A NEW FORMULATION OF THE EQUATIONS GOVERNING THE PULSAR MAGNETOSPHERE

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The relatively slow progress in both analytical and numerical work on the structure of the pulsar magnetosphere calls for a reformulation of the equations governing the magnetosphere. Since the set of equations for the pulsar magnetosphere are nonlinear anyway, one should not hesitate to use nonlinear representations for the field intensities and the current from the very beginning. Those nonlinear representations can be chosen to fulfill certain equations automatically and are therefore more adapted to the problem. The aligned rotator was formulated to some extent in this way by Schmalz et al. (1979). We formulate here the general problem and use for this the calculus of differential forms and some theorems concerning the normal form of forms (= representation of a form with a minimal number of scalar functions and their gradients). In the language of forms Maxwell's equations read

$$dF = 0 \qquad , \qquad d{}^{*}F = 4\pi j \tag{1}$$

where F is the field intensity 2-form and j the current 3-form. The first equation states that F can be derived from a 1-form A: F = dA, the usual vector potential. A consequence of the second equation is dj = 0, the continuity equation for the current. Now since j is a simple closed 3-form, it can be represented by three scalar functions ϕ, χ, ψ in the following way

$$j = d\phi \wedge d\chi \wedge d\psi \tag{2}$$

which obviously satisfies dj = 0. This is the normal form of the 3-form with the properties mentioned above.

Next consider the equation of motion in terms of the four-velocity 3-form u

$$^{*}[{}^{*}d(A - \frac{m_o}{e}\,{}^{*}u)\wedge{}^{*}u] = {}^{*}[{}^{*}\tilde{F}\wedge{}^{*}u] = 0 \ . \tag{3}$$

The introduction of the 2-form $\tilde{F}$ puts the equation of motion into a form

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which is similar to the force-free equation (Heintzmann and Schrüfer, 1977). j and u are related by $j = u\|j\|$, $\|j\| = \{-{}^*(j\wedge{}^*j)\}^{1/2}$ for a one-component plasma considered here for simplicity, so that (3) can also be written as ${}^*({}^*\tilde{F}\wedge{}^*j) = 0$. Expansion of this equation gives

$$d\phi\,{}^*[\tilde{F}\wedge d\chi\wedge d\psi] - d\chi\,{}^*[\tilde{F}\wedge d\phi\wedge d\psi] + d\psi\,{}^*[\tilde{F}\wedge d\phi\wedge d\chi] = 0 \; . \qquad (3a)$$

Since ϕ,χ,ψ are independent functions the square brackets have to vanish, resulting in three scalar equations of motion (see also Dougherty, 1974). From eq. (3) it follows that $\tilde{F}$ has rank 2 and can therefore be written in its normal form (Darboux theorem)

$$\tilde{F} = dG\wedge dH \qquad \text{or} \qquad \tilde{A} = GdH + d\Gamma \; . \qquad (4)$$

Putting this into eq. (3a), we have the following relations $dG\wedge dH\wedge d\chi\wedge d\psi = 0$, $dG\wedge dH\wedge d\phi\wedge d\psi = 0$, $dG\wedge dH\wedge d\phi\wedge d\chi = 0$, which mean that G and H are dependent only on ϕ,ψ,χ. The vector potential can now be written as

$$A = \tilde{A} + \frac{m_o}{e}\,{}^*j/\|j\| = G(\phi,\chi,\psi)dH(\phi,\chi,\psi) + d\Gamma + \frac{m_o}{e}\,\frac{{}^*(d\phi\wedge d\chi\wedge d\psi)}{\|d\phi\wedge d\chi\wedge d\psi\|} \; . \qquad (5)$$

This functional form for the vector potential guarantees that the equation of motion, the continuity equation and the first Maxwell equation are automatically fulfilled.

The remaining inhomogenous Maxwell equation can just be obtained in terms of ϕ,χ,ψ by direct insertion. However, one can also derive equivalent equations from a Lagrangian 4-form L with dynamical variables ϕ,ψ,χ. Three unconstrained scalar equations result. The variational formulation has the advantage that the full apparatus of field theory is available. By Noether's theorem, for example, first integrals can be constructed to any given symmetry. For the numerical treatment variational finite element methods can be directly applied.

The whole problem is now formulated in terms of ϕ,χ,ψ. We can add another independent function Θ to these three functions, which together can serve as coordinates. The role of the space-time coordinates and the set of functions ϕ,χ,ψ,Θ can therefore be exchanged. The transformed Lagrangian then leads to differential equations with solutions $x^\mu = x^\mu(\phi,\chi,\psi,\Theta)$. This feature is very attractive in view of the possibility of an unconnected or finite pulsar magnetosphere that would otherwise be very hard to calculate.

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