VON NEUMANN'S MANUSCRIPT ON INDUCTIVE LIMITS OF REGULAR RINGS

ISRAEL HALPERIN

1. Introduction. It is now known **(3)** that if $\mathfrak{R}$ is a regular rank ring, then the rank function can be extended to the matrix ring $\mathfrak{R}_n$ in such a way that $R(a) = R(a \otimes n)$; here, a is an arbitrary element of $\mathfrak{R}$, $a \otimes n$ is the $n \times n$ diagonal matrix with a for each entry on the diagonal, and R denotes rank in $\mathfrak{R}$ and also in $\mathfrak{R}_n$. It is also known **(2)** that every regular rank ring has a rank-metric completion which is again a regular rank ring. Thus von Neumann's procedure of forming inductive limits applies to an arbitrary regular rank ring $\mathfrak{R}$; one begins with an arbitrary factor sequence $\mu = (n_i)$ and constructs first the matrix ring $\mathfrak{R}_t$ for each $t \in \mu$, then the inductive limit $\mathfrak{R}_\mu$, then the completion $(\mathfrak{R}_\mu)^\wedge$.

In a manuscript written in 1936–37, J. von Neumann proved the following two theorems *for the case when* $\mathfrak{R}$ *is a division ring* (*skew-field*):

THEOREM 1. *If* μ *and* γ *are factor sequences, then the rings* $(\mathfrak{R}_\mu)^\wedge$ *and* $(\mathfrak{R}_\gamma)^\wedge$ *are isomorphic.*

THEOREM 2. *If* μ *is a factor sequence and* $e \neq 0$ *is an idempotent in* $(\mathfrak{R}_\mu)^\wedge$, *then the rings* $e((\mathfrak{R}_\mu)^\wedge)e$ *and* $(\mathfrak{R}_\mu)^\wedge$ *are isomorphic.*

Throughout this note, *isomorphism* means ring isomorphism.

Von Neumann's proof of Theorems 1 and 2 (for the case of a division ring) has not been published previously although an abstract appeared in **(5)**. The present note will give a detailed exposition of von Neumann's proof; however, the writer has freely changed the arrangement and notation, has made minor alterations in the argument, and has inserted the Lemmas 3 and 4 and the Remarks 1 and 2 below. Theorems 1 and 2 will be proved here for every regular rank ring $\mathfrak{R}$ with unit for which $\mathfrak{R}^\wedge$ is irreducible (then necessarily $\mathfrak{R}^\wedge$ is an irreducible, discrete or continuous rank ring).

2. Preliminaries. An infinite sequence of integers $\mu = (n_i)$ will be called a *factor sequence* if

(i) $n_1 \geqslant 1$,

(ii) for each $i \geqslant 1$: $n_{i+1} = k_i n_i$ for some integer $k_i \geqslant 1$,

(iii) $n_i \to \infty$ when $i \to \infty$.

μ will be called *complete* if also every integer $m \geqslant 1$ is a divisor of some member of μ.

Received August 15, 1966.

If $\mathfrak{R}$ is an arbitrary (associative) ring, then the matrix ring $\mathfrak{R}_n$, the product $A \otimes q$, the mapping $\phi_{kn,n}$: $\mathfrak{R}_n \to \mathfrak{R}_{kn}$ and the inductive limit

$$\mathfrak{R}_\mu = \varinjlim (\mathfrak{R}_{n_i}, \phi_{k_i n_i, n_i})$$

are defined as in **(1)**. If n is a member of a factor sequence μ and $A \in \mathfrak{R}_n$, we shall write $A(\mu)$ to denote the element in $\mathfrak{R}_\mu$ determined by A (that is, the equivalence class of A).

If μ is a subsequence of a factor sequence γ, the rule $A(\mu) \to A(\gamma)$ clearly determines an isomorphism of $\mathfrak{R}_\mu$ onto $\mathfrak{R}_\gamma$. Thus $\mathfrak{R}_\mu$ and $\mathfrak{R}_\gamma$ are isomorphic whenever μ and γ possess a common infinite subsequence, hence, whenever: each member of μ is a divisor of some member of γ and conversely each member of γ is a divisor of some member of μ, in particular, whenever both μ and γ are complete.

If $\mathfrak{R}$ is a regular rank ring, as defined in **(4)**, then each $\mathfrak{R}_n$ is a regular rank ring with a rank function which is an extension to $\mathfrak{R}_n$ of the rank function of $\mathfrak{R}$ (see **3**); with respect to this rank function, each $\phi_{kn,n}$ is rank preserving and $\mathfrak{R}_\mu$ is a regular rank ring. As shown in **(2)**, $\mathfrak{R}_\mu$ has a metric completion, denoted $(\mathfrak{R}_\mu)^\wedge$, which is a complete regular rank ring. If $\mathfrak{R}$ has a unit element, then so do all $\mathfrak{R}_n$, $\mathfrak{R}_\mu$, and $(\mathfrak{R}_\mu)^\wedge$, and we may suppose that all rank functions are normalized by the condition $R(1) = 1$. We shall, without fear of ambiguity, use the same letter R to denote the different rank functions.

3. Outline of the proof. In §4, we shall prove the following four lemmas ($\mathfrak{R}$ will denote a regular rank ring *with unit* and θ will denote an arbitrary real number satisfying $0 < \theta < 1$):

LEMMA 1. *Suppose that $\mu = (n_i)$ and $p = (p_i)$ are factor sequences satisfying*

$$1 \geqslant \frac{p_1}{n_1} \geqslant \frac{p_2}{n_2} \geqslant \ldots \geqslant \frac{p_i}{n_i} \geqslant \frac{p_{i+1}}{n_{i+1}} \geqslant \ldots,$$

$$\lim_{i\to\infty} \frac{p_i}{n_i} = \theta,$$

then there exists an idempotent $e \in (\mathfrak{R}_\mu)^\wedge$ with $R(e) = \theta$ and an isomorphism of $(\mathfrak{R}_p)^\wedge$ onto $e((\mathfrak{R}_\mu)^\wedge)e$.

LEMMA 2. If $\mu = (n_i)$ *is a factor sequence, there exists a subsequence* γ of μ, *say $\gamma = (m_i)$, and a complete sequence $p = (p_i)$ such that*

$$1 \geqslant \frac{p_1}{m_1} \geqslant \frac{p_2}{m_2} \geqslant \ldots \geqslant \frac{p_i}{m_i} \geqslant \frac{p_{i+1}}{m_{i+1}} \geqslant \ldots$$

and
$$\lim_{i\to\infty} \frac{p_i}{m_i} = \theta.$$

LEMMA 3.

(i) *If* $\mathfrak{R}$ *is irreducible and complete in the rank metric, then* $\mathfrak{R}$ *is an irreducible, discrete or continuous rank ring.*

(ii) $\mathfrak{R}^\wedge$ *is irreducible if and only if* $\mathfrak{R}$ *has the property:*

(P) *if* f *is an idempotent in* $\mathfrak{R}$ *and* $R(f) \leqslant R(1-f)$, *then*

$$\sup_{x \in \mathfrak{R}} R(fx(1-f)) = R(f).$$

(iii) *If* $\mathfrak{R}^\wedge$ *is irreducible (in particular, if* $\mathfrak{R}$ *is any division ring, more generally any irreducible, discrete or continuous rank ring) and* μ *is a factor sequence, then* $(\mathfrak{R}_\mu)^\wedge$ *is an irreducible continuous rank ring.*

LEMMA 4. *Suppose that* $\mathfrak{R}$ *is an irreducible, discrete or continuous rank ring. Then*

(i) *If* f *is an idempotent in* $\mathfrak{R}$ *with* $R(f) = \frac{1}{2}$, *then* $\mathfrak{R}$ *is isomorphic to* $(f\mathfrak{R}f)_2$ *(this is the ring of* 2×2 *matrices with entries in* $f\mathfrak{R}f$*).*

(ii) *If* f_1 *and* f_2 *are idempotents of equal rank, then* $f_1\mathfrak{R}f_1$ *and* $f_2\mathfrak{R}f_2$ *are isomorphic.*

Assuming that these four lemmas have been established, we consider a factor sequence μ and an arbitrary regular rank ring $\mathfrak{R}$ with unit for which $\mathfrak{R}^\wedge$ is an irreducible ring (hence, by Lemma 3(iii), $(\mathfrak{R}_\mu)^\wedge$ is an irreducible continuous ring) and we argue as follows:

1. For some idempotent f in $(\mathfrak{R}_\mu)^\wedge$ with $R(f) = \frac{1}{2}$, the ring $f((\mathfrak{R}_\mu)^\wedge)f$ is isomorphic to $(\mathfrak{R}_p)^\wedge$ for some (and hence every) complete factor sequence p (use Lemma 2 and Lemma 1 with $\theta = \frac{1}{2}$).

2. Since $(\mathfrak{R}_\mu)^\wedge$ is an irreducible continuous ring, and $R(f) = \frac{1}{2}$, $(\mathfrak{R}_\mu)^\wedge$ is isomorphic to $(f((\mathfrak{R}_\mu)^\wedge)f)_2$ (use Lemma 4(i)) and hence to $((\mathfrak{R}_p)^\wedge)_2$. This implies that Theorem 1 holds for such $\mathfrak{R}$.

3. If $e \neq 0, 1$, then $(\mathfrak{R}_p)^\wedge$ is isomorphic to $f((\mathfrak{R}_\mu)^\wedge)f$ for some idempotent f in $(\mathfrak{R}_\mu)^\wedge$ with $R(f) = R(e)$ (use Lemma 2 and Lemma 1 with $\theta = R(e)$). Then $e((\mathfrak{R}_\mu)^\wedge)e$ is isomorphic to $f((\mathfrak{R}_\mu)^\wedge)f$ (use Lemma 4(ii)), hence to $(\mathfrak{R}_p)^\wedge$, hence to $(\mathfrak{R}_\mu)^\wedge$ (use Theorem 1). This proves Theorem 2 for the case $e \neq 1$ (when $e = 1$, Theorem 2 holds trivially).

4. Proofs of Lemmas 1, 2, 3, 4. We shall prove these lemmas in reverse order.

Proof of Lemma 4(i). The principal right ideals $f\mathfrak{R}$ and $(1-f)\mathfrak{R}$ form a homogeneous basis of order 2 in the lattice of all principal right ideals of $\mathfrak{R}$. The conclusion of Lemma 4(i) now follows from (**4**, Part II, Lemma 3.6).

Proof of Lemma 4(ii). The principal right ideals $f_1\mathfrak{R}$ and $f_2\mathfrak{R}$ have equal dimension, and hence are perspective. The proof of (**4**, Part II, Theorem 15.3(a)) now shows that there exist idempotents e, f in $\mathfrak{R}$ such that

$$e\mathfrak{R} = f_1\mathfrak{R}, \qquad f\mathfrak{R} = f_2\mathfrak{R}, \qquad \text{and } ef = e, fe = f.$$

Then

$$(ef_2)(ff_1) = e(f_2f)f_1 = eff_1 = ef_1 = f_1,$$
$$(ff_1)(ef_2) = f(f_1e)f_2 = fef_2 = ff_2 = f_2.$$

Hence $f_1 \mathfrak{R} f_1$ and $f_2 \mathfrak{R} f_2$ are isomorphic by (**4**, Part II, Lemma 15.8).

Proof of Lemma 3(i). This is shown in (**4**, Part II, Theorem 18.1).

Proof of Lemma 3(ii). Suppose first that property (P) holds in $\mathfrak{R}$. If $\mathfrak{R}^\wedge$ is *not* irreducible, then in $\mathfrak{R}^\wedge$ there exists some central idempotent $e \neq 0, 1$. By (**2**, Proof of Theorem 3.7(ii)) there exists an idempotent $f \in \mathfrak{R}$ such that $R(f - e) < R(e)/5$. Then there exists an x in $\mathfrak{R}$ such that at least one of the following relations holds:

$$R(fx(1 - f)) > \tfrac{1}{2}R(f) \quad \text{or} \quad R((1 - f)xf) > \tfrac{1}{2}R(1 - f).$$

Therefore, either

$$\begin{aligned} R(ex(1 - e)) &\geqslant R(fx(1 - f)) - R((f - e)x(1 - f)) - R(ex((1 - f) - (1 - e))) \\ &> \tfrac{1}{2}R(f) - 2R(f - e) \geqslant \tfrac{1}{2}\{R(e) - 5R(f - e)\} > 0, \end{aligned}$$

or (similar calculation) $R((1 - e)xe) > 0$. This contradicts the fact that $ex(1 - e) = 0 = (1 - e)xe$ (since e is in the centre of $\mathfrak{R}^\wedge$). This shows that $\mathfrak{R}^\wedge$ must be irreducible.

Next suppose that $\mathfrak{R}^\wedge$ is irreducible, and that f is an idempotent in $\mathfrak{R}$ such that $R(f) \leqslant R(1 - f)$. Then f is an idempotent in the irreducible, discrete or continuous rank ring $\mathfrak{R}^\wedge$. It follows from (**4**, Part I, Theorem 6.9(iii)) that $R(f) = R(g)$ for some idempotent g such that $g = (1 - f)g(1 - f)$. Then, as in the proof of Lemma 4(ii) above, there exist idempotents p, q in $\mathfrak{R}^\wedge$ such that

$$p\mathfrak{R} = f\mathfrak{R}, \qquad q\mathfrak{R} = g\mathfrak{R}, \qquad \text{and } pq = p,\ qp = q.$$

Then $fp(1 - f)gq = fpgq = fpq = p$. Therefore

$$R(f) \geqslant R(fp(1 - f)) \geqslant R(fp(1 - f)gq) = R(p) = R(f).$$

Hence $R(fx(1 - f)) = R(f)$ if $x = p \in \mathfrak{R}^\wedge$. Since $x = \lim x_n$ for suitable $x_n \in \mathfrak{R}$, it follows that

$$\sup_{x \in \mathfrak{R}} R(fx(1 - f)) = R(f).$$

Proof of Lemma 3(iii). If $(\mathfrak{R}_\mu)^\wedge$ is irreducible, then by Lemma 3(i) above, it is an irreducible, discrete or continuous rank ring. The first alternative (discrete) cannot hold since $(\mathfrak{R}_\mu)^\wedge$ contains an infinite sequence of orthogonal non-zero idempotents.

To show that $(\mathfrak{R}_\mu)^\wedge$ is irreducible, it is sufficient, by Lemma 3(ii), to show that $\mathfrak{R}_\mu$ has property (P). Hence it is sufficient to show that for each integer n, $\mathfrak{R}_n$ has property (P).

Since $(\mathfrak{R}_n)^\wedge$ is isomorphic to $(\mathfrak{R}^\wedge)_n$, it is sufficient to show that $(\mathfrak{R}^\wedge)_n$ is irreducible; but it is easy to see that $\mathfrak{S}_n$ is irreducible whenever $\mathfrak{S}$ is an irreducible regular ring. This completes the proof of Lemma 3(iii).

Proof of Lemma 2. Set $\sigma(1) = 1$, $m_1 = n_{\sigma(1)} = n_1$, and $p_1 = 1$. To prove Lemma 2 it is clearly sufficient to define $\sigma(i)$, $m_i = n_{\sigma(i)}$ and p_i, for $i > 1$, to satisfy the conditions:

$$\sigma(i) > \sigma(i-1),$$

$$ip_{i-1} \text{ is a divisor of } p_i,$$

$$\frac{1}{2}\left(\frac{p_{i-1}}{m_{i-1}} - \theta\right) > \frac{p_i}{m_i} - \theta > 0.$$

For this purpose we use induction on i. First we choose $\sigma(i)$ so large that

$$\sigma(i) > \sigma(i-1) \quad \text{and} \quad \frac{1}{n_{\sigma(i)}} < \frac{1}{2ip_{i-1}}\left(\frac{p_{i-1}}{m_{i-1}} - \theta\right).$$

Then we choose $m_i = n_{\sigma(i)}$, and h_i to be the smallest integer for which $ip_{i-1}h_i/m_i - \theta > 0$ and $p_i = ip_{i-1}h_i$. This completes the proof of Lemma 2.

Proof of Lemma 1. We define the symbol $[g, k]$, for positive integers $g \leqslant k$, to denote the $k \times k$ matrix which has 1 for (t, t)-entry $(1 \leqslant t \leqslant g)$ and 0 for all other entries.

We set $g_0 = p_1$, $k_0 = n_1$, and for $i \geqslant 1$: $g_i = p_{i+1}/p_i$, $k_i = n_{i+1}/n_i$. We remark that every integer $t \geqslant 1$ has a unique representation of the form

$$t = 1 + \alpha_0 + \alpha_1 p_1 + \ldots + \alpha_i p_i + \ldots \qquad \text{with } 0 \leqslant \alpha_i < g_i \text{ for } i \geqslant 0$$

and we define $\psi(t)$ to be the integer

$$\psi(t) = 1 + \alpha_0 + \alpha_1 n_1 + \ldots + \alpha_i n_i + \ldots .$$

It is clear that for each $i \geqslant 1$, ψ maps the set of integers $1 \leqslant t \leqslant p_i$ injectively onto a certain subset of the set $1 \leqslant t \leqslant n_i$.

If now $A \in \mathfrak{R}_{p_i}$ for some $i \geqslant 1$, we define $\bar{\psi}(A)$ to be the following $n_i \times n_i$ matrix: if $1 \leqslant s, t \leqslant p_i$, then $\bar{\psi}(A)$ shall have the (s, t)-entry of A for its $(\psi(s), \psi(t))$-entry and $\bar{\psi}(A)$ shall have 0 for all other entries.

It is easy to see that if $A \in \mathfrak{R}_{p_i}$, then

$$R(\bar{\psi}(A)) = \frac{p_i}{n_i} R(A), \qquad \bar{\psi}(A \otimes g_i) = \bar{\psi}(A) \otimes [g_i, k_i],$$

and if $j \geqslant h \geqslant i$, then

$$\begin{aligned} R(\bar{\psi}(A \otimes g_i \otimes \ldots \otimes g_{j-1})) &= R(\bar{\psi}(A)) \frac{g_i}{k_i} \cdots \frac{g_{j-1}}{k_{j-1}} \\ &= R(A) \frac{p_i}{n_i} \frac{g_i}{k_i} \cdots \frac{g_{j-1}}{k_{j-1}} = \frac{p_j}{n_j} R(A) \end{aligned}$$

and

$$\begin{aligned} R(((\bar{\psi}(A \otimes g_i \otimes \ldots \otimes g_{h-1}))(\mu) - (\bar{\psi}(A \otimes g_i \otimes \ldots \otimes g_{j-1}))(\mu)) \\ = \left(\frac{p_h}{n_h} - \frac{p_j}{n_j}\right) R(A). \end{aligned}$$

Hence, when $j \to \infty$, $(\bar{\psi}(A \otimes g_i \otimes \ldots \otimes g_{j-1}))(\mu)$ converges to a limit in $(\mathfrak{R}_\mu)^\wedge$, to be denoted by $\bar{\bar{\psi}}(A)$.

It is clear that $R(\bar{\bar{\psi}}(A)) = \theta R(A)$. It is also clear that $\bar{\bar{\psi}}(A) = \bar{\bar{\psi}}(B)$ if $A(p) = B(p)$. Hence we may define a mapping ϕ: $\mathfrak{R}_p \to (\mathfrak{R}_\mu)^\wedge$ by the rule $\phi(x) = \bar{\bar{\psi}}(A)$ if $A(p) = x \in \mathfrak{R}_p$. Since $\bar{\bar{\psi}}$ preserves differences and products, it follows that ϕ is a ring homomorphism. Since $R(\phi(x)) = \theta R(x)$ and $\theta < \infty$, it follows that the mapping ϕ has a unique extension (to be denoted again by ϕ) which is a ring *isomorphism* of $(\mathfrak{R}_p)^\wedge$ onto a subring $\mathfrak{S}$ of $(\mathfrak{R}_\mu)^\wedge$.

To complete the proof of Lemma 1 we need only show that $\mathfrak{S} = e((\mathfrak{R}_\mu)^\wedge)e$ for some idempotent e in $(\mathfrak{R}_\mu)^\wedge$ satisfying $R(e) = \theta$.

We define e_1 to be the matrix $[g_0, k_0]$ and for $i \geqslant 1$ we define e_{i+1} by induction to satisfy

$$e_{i+1} = e_i \otimes [g_i, k_i].$$

It is easy to see that each e_i is an indempotent, $R(e_i) = p_i/n_i$, and if $j > i$, then

$$R(e_i(\mu) - e_j(\mu)) = \frac{p_i}{n_i} - \frac{p_j}{n_j}.$$

Hence $e_i(\mu)$ is convergent in $(\mathfrak{R}_\mu)^\wedge$ to some idempotent e such that $ee_i(\mu) = e_i(\mu)e = e$ for all i, $R(e) = \lim_{i\to\infty} R(e_i) = \theta$, $e = \bar{\bar{\psi}}(A)$ if A is the unit matrix in $\mathfrak{R}_{p_i}$,

$$\phi(x) = e\phi(x) = \phi(x)e \qquad \text{for all } x \in (\mathfrak{R}_p)^\wedge.$$

Thus $\mathfrak{S} \subset e((\mathfrak{R}_\mu)^\wedge)e$, and it is sufficient to show that $\phi(\mathfrak{R}_p)$ is dense in $e((\mathfrak{R}_\mu)^\wedge)e$.

Suppose that $y \in (\mathfrak{R}_\mu)^\wedge$. For any $\epsilon > 0$ we can choose i so large that $R(y - A(\mu)) < \epsilon/4$ for some $A \in \mathfrak{R}_{n_i}$. We can also suppose that i is so large that $R(e_i(\mu) - e) < \epsilon/4$. Then

$$\begin{aligned} R(eye - (e_i A e_i)(\mu)) &\leqslant R(eye - e_i(\mu)ye_i(\mu)) \\ &\quad + R(e_i(\mu)ye_i(\mu) - (e_i A e_i)(\mu)) \\ &\leqslant 2R(e_i(\mu) - e) + R(y - A(\mu)) \leqslant \tfrac{2}{4}\epsilon + \tfrac{1}{4}\epsilon = \tfrac{3}{4}\epsilon. \end{aligned}$$

On the other hand, $e_i A e_i$ is of the form $\bar{\psi}(B)$ for some $B \in \mathfrak{R}_{p_i}$ and

$$\begin{aligned} R((e_i A e_i)(\mu) - \phi(B(p))) &\leqslant \left(\frac{p_i}{n_i} - \theta\right) R(B) \leqslant \frac{p_i}{n_i} - \theta \\ &= R(e_i(\mu) - e) < \epsilon/4. \end{aligned}$$

Thus $R(eye - \phi(B(p))) < \epsilon$. This shows that $\phi(\mathfrak{R}_p)$ is dense in $e((\mathfrak{R}_\mu)^\wedge)e$ and completes the proof of Lemma 1.

5. Remarks.

Remark 1. Each $\mathfrak{R}_n$, $\mathfrak{R}_\mu$, and $(\mathfrak{R}_\mu)^\wedge$ is a left $\mathfrak{R}$-module and a right $\mathfrak{R}$-module. In Lemma 1 above, the ring $e((\mathfrak{R}_\mu)^\wedge)e$ is also a left and right $\mathfrak{R}$-module and the ring isomorphism of $(\mathfrak{R}_p)^\wedge$ onto $e((\mathfrak{R}_\mu)^\wedge)e$ is also an isomorphism of left and right $\mathfrak{R}$-modules. But it is not known whether in Theorem 1 the ring isomorphism can be chosen to be an isomorphism of left and right $\mathfrak{R}$-modules.

Remark 2. Suppose that $\mathfrak{R}$ is a finite or countably infinite direct sum of regular rank rings with unit, say $\mathfrak{R} = \Sigma_t \oplus \mathfrak{R}_t$. Suppose that $R(1) = 1$ for the unit in each $\mathfrak{R}_t$ and let $0 < d_t \leqslant 1$ be such that $\Sigma_t d_t = 1$. For each $x = \Sigma_t \oplus x_t$ in $\mathfrak{R}$ let $R(x) = \Sigma_t d_t R(x_t)$. Then $\mathfrak{R}$ is a regular rank ring with unit, $\mathfrak{R}\hat{\ } = \Sigma_t \oplus (\mathfrak{R}_t)\hat{\ }$, and $(\mathfrak{R}_\mu)\hat{\ } = \Sigma_t \oplus ((\mathfrak{R}_t)_\mu)\hat{\ }$. It follows that Theorem 1 holds for this ring $\mathfrak{R}$ if each $(\mathfrak{R}_t)\hat{\ }$ is irreducible. It is not known whether Theorem 1 holds for *every* regular rank ring with unit.

Added in proof (February 8, 1968). Theorems 1 and 2 for the case $\mathfrak{R}$ is a division ring are treated in the paper by B. Chernishov, CR-*rings and their isomorphisms*, Siberian Math. J., *7* (1966), 1168–1193.

References

1. Brian P. Dawkins and Israel Halperin, *The isomorphism of certain continuous rings*, Can. J. Math., *18* (1966), 1333–1344.

2. Israel Halperin, *Regular rank rings*, Can. J. Math., *17* (1965), 709–719.

3. ——— *Extension of the rank function*, Studia Math. *27* (1966), 325–335.

4. John von Neumann, *Continuous geometry* (Princeton, 1960).

5. ——— *Independence of F_∞ from the sequence γ*, unpublished manuscript written in 1936–37 (review by I. Halperin in Vol. IV of the Collected Works of John von Neumann (Pergamon, 1962)).

University of Toronto