dots, n)$ we have

$$\begin{aligned} &\Big[b_i(t) - \sum_{j=1}^{n} c_{ij}(t)\big(k_{ij} * u_j^*\big)(t) - a_{ii}(t)u_i^-(t)\Big] \\ &\quad = \Big[b_i(t) - \sum_{j=1}^{n} c_{ij}(t)\big(k_{ij} * u_j^-\big)(t) - a_{ii}(t)u_i^+(t)\Big]; \quad i = 1, 2, \dots, n. \end{aligned} \tag{3.15}$$

Let $v_i(t) = u_i^+(t) - u_i^-(t)$; $i = 1, 2, \dots, n$. From (3.15) it follows that

$$\Big[\sum_{j=1}^{n} c_{ij}(t)(k_{ij} * v_j)(t)\Big] = [a_{ii}(t)v_i(t)] \quad i = 1, 2, \dots, n. \tag{3.16}$$

As a consequence of Lemma 2.3 we have,

$$\left[\sum_{j=1}^{n} v_j(t)(k_{ij} * c_{ij})(t)\right] = [a_{ii}(t)v_i(t)]\,; \quad i = 1, 2, \ldots, n. \tag{3.17}$$

It follows that

(3.18)

$$[b_{ii}(t)v_i(t)] = [v_i(t)(k_{il} * c_{il})(t)] + \ldots + [v_i(t)(k_{ii} * c_{ii})(t)] + \ldots + [v_n(t)(k_{in} * c_{in})(t)];$$
$$i = 1, 2, \ldots, n.$$

Condition (3.2) implies

$$a_{ii}(t) > \sum_{j=1}^{n} \max_{t\in[0,p]} c_{ii}(t) > \sum_{j=1}^{n} (k_{ij} * c_{ij})(t) \quad 0 \leqslant t \leqslant p; \quad i = 1, 2, \ldots, n$$

which further implies that

$$a_{ii}(t) > (k_{ii} * c_{ii})(t) \Rightarrow a_{ii}(t)v_i(t) > v_i(t)(k_{ii} * c_{ii})(t).$$

Since $v_i(t) \geqslant 0$, $i = 1, 2, \ldots, n$ we have

$$[a_{ii}(t)v_i(t)] > [v_i(t)(k_{ii} * c_{ii}(t))], \quad i = 1, 2, \ldots, n. \tag{3.19}$$

But (3.19) contradicts the equality in (3.18). Therefore it follows that

$$v_i(t) \equiv 0 \text{ for } i = 1, 2, \ldots, n$$

and hence

$$u^+(t) = u^-(t), \quad i = 1, 2, \ldots, n.$$

This completes the proof of the existence of a fixed point of the operator B and hence the existence of a positive periodic solution of (1.1). ∎

PART 2.

PROOF OF THE GLOBAL ASYMPTOTIC STABILITY OF THE SOLUTION.

Let the unique positive (componentwise) fixed point of the operator B be denoted by

$$x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t)).$$

It will now be shown that $x^*(t)$ is a globally asymptotically stable solution of the system (1.1) together with (1.2) and (1.3).

Let $x(t) = x_1(t), x_2(t), \dots, x_n(t)$ be any solution of the system (1.1) with $x(t) = \phi(t)$ for $t \leqslant 0$, where $\phi(t) = \{\phi_i(t)\}_{i=1}^n$ with the $\phi_i(t)$ continuous functions on $(-\infty, 0)$. Since all solutions of (1.1) with strictly positive initial conditions remain strictly positive, we let

$$X_i^*(t) = \log x_i^*(t) \text{ and } X_i(t) = \log x_i(t), \quad i = 1, 2, \dots, n. \tag{3.20}$$

Consequently, from (1.1), it follows that:

(3.21)
$$\begin{aligned}\frac{d}{dt}(X_i^*(t) - X_i(t)) &= -a_{ii}(t)\big(\exp(X_i^*) - \exp(X_i(t))\big)\\ &\quad - \sum_{j=1}^n c_{ij}(t)\int_{-\infty}^t k_{ij}(t-s)\big(\exp\big(X_j^*(s)\big) - \exp(X_j(s))\big)ds\\ &\qquad i = 1, 2, \dots, n, \quad t > 0.\end{aligned}$$

We define a Lyapunov function as follows

(3.22)
$$\begin{aligned}v(t) = \sum_{i=1}^n \Big[&|X_i^*(t) - X_i(t)|\\ &+ \sum_{j=1}^n \int_0^\infty k_{ij}(s)\int_{t-s}^t c_{ij}(u+s)\big|\exp\big(X_j^*(u)\big) - \exp(X_j(u))\big|\,du\,ds\Big]\end{aligned}$$

with $t \geqslant 0$.

It follows from (3.22) that $v(t) \geqslant 0$. Since v is continuous and globally Lipschitzian on $\mathbb{R}_n$, the upper right Dini-derivative of v along the trajectories of (3.21) exists. In order to utilise the properties of upper-Dini derivatives we introduce a functional σ as follows. Let z be a continuously differentiable scalar function defined on $[0, \infty]$, then define $\sigma(z)(t)$ as follows:

$$\sigma(z)(t) = \begin{cases} 1, & z(t) \geqslant 0, \quad \dfrac{dz(t)}{dt} > 0\\ 0, & z(t) = 0, \quad \dfrac{dz(t)}{dt} = 0\\ -1, & z(t) \leqslant 0, \quad \dfrac{dz(t)}{dt} < 0.\end{cases} \tag{3.23}$$

It is not difficulty to verify that

$$z(t)\sigma(z)(t) = |z(t)|$$

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and

$$D^{+}|z(t)| = \sigma(z)(t)\frac{dz}{dt} \tag{3.24}$$

where D^{+} denotes the upper right Dini-derivative.

By evaluating the upper right Dini derivative along the solutions of (3.21) we get

$$\begin{aligned}
D^{+}v(t) = \sum_{i=1}^{n}\Big[&\sigma(\exp(X_i^*) - \exp(X_i))(-a_{ii}(t))\big(\exp(X_i^*(t)) \\
&- \exp(X_i(t))\big) - \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)\big(\exp\big(X_j^*(s)\big) - \exp(X_j(s))\big) \\
&\sigma\big(\exp\big(X_j^*(s)\big) - \exp(X_j(s))\big)ds + \sum_{j=1}^{n}\int_0^{\infty} k_{ij}(s)c_{ij}(u+s) \\
&\times \sigma\big(\exp X_j^*(t) - \exp(X_j(t))\big)\Big(\exp\big(X_j^*(t) - \exp(X_j(t))\big)\Big)ds \\
&- \sum_{j=1}^{n}\int_{j=1}^{\infty} k_{ij}(s)c_{ij}(t)\sigma\Big(\exp\big(X_j^*(t-s) - \exp(X_j(t-s))\big)\Big) \\
&\times\big(\exp\big(X_j^*(t-s)\big) - \exp(X_j(t-s))\big)ds\Big].
\end{aligned}$$

Using (3.24),

$$\begin{aligned}
D^{+}v(t) < -\sum_{i=1}^{n}\Big[&|a_{ii}|\,|\exp(X_i^*(t)) - \exp(X_i(t))| \\
&+ \sum_{j=1}^{n} c_{ij}(t)\int_0^{\infty} k_{ij}(s)\left|\exp\big(X_j^*(t-s)\big) - \exp(X_j(t-s))\right| ds \\
&- \sum_{j=1}^{n}\int_0^{\infty} k_{ij}(s)c_{ij}(t+s)\left|\exp\big(X_j^*(t)\big) - \exp(X_j(t))\right| ds \\
&- \sum_{j=1}^{n}\int_0^{\infty} k_{ij}(s)c_{ij}(t)\left|\exp\big(X_j^*(t-s)\big) - \exp(X_j(t-s))\right| ds\Big] \\
= -\sum_{i=1}^{n}\Big[&|a_{ii}|\,|\exp(X_i^*(t)) - \exp(X_i(t))| \\
&- \sum_{j=1}^{n}\left|\exp\big(X_j^*(t)\big) - \exp(X_j(t))\right| \times \int_0^{\infty} k_{ij}(s)c_{ij}(t+s)ds\Big].
\end{aligned}$$

As a consequence of (3.2) and (1.6), we have

$$D^{+}v(t) < -\alpha_1\sum_{i=1}^{n}|x_i^*(t) - x_i(t)|. \tag{3.25}$$

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From the non-negativity of $v(t)$ for $t \geqslant 0$, by integrating (3.25) we have

$$v(t) - v(0) \leqslant -\alpha_1 \int_0^t \sum_{i=1}^n |x_i^*(s) - x_i(s)|\, ds,$$

which implies

(3.26)

$$\sum_{i=1}^n \Big[|X_i^*(t) - X_i(t)| + \sum_{j=1}^n \int_0^\infty k_{ij}(s) \int_{t-s}^t c_{ij}(u+s) \left|\exp\left(X_j^*(u)\right) - \exp\left(X_j(u)\right)\right| du\, ds \Big] + \alpha_1 \int_0^t \sum_{i=1}^n |x_i^*(s) - x_i(s)|\, ds \leqslant v(0) < \infty.$$

It follows from (3.26) that

$$v(t) = \sum_{i=1}^n \Big[|X_i^*(t) - X_i(t)| + \sum_{j=1}^n \int_0^\infty k_{ij}(s) \int_{t-s}^t c_{ij}(u+s) \left|\exp\left(X_j^*(u)\right) - \exp\left(X_j(u)\right)\right| du\, ds \Big]$$

is bounded on $[0, \infty)$, from which is follows that $v(t)$ is uniformly continuous on $[0, \infty)$. It is now claimed that (3.22) implies that $v(t) \to 0$ as $t \to \infty$, implying (3.1).

Now if it is supposed that (3.1) is not true, then there exists a sequence $\{t_m\}$, $(m = 0, 1, 2, \ldots, ;\ t_m \to \infty)$, such that

$$\sum_{i=1}^n |x_i^*(t_m) - x_i(t_m)| > \frac{\varepsilon}{\alpha}; \quad m = 0, 1, 2, \ldots, \tag{3.27}$$

for some small positive number ε. It then follows from (3.25) that

$$D^+ v(t_m) < -\varepsilon \text{ for } m = 0, 1, 2, \ldots. \tag{3.28}$$

If η is a sufficiently small positive number such that

$$t_{m-1} < t_m - \eta < t_m$$

then

$$D^+ v(t) \leqslant (-\varepsilon/2) \quad \text{for } t \in (t_m - \eta, t_m) \text{ and } m = 0, 1, 2, \ldots. \tag{3.29}$$

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Since $v(t)$ is uniformly continuous on $[0, \infty)$ this ensures that η can be chosen to be independent of t_m. It will then follow from (3.29) that

$$v(t_m) - v(t_m - \eta) \leqslant \int_{t_m - \eta}^{t_m} D^+ v(s) ds \leqslant - \left[\frac{\varepsilon}{2}\right] \eta. \tag{3.30}$$

Since v is non-increasing with respect to t, $t \geqslant 0$, from (3.30)

$$\begin{aligned} v(t_m) &\leqslant v(t_{m-1}) - \left[\frac{\varepsilon}{2}\right] \eta \\ &\leqslant v(t_{m-2}) - 2\left[\frac{\varepsilon}{2}\right] \eta \\ &\leqslant v(t_0) - m\left[\frac{\varepsilon}{2}\right] \eta. \end{aligned} \tag{3.31}$$

Equation (3.31) implies that, for large m, $v(t_m)$ will become negative, since $v(t_0)$ is finite. Since $v(t) \geqslant 0$, the implied negativity is a contradiction. Hence it follows that such a sequence $\{t_m\}$ cannot exist. Thus, (3.1) follows, and the proof is complete.

4. Discussion

Since the coefficients $-a_{ii}(t)$ $(i = 1, 2, 3, \dots, n)$ are usually considered to represent the strength of the self-regulating negative feedback effects (intraspecific competition) and $b_i(t)$ $(i = 1, 2, \dots, n)$ the potential growth rates of the n-competing species at time t, we have shown the following: in the case of an n-species Lotka Volterra model with hereditary effects in a periodic environment, the system will be globally asymptotically stable if, besides condition (1.4), the following hold:

(i) The self regulating periodic intraspecific negative feedback effects $-a_{ii}(t)$ are instantaneous and dominate the periodic interspecific interaction effects of all the species (3.2).

(ii) The minimum possible growth rates of the i-th species are stronger than the maximum interspecific effects weighted with the respective sizes of all species when they are near their potential maximum sizes.

In this sense, if

$$\min_{t \in [0,p]} b_i(t) > \sum_{j=1}^{n} \max_{t \in [0,p]} c_{ij}(t) \left[\frac{\max_{t \in [0,p]} b_j(t)}{\min_{t \in [0,p]} a_{jj}(t)}\right]$$

then the i-th species can successfully recover when its population is low in the presence of delayed periodic interspecific interactions of all species when all are at their maximum sizes.

In the case of a constant environment the periodic coefficients become constant, which satisfies the conditions (1.4) – (1.6) and (3.2) – (3.3). Our result implies, in this case, the existence and global asymptotic stability of a positive steady state of the system of integrodifferential equations considered by Gopalsamy [4]:

$$\frac{dx_i}{dt} = x_i(t)\left[b_i - a_{ii}x_i(t) - \sum_{j=1}^{n} c_{ij}\int_{-\infty}^{t} k_{ij}(t-s)x_j(s)ds\right],$$

which is a particular case of the system (1.1).

5. Example

The coupled first order differential equations given by (1.1) for the case $n = 3$ were integrated numerically by using a standard Runge-Kutta-Fehlberg method. The numerical values used are listed below:

$$\begin{aligned}
\frac{dx_1}{dt} = x_1(t)\Big[&(14 + 0.5\cos\pi t) - (7 + 0.5\cos\pi t)x_1(t)\\
&- (0.7 + 0.5\cos\pi t)\int_{-\infty}^{t} \exp(-(t-s))x_1(s)ds\\
&- (0.8 + 0.5\sin\pi t)\int_{-\infty}^{t} \exp(-(t-s))x_2(s)ds\\
&-(0.6 + 035\sin\pi t)\int_{-\infty}^{t} \exp(-(t-s))x_3(s)ds\Big]\\
\frac{dx_2}{dt} = x_2(t)\Big[&(12 + 0.5\sin\pi t) - (6 + 0.4\cos\pi t)x_2(t)\\
&- (0.8 + 0.5\cos\pi t)\int_{-\infty}^{t} \exp(-(t-s))x_1(s)ds\\
&- (0.7 + 0.4\sin\pi t)\int_{-\infty}^{t} \exp(-(t-s))x_2(s)ds\\
&- (0.8 + 0.5\cos\pi t)\int_{-\infty}^{t} \exp(-(t-s))x_3(s)ds\Big]\\
\frac{dx_3}{dt} = x_3(t)\Big[&(9 + 0.5\sin\pi t) - (6.5 + 0.3\sin\pi t)x_3(t)\\
&- (0.4 + 0.3\sin\pi t)\int_{-\infty}^{t} \exp(-(t-s))x_1(s)ds\\
&- (0.5 + 0.4\sin\pi t)\int_{-\infty}^{t} \exp(-(t-s))x_2(s)ds
\end{aligned}$$

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$$-(0.4+0.3\sin\pi t)\int_{-\infty}^{t}\exp\left(-(t-s)\right)x_3(s)ds\Big].$$

The initial conditions, chosen arbitrarily were:

$$x_1(s)=0.6 \quad (-\infty<s<0)$$
$$x_2(s)=0.62$$

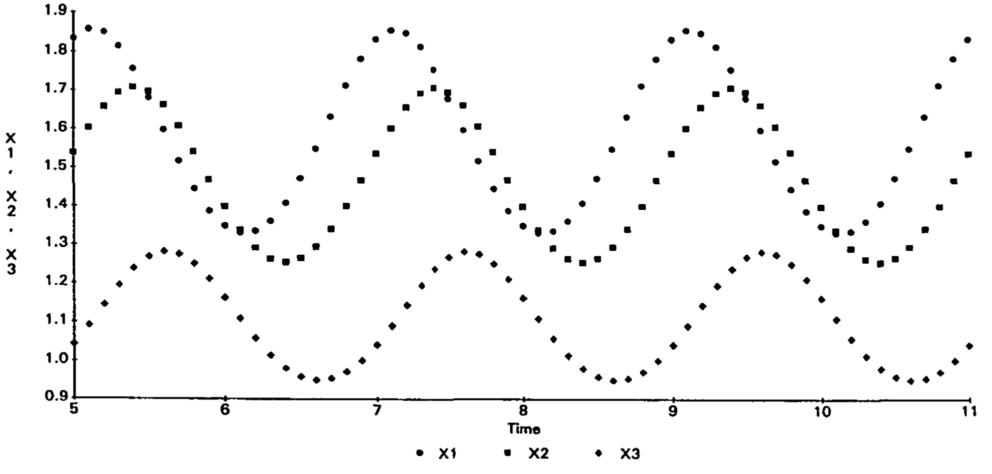

Figure 1. $X1$, $X2$, $X3$, as a function of time.

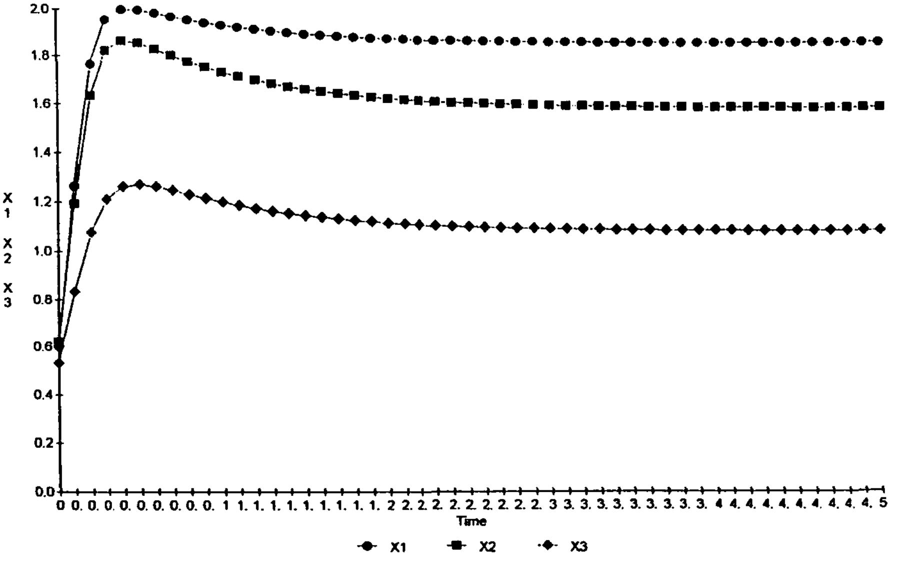

Figure 2. $X1$, $X2$, $X3$, as a function of time (constant coefficients case).

Figure 1 displays the results and shows the globally asymptotically stable nature of the system comprised of individually periodic solutions. Figure 2 illustrates the case for the same system of differential equations with constant coefficients, obtained from the original system with the period being set equal to zero.

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