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The Two-Step Nilpotent Representations of the Extended Affine Hecke Algebra of Type A

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Abstract. We define a set of 'cell modules' for the extended affine Hecke algebra of type A which are parametrised by $SL_n(\mathbb{C})$-conjugacy classes of pairs (s, N), where $s \in SL_n(\mathbb{C})$ is semisimple and N is a nilpotent element of the Lie algebra which has at most two Jordan blocks and satisfies $\mathrm{Ad}(s) \cdot N = q^2 N$. When $q^2 \neq -1$, each of these has irreducible head, and the irreducible representations of the affine Hecke algebra so obtained are precisely those which factor through its Temperley–Lieb quotient. When $q^2 = -1$, the above remarks apply to a subset of the cell modules. Using our work on the cellular nature of those quotients, we are able to obtain complete information on the decomposition of the cell modules in all cases, even when q is a root of unity. They turn out to be multiplicity free, and the composition factors may be precisely described in terms of a partial order on the pairs (s, N). These results give explicit formulae for the dimensions of the irreducibles. Assuming our modules are identified with the 'standard modules' earlier defined by Bernstein–Zelevinski, Kazhdan–Lusztig and others, our results may be interpreted as the determination of certain Kazhdan–Lusztig polynomials. [This has now been proved and will appear in a subsequent work of the authors.]

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Introduction

The extended affine Hecke algebra $\widetilde{H_n^a}(q)$ of type A has a quotient which is analogous to the Temperley–Lieb algebra, in that it is obtained from $\widetilde{H_n^a}(q)$ by taking the quotient by the principal ideal generated by the central idempotent in any Hecke subalgebra of type A_2 (which is 6-dimensional) which corresponds to the trivial representation of that subalgebra. Denote this quotient by $\widetilde{TL_n^a}(q)$ (the 'extended affine Temperley–Lieb algebra'). The results of [GL2] may be applied to classify all irreducible representations of $\widetilde{TL_n^a}(q)$ for arbitrary values of the parameter q, including arbitrary roots of unity. This provides a class of irreducible representations for $\widetilde{H_n^a}(q)$, and for these, we are able to give explicit dimension formulae, as well as

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decomposition numbers for the 'cell modules' which are involved in their construction. Although it is not proved here, there is strong evidence that our cell modules coincide with the 'standard modules' defined earlier by several authors (see below).

Our parametrisation of both the cell modules and the irreducibles may be stated (see Theorem (5.5) below) in terms of semisimple-nilpotent pairs (s, N) associated with $SL_n(\mathbb{C})$, just as in the general 'standard module' theory of Lusztig, Ginzburg and Kazhdan and Lusztig (see [CG], [KL] or [X]) or of Bernstein–Zelevinsky (cf. [R], [BZ], [Z2]), except that we deal only with the case where N is a two-step nilpotent matrix (i.e. it has just 1 or 2 Jordan blocks). If we assume that our cell modules coincide with either of the standard modules above, then given the results of [Z2] and [R] or [G], our results also give explicit computations of some Kazhdan–Lusztig polynomials.

One of the results we prove in this work is that in the Grothendieck ring of finite dimensional representations of the algebras $T^a(n)$ of [GL2], each irreducible representation is a linear combination of the 'cell modules' (analogous to Weyl modules), in which the nonzero coefficients are 1, extending the results of [GL, §5]. We also discuss consequences of this result for the algebras $\widetilde{H_n^a}(q)$.

1. Dramatis Personae; Various Algebras of 'Type A'

Let R be a commutative ring and let q be an invertible element of R. The affine Hecke algebra of type A_{n-1} over R, denoted $H_n^a(q)$, has generators $T_1, \dots, T_n$, which generate $H_n^a(q)$ as associative algebra subject to the relations

$$\begin{aligned} &T_iT_j = T_jT_i \quad \text{if } |i-j| \geqslant 2 \quad \text{and} \quad \{i, j\} \neq \{1, n\}, \\ &T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \end{aligned} \tag{1.1}$$

and

$$(T_i - q)(T_i + q^{-1}) = 0, \tag{1.2}$$

where in (1.1) the indices are taken mod n.

The algebra $H_n^a(q)$ corresponds to the affine Weyl group W^a of type $\tilde{A}_{n-1}$, which is a Coxeter group; it has an R-basis consisting of elements $\{T_w, w \in W^a\}$, where if $w = r_{i_1}, \dots, r_{i_\ell}$ is a reduced expression for $w \in W^a$ (the simple reflections in W^a being denoted $r_1, \dots, r_n$), $T_w = T_{i_1}, \dots, T_{i_\ell}$.

The *extended affine Hecke algebra* (cf. [X]), which we denote $\widetilde{H_n^a}(q)$ may be defined by

$$\widetilde{H_n^a}(q) = R[\Omega] \otimes_R H_n^a(q), \tag{1.3}$$

where Ω is the cyclic group of order n, thought of as the group of automorphisms of $H_n^a(q)$ which permute the T_i cyclically, and the tensor product is 'twisted' in the sense that if $\omega_1, \omega_2 \in \Omega$ and $h_1, h_2 \in H_n^a(q)$, we have

$$\omega_1 \otimes h_1 \cdot \omega_2 \otimes h_2 = \omega_1\omega_2 \otimes \omega_2^{-1}(h_1)h_2. \tag{1.4}$$

The algebra $\widetilde{H_n^a}(q)$ corresponds to the extension of the finite Weyl group by the lattice of weights in the same sense that $H_n^a(q)$ corresponds to W^a. Clearly, $H_n^a(q)$ is

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a subalgebra of $\widetilde{H_n^a}(q)$. Let $W(i)$ be the subgroup of W^a which is generated by the simple reflections r_i and r_{i+1} $(i = 1, 2, \ldots, n)$. Then $W(i) \cong \mathrm{Sym}_3$ (the symmetric group of degree 3, which has order 6), and in $\widetilde{H_n^a}(q)$ we have elements

$$E_i = \sum_{w \in W(i)} q^{\ell(w)} T_w \quad (i = 1, \ldots, n-1), \tag{1.5}$$

where $\ell(w)$ is the length function in the Coxeter group W^a. These satisfy

$$E_i^2 = (1+q^2)(1+q^2+q^4)E_i.$$

As in [GL2], we define the Temperley–Lieb quotient of $H_n^a(q)$ as

$$TL_n^a(q) := H_n^a(q)/I_n^a(q), \tag{1.6}$$

where $I_n^a(q)$ is the ideal of $H_n^a(q)$ generated by $E_1, \ldots, E_n$ (of course only one E_i is needed, since they are conjugate in $H_n^a(q)$).

We refer to the extended analogue $\widetilde{TL_n^a}(q)$ of $TL_n^a(q)$ as the extended affine Temperley–Lieb algebra of type $\tilde{A}_{n-1}$. It is defined as

$$\widetilde{TL_n^a}(q) = \widetilde{H_n^a}(q)/\widetilde{I_n^a}(q), \tag{1.7}$$

where $\widetilde{I_n^a}(q)$ is the ideal of $\widetilde{H_n^a}(q)$ generated by $E_1, \ldots, E_n$.

(1.8) LEMMA. *We have*

$$\widetilde{TL_n^a}(q) \cong R[\Omega] \otimes_R TL_n^a(q).$$

Proof. First, observe that since $I_n^a(q)$ is stable under Ω (it is generated by an Ω-stable set), it follows that $\widetilde{I_n^a}(q) = R[\Omega] \otimes_R I_n^a(q)$. Since the right side is an ideal, it contains $I_n^a(q)$ and it is clearly the smallest ideal satisfying this condition. The lemma follows. □

It follows from (1.8) that if we write $\sigma = \omega \otimes 1 \in \widetilde{TL_n^a}(q)$, where ω is a (fixed) generator of Ω, then $\widetilde{TL_n^a}(q)$ is generated by $TL_n^a(q)$ together with σ. But $TL_n^a(q)$ has the following presentation described in [GL2, (2.9)]:

$$TL_n^a(q) = \begin{cases} \langle f_1, \ldots, f_n | f_i f_j = f_j f_i = \text{if } |k-j| \geqslant 2 \text{ and } (i,j) \neq (1,n), \\ f_i f_{i+1} f_i - f_i = f_{i+1} - f_{i+1} = 0 \quad (i = 1, \ldots, n), \\ f_i^2 = \delta f_i \rangle, \end{cases} \tag{1.9}$$

where $\delta = -(q + q^{-1})$ and the indices in the second relation are taken modulo n.

The element σ above defies an automorphism of $TL_n^a(q)$:

$$\sigma f_i \sigma^{-1} = f_{i+1} \quad (i = 1, \ldots, n), \tag{1.10}$$

where the index i is again taken mod n.

It follows that $\widetilde{TL_n^a}(q)$ is generated by $\{\sigma, f_1, \ldots, f_n\}$ subject to the relations (1.9), (1.10) and $\sigma^n = 1$.

Now in addition to the algebras $\widetilde{TL_n^a}(q)$ and $TL_n^a(q)$, we shall require the algebra$T^a(n)$ which was defined in [GL2, (2.7)] and referred to there (loc. cit.) as

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'the affine Temperley–Lieb algebra'. This is defined as an algebra of diagrams or, more accurately, as the algebra of morphisms: $n \longrightarrow n$ in the category T^a (see [GL2, (2.5)]) and $TL_n^a(q)$ is identified [GL2, (2.9)] as the subalgebra of $T^a(n)$ spanned by the 'nonmonic diagrams: $n \longrightarrow n$ of even rank', together with the identity. It also occurs independently in the work of Green [Gr] and Fan and Green [FG]. We shall need to make some use of the diagrammatic description in this work; details may be found in [GL2], but a good approximation to the picture is obtained if one thinks of affine diagrams as arcs drawn on the surface of a cylinder joining $2n$ marked points, n on each circle component of the boundary, in pairs. The arcs must not intersect, and diagrams are multiplied by concatenation in the usual way. These diagrams are represented by periodic diagrams drawn between two horizontal lines, each diagram being determined by the 'fundamental rectangle', from which the cylinder is obtained by identifying vertical edges. In this interpretation, the generators f_i of $TL_n^a(q)$ are represented by the diagrams shown in Figure 1.

The *rank* of such a diagram is the smallest number of intersections of arcs with the left vertical edge.

Now $T^a(n)$ contains the 'twist' $\tau = \tau_n$, which has rank 1, and $T^a(n)$ is generated by $\{TL_n^a(q), \tau\}$; however it is not true that $T^a(n) \cong R[\langle\tau\rangle] \otimes TL_n^a(q)$. Instead we have

(1.11) LEMMA. *Let A be the augmentation ideal (i.e. the ideal generated by $f_1, f_2, \ldots, f_n$) of $TL_n^a(q) \subseteq T^a(n)$, so that $TL_n^a(q) = R1 \oplus A$. Then $\tau^2 A = A\tau^2 = A$ and we have $T^a(n) = R[\langle\tau\rangle] \oplus A \oplus A\tau$.*

Proof. This may be found in [Gr, §2]. We sketch it here for the sake of our exposition. Since $T^a(n)$ is generated as associative algebra by $TL_n^a(q)$ and τ, the result follows easily if we know that $A\tau^2 = \tau^2 A = A$. To see this latter fact, observe that

$$f_1\tau^2 = \tau^2 f_{n-1} = f_1 f_2 \cdots f_{n-1} \in A.$$

Conjugating this relation by τ, we see that $f_i\tau^2$ and $\tau^2 f_i$ are both in A for any i, whence $A\tau^2 \subseteq A, \tau^2 A \subseteq A$. Similarly, $\tau^{-2}A \subseteq A$ and $A\tau^{-2}A$ (use the mirror image of the above relation: $\tau^{-2}f_1 = f_{n-1}\tau^{-2} = f_{n-1}f_{n-2}\cdots f_1$).

Thus $A = (\tau^{-2}A)\tau^2 \subseteq A\tau^2$, whence we have equality in each case. □

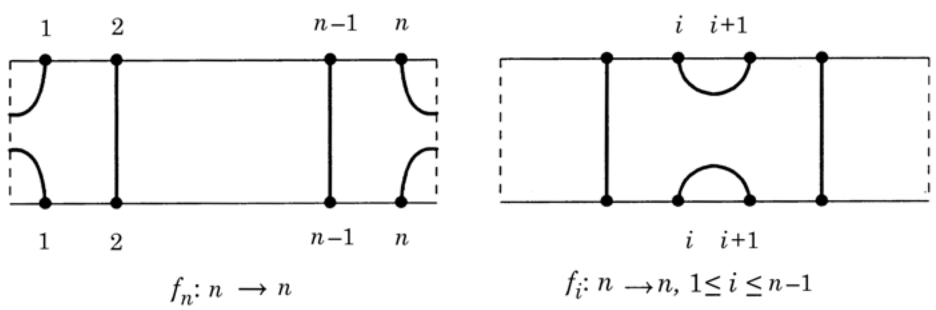

Figure 1.

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(1.12) COROLLARY. *We have* $T^a(n) \cong TL_n^a(q) \oplus I$, *where* $I = \oplus_{i\neq 0} R\tau^i \oplus A\tau$.

This is immediate from (1.10).

The algebra $\widetilde{TL_n^a}(q)$ is not a quotient of $T^a(n)$, but the representation theory of $\widetilde{TL_n^a}(q)$ may be discussed via that of $TL_n^a(q)$, which in turn comes from that of $T^a(n)$, whose representations are completely described in [GL2]. Note also that there is an obvious surjective homomorphism ϕ: $\widetilde{TL_n^a}(q) \to TL_n^a(q)$, defined by $\omega \otimes \mu \mapsto \tau_n\mu$ (for $\omega \in \Omega$ and $\mu \in TL_n^a(q)$).

2. Representations

We shall determine all irreducible representations of $\widetilde{TL_n^a}(q)$ using the classification of those of $TL_n^a(q)$ determined in [GL2]. In this section, we assume that R is an algebraically closed field.

To discuss the representations of $\widetilde{TL_n^a}(q) \cong \mathbb{C}[\Omega] \otimes TL_n^a(q)$, let us first recall the description given in [GL2, (2.9.1)] of the representations of $TL_n^a(q)$. As was recalled above, $TL_n^a(q)$ may be realised as an algebra spanned by diagrams on the surface of a cylinder or 'affine diagrams α: $n \longrightarrow n$' (see [GL2, (1.3), (1.4), (2.5)]). In general an affine diagram α: $t \longrightarrow n (t, n \in \mathbb{Z}_{\geqslant 0})$ is depicted by its restriction to the fundamental rectangle, which has t marked points on the bottom edge, n on the top, and distinct points joined in pairs by non-intersecting arcs (after the vertical edges are identified). If α has no 'through arcs', i.e. if top (resp. bottom) vertices are joined to top (resp. bottom) vertices, then α may also have a certain number, $y(\alpha)$ say, of arcs which wrap around the cylinder. This description of $TL_n^a(q)$ realises $TL_n^a(q)$ as a subalgebra of $T^a(n)$, whose representation theory is discussed in [GL2]. The irreducible representations of $TL_n^a(q)$ turn out to be largely restrictions of those of $T^a(n)$.

We recall the definition of the cell modules for $T^a(n)$, which by restriction are also modules for $TL_n^a(q)$ (cf. [GL2, (2.6)]). Let $t \in \mathbb{Z}_{\geqslant 0}$, $0 \leqslant t \leqslant n$, with $t \equiv n \pmod 2$ and let z be any invertible element of R. Let $W'_{t,z}(n)$ be the R-module with basis all monic diagrams (see [GL2, (1.6)]; this means all t arcs from the bottom vertices are through arcs), then $W_{t,z}(n) = W'_{t,z}(n)/V$, where V is the subspace generated by elements of the form $\alpha\tau_t - z\alpha$ $(t > 0)$ or $\alpha\tau_t - (z + z^{-1})\alpha$ $(t = 0)$, where τ_t is the 't-twist' depicted in Figure 2.

The module $W_{t,z}(n)$ has an R-basis consisting of 'standard diagrams' (see [GL2, Definition (1.7)]), which are monic diagrams whose through strings all lie inside the fundamental rectangle. It is easily seen, by counting these, that

$$\dim W_{t,z}(n) = \binom{n}{\frac{n-t}{2}}. \tag{2.1}$$

The space $W_{t,z}(n)$ is a $T^a(n)$-module, with the algebra acting by concatenating diagrams $t \longrightarrow n$ $(\in W_{t,z}(n))$ with α: $n \longrightarrow n$ $(\in TL_n^a(q))$ according to the description in [GL2, (2.1), (2.6)]. We write $\alpha * \mu$ for this action. There is a bilinear map $\phi_{t,z}$: $W_{t,z}(n) \times W_{t,z^{-1}}(n) \longrightarrow R$ which ([GL2, (2.7)]) is invariant in the sense that $\phi_{t,z}(\alpha * \mu, \nu) = \phi_{t,z}(\mu, \alpha^* * \nu)$, $(\mu, \nu \in W_{t,z}(n), \alpha \in T^a(n))$ where α^* denotes the

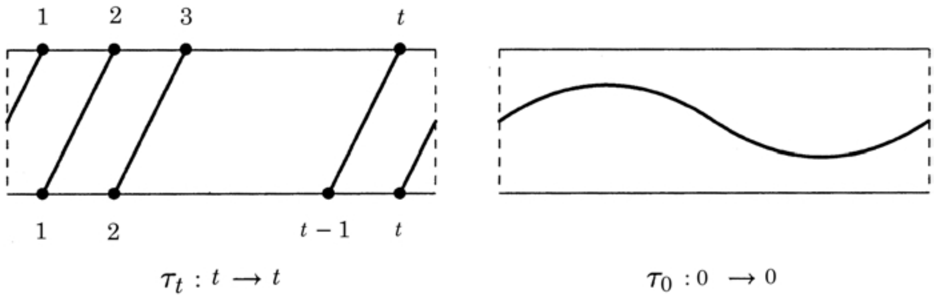

Figure 2.

reflection of α in a horizontal line. Note that $\alpha \mapsto \alpha^*$ is an anti-automorphism of $T^a(n)$, which preserves $TL_n^a(q)$. If R is a field, the irreducible $T^a(n)$-modules have a simple description ([GL2, (2.8)]). In order to give it, we shall require a certain set of parameters. For convenience, we collect its definition, as well as some others which we shall require, here.

(2.2) DEFINITION. Let $\Lambda^a(n)^+$ be the set

$$\Lambda^a(n)^+ = \{(t, z) \mid t \in \mathbb{Z}_{\geqslant 0},\ 0 \leqslant t \leqslant n,\ n - t \in 2\mathbb{Z};\ z \in R^*\}.$$

Define $\Lambda^a(n)^+$ by

$$\Lambda^a(n) = \begin{cases} \Lambda^a(n)^+ & \text{if } q^2 \neq -1, \\ \Lambda^a(n)^+ \setminus \{(0, \pm q)\} & \text{if } q^2 = -1. \end{cases} \tag{2.2.1}$$

Define the equivalence relation $\approx$ on $\Lambda^a(n)^+$ as that which identifies $(0, z)$ and $(0, z^{-1})$ for all $z \in R^*$, and write

$$\Lambda^a(n)^{+0} = \Lambda^a(n)^+/\approx \ \text{ and } \Lambda^a(n)^0 = \Lambda^a(n)/\approx. \tag{2.2.2}$$

Next, let $\Lambda^a(n)'$ be the set

$$\Lambda^a(n)' = \begin{cases} \{(t, z) \in \Lambda^a(n) \mid z^2 \neq -1 \quad \text{if } t = 0\} \coprod \{(0, i)^+, (0, i)^-\} & \text{if } q^2 \neq -1, \\ \{(t, z) \in \Lambda^a(n) \mid z^2 \neq -1 \quad \text{if } t = 0\} \quad \text{if } q^2 = -1, \end{cases}$$

where i denotes a fixed element of R such that $i^2 = -1$. Define the equivalence relation $\sim$ on $\Lambda^a(n)'$ by $(t, z) \sim (t', z')$, if and only if $t = t' = n$ or $t = t'$ and $z' = \pm z$ or $(t, z) \approx (t', z')$. Define

$$\overline{\Lambda^a(n)} = \Lambda^a(n)'/\sim. \tag{2.2.3}$$

Let $\equiv$ be the equivalence relation on $\Lambda^a(n)$ which is given by $(t, z) \equiv (t', z')$, if and only if $(t, z) \approx (t', z')$ or $t = t' = n$ or $t = t'$ and $z' = \pm z$. Define

$$\widetilde{\Lambda^a(n)} = \Lambda^a(n)/\equiv. \tag{2.2.4}$$

We denote the elements of this set by $\widetilde{(t, z)}$, where $(t, z) \in \Lambda^a(n)$.

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It is clear that $\widetilde{\Lambda^a(n)}$ is in natural bijection with the quotient of $\overline{\Lambda^a(n)}$ obtained by identifying $(0, i)^+$ and $(0, i)^-$.

(2.2A) *Remark*. We shall see shortly that the sets $\Lambda^a(n)^0$ and $\overline{\Lambda^a(n)}$, respectively, parametrise the isomorphism classes of irreducible modules for the algebras $T^a(n)$ and $TL_n^a(q)$. We often abuse notation by lifting the parameters to $\Lambda^a(n)$, assuming the appropriate identifications.

THEOREM ([GL2, (2.8)]). *Let R be an algebraically closed field and maintain the above notation. For $(t, z) \in \Lambda^a(n)^0$, $L_{t,z}(n) := W_{t,z}(n)/\text{rad}\,\phi_{t,z}$ is an irreducible $T^a(n)$ module. All irreducible $T^a(n)$ modules are realised thus, and if $(t_1, z_1) \not\approx (t_2, z_2)$, then $L_{t_1,z_1}(n) \ncong L_{t_2,z_2}(n)$.*

The main facts concerning the restrictions of the above modules to $TL_n^a(q)$ are as follows.

(2.3) THEOREM. *Assume that R is an algebraically closed field.*

(i) *As $TL_n^a(q)$-modules, $W_{t,z}(n) \cong W_{t,y}(n)$ if $t = n$ (any y, z) or if $y + z = 0$ (any t).*
(ii) *If $t \neq 0$, the module $L_{t,z}(n) := W_{t,z}(n)/\text{rad}\,\phi_{t,z}$ is an irreducible $TL_n^a(q)$-module.*
(iii) *If $t = 0$ and $z^2 \neq -1$, then $L_{0,z}(n)$ (defined as above) is irreducible as a $TL_n^a(q)$-module.*
(iv) *If $t = 0$ and $z^2 = -1$, then $W_{0,z}(n)$ is the direct sum of two submodules $W_{0,z}^+(n)$ and $W_{0,z}^-(n)$ which are spanned respectively by diagrams of even and odd rank.*
(v) *If $t = 0$, $z^2 = -1$ and $q^2 \neq -1$ then $W_{0,z}^\pm(n)$ have nonisomorphic irreducible heads $L_{0,z}^\pm(n)$.*
(vi) *The set $\overline{\Lambda^a(n)}$ defined in (2.2.3) parametrises the irreducible $TL_n^a(q)$-modules, with each one occurring once.*

Proof. First observe that $W_{n,z}(n)$ is one-dimensional, spanned by the identity diagram $n \to n$. Its structure as $T^a(n)$ module is therefore clearly independent of n (all f_i act trivially). Next, since $TL_n^a(q)$ is spanned by certain diagrams of even rank, it follows that for any diagram $\alpha \in TL_n^a(q)$ and standard diagram μ: $t \longrightarrow n$, we have (by [GL2, (1.5) (4)])

$$\text{rank}\,(\alpha \circ \mu) \equiv \text{rank}\,(\mu) \pmod 2.$$

Consider the linear transformation U: $W_{t,z}(n) \longrightarrow W_{t,z}(n)$ defined by $U(\mu) = (-1)^{\text{rank}\,(\mu)}\mu$ for any standard diagram μ: $t \longrightarrow n$. We shall show that for $\alpha \in TL_n^a(q)$, we have

$$\alpha \cdot_{(z)} U(\mu) = U(\alpha \cdot_{(-z)} \mu), \tag{2.3.2}$$

where $\cdot_{(y)}$ denotes the action of α on the module $W_{t,y}(n)$ (any $y \in R^*$). This will show that U intertwines the actions of $TL_n^a(q)$ on $W_{t,z}(n)$ and $W_{t,-z}(n)$. Assume first that $t \neq 0$.

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To prove (2.3.2), observe that both sides are 0 unless $\alpha \circ \mu$ is monic, in which case $\alpha \circ \mu = \mu' \circ \tau_t^k$ for some $k \in \mathbb{Z}$. It follows that the left side of (2.3.2) is equal to $(-1)^{\text{rank}\,\mu} z^k \mu'$. The right side is $U((-z)^k \mu') = (-z)^k (-1)^{\text{rank}\,\mu'} \mu'$. But by (2.3.1) we have rank $(\mu') + k \cong$ rank $\mu (\text{mod}\, 2)$, whence the result.

If $t = 0$, the proof is the same, with z replaced by $z + z^{-1}$. Notice that the proof is valid even if $z + z^{-1} = 0$ (i.e. $z^2 = -1$). This proves (i).

The proofs of (ii) and (iii) follow the arguments of [GL1, (2.6)]. For (iv) note that if $t = 0$ and $z^2 = -1$ the splitting of $W_{0,z}$ takes place because $z + z^{-1} = 0$, so that nonstandard monic diagrams μ: $0 \longrightarrow n$ are all zero. Hence, operation by $TL_n^a(q)$ never effects a parity change in the rank. The rest of the proof is a straightforward adaptation of [GL2, (2.8), (2.9.1)]. □

(2.4) PROPOSITION. *Let ω be the automorphism of $TL_n^a(q)$ defined by $\omega(f_i) = f_{i+1}$ ($i = 1, 2, \ldots, n$), where the index i is taken mod n. For any representation ρ of $TL_n^a(q)$ define the representation ρ^ω by $\rho^\omega = \rho \circ \omega$. Then for any pair $(t, z) \in \Lambda^a(n)$, we have an isomorphism of $TL_n^a(q)$ modules $W_{t,z}(n)^\omega \cong W_{t,z}(n)$.*

Proof. The vector space $W_{t,z}(n)$ is a module for the algebra $T^a(n) \supset TL_n^a(q)$. But $\tau_n \in T^a(n)$ satisfies $\tau_n f_i \tau_n^{-1} = f_{i+1}$ (all $i (\text{mod}\, n)$). Hence, the linear transformation of $W_{t,z}(n)$ which is defined by the action of τ_n on the $T^a(n)$ module $W_{t,z}(n)$ intertwines the $TL_n^a(q)$ modules $W_{t,z}(n)$ and $W_{t,z}(n)^\omega$. □

We henceforth take R to be an algebraically closed field.

(2.5) COROLLARY. (i) *All irreducible $TL_n^a(q)$-modules M satisfy $M^\omega \cong M$, except if $q^2 \neq -1$ and $M \cong L_{0,i}^{\pm}(n)$.*

(ii) *The automorphism ω interchanges the irreducible modules $L_{0,i}^{\pm}(n)$.*

Proof. (i) In this case $M \cong L_{t,z}(n)$ for some $(t, z) \in \Lambda^a(n)/\{(0, \pm i)\}$. Since $W_{t,z}(n)$ is invariant (up to isomorphism) under ω and $L_{t,z}(n)$ is its head, the result is immediate.

(ii) The map $\mu \mapsto \tau_n \circ \mu$ clearly interchanges the modules $W_{0,i}^{\pm}$, and hence interchanges their heads, which are the irreducible modules $L_{0,i}^{\pm}(n)$. □

The next result is required to complete the classification of the irreducible $\widetilde{TL_n^a}(q)$ modules. The authors wish to thank R.B. Howlett for discussions concerning its proof.

(2.6) THEOREM. *Let B be an associative algebra with identity over the algebraically closed field R. Suppose σ is an automorphism of B of finite order n, assume that n is not divisible by the characteristic of R, and let A be the associative algebra $A = R[\langle\sigma\rangle] \otimes B$, where $\langle\sigma\rangle$ is the cyclic group generated by σ and the tensor product is 'twisted' in the sense of (1.4). Let M be a finite-dimensional irreducible A module and let $M_1 \subset M$ be an irreducible (B-) submodule of the restriction of M from A to B. Let r be the smallest positive integer such that the twist $M_1^{\sigma^r} \cong M_1$ as B module; clearly r divides n. Then*

$$M \cong M_1 \oplus \sigma M_1 \oplus \cdots \oplus \sigma^{r-1} M_1$$

where σ is identified with the element $\sigma \otimes 1 \in A$.

Proof. Write ρ: $A \to \mathrm{End}_R(M)$ and ρ_i: $B \to \mathrm{End}_R(M_1^{\sigma^{i-1}})$ $(i = 1, 2, \dots, r)$ for the representations associated with the corresponding modules. We shall soon identify these with matrix representations. Observe first that there is a positive integer k such that

$$M \cong M_1 \oplus \sigma M_1 \oplus \cdots \oplus \sigma^{k-1} M_1. \tag{2.6.1}$$

This is because if the sum $M_1 + \sigma M_1 + \cdots + \sigma^i M_1$ is direct (this is trivial for $i = 0$) then $\sigma^{i+1} M_1 \cap (M_1 + \sigma M_1 + \cdots + \sigma^i M_1)$ is either 0 or $\sigma^{i+1} M_1$, since $\sigma^{i+1} M_1$ is an irreducible B module. In the latter case, (2.6.1) follows immediately, while in the former, we repeat the argument replacing i by $i+1$, proving (2.6.1) by induction.

Clearly, since $M_1, \sigma M_1, \dots, \sigma^{r-1} M_1$ is a complete list of nonisomorphic σ-twists of the B module M_1, and M is invariant under σ, r divides k, so that we may write

$$k = hr \tag{2.6.2}$$

for some positive integer h. It follows that

$$M \cong M_1 \oplus M_1 \oplus \cdots \oplus M_1 \oplus \sigma M_1 \oplus \cdots \oplus \sigma M_1 \oplus \cdots \oplus \sigma^{r-1} M_1, \tag{2.6.3}$$

where the multiplicity of each summand $\sigma^{i-1} M_1$ is h.

We shall consider the matrix representation of A on M with respect to a basis adapted to the decomposition (2.6.3). Note first that one obtains a basis of $\sigma^i M_1$ by applying σ to a basis of $\sigma^{i-1} M_1$, for $i = 1, 2, \dots, r-1$. Hence, we may assume that for any element $b \in B$, we have

$$\rho_{i+1}(b) = \rho_i(\sigma b \sigma^{-1}) \ (b \in B, i = 1, 2, \dots, r-1). \tag{2.6.4}$$

Moreover since $M_1^{\sigma^r} \cong M_1$ as B module, there is an element $U \in \mathrm{End}_R(M_1)$, unique up to multiplication by a scalar, such that

$$\rho_1(\sigma^r b \sigma^{-r}) U = U \rho_1(b) \quad \text{for all } b \in B. \tag{2.6.5}$$

It follows that $\rho(b)$ acts on M with matrix of block diagonal form as depicted below

$$\rho(b) = \begin{bmatrix} \rho_1(b) & & & & & & \\ & \rho_1(b) & & & & & \\ & & \ddots & & & & \\ & & & \rho_1(b) & & & \\ & & & & \rho_2(b) & & \\ & & & & & \ddots & \\ & & & & & & \rho_r(b) \end{bmatrix}, \tag{2.6.6}$$

where the number of blocks $\rho_j(b)$, each of which is square of size $m = \dim M_1$, is h (for each $j = 1, \dots, r$).

We now consider the shape of the block matrix $\rho(\sigma) = [\rho_{ij}(\sigma)]$ $(1 \leqslant i, j \leqslant k = hr)$. In view of the decomposition (2.6.3), we shall consider larger submatrices, consisting of h^2 of the blocks $\rho_{ij}(\sigma)$ of size m. These will generally be denoted $\mathbf{D}_{pq}$ $(1 \leqslant p, q \leqslant r)$;

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they are square of size hm. In order to describe them, for any positive integer i such that $1 \leqslant i \leqslant k$, write

$$i = q(i)h + r(i), \text{ where } 0 \leqslant r(i) < h, \; 0 \leqslant q(i) \leqslant r - 1. \tag{2.6.7}$$

Since $\rho(\sigma b \sigma^{-1}) = \rho(\sigma)\rho(b)\rho(\sigma^{-1})$ for any element $b \in B$, we may use (2.6.6), (2.6.4) and (2.6.5) to deduce the shape of $\rho(\sigma)$. In terms of the block decomposition of $\rho(\sigma)$ as explained above, the result is, in the notation of (2.6.7)

$$\rho_{q(i)+2}(b)\rho_{ij}(\sigma) = \rho_{ij}(\sigma)\rho_{q(j)+1}(b) \quad \text{for } 1 \leqslant i, j \leqslant k, \tag{2.6.7}$$

where

$$\rho_{r+1}(b) = \rho_r(\sigma b \sigma^{-1}) = U\rho_1(b)U^{-1},$$

U being the intertwining matrix of (2.6.5).

It follows by Schur's Lemma that $\rho_{ij}(\sigma) = 0$ unless either

(a) $q(i) + 2 = q(j) + 1$, or
(b) $q(j) = 0$ and $q(i) = r - 1$.

In case (a), i.e. $q(i) + 1 = q(j)$, we have $\rho_{ij}(\sigma) = \lambda_{ij} I_m$, where $\lambda_{ij} \in R$ and I_m is the identity matrix of size m. In case (b), i.e. when $q(j) = 0$ and $q(i) = r - 1$, we have $\rho_{ij}(\sigma) = \lambda_{ij} U$. Hence the block decomposition of $\rho(\sigma)$ is of the form

$$\rho(\sigma) = \begin{bmatrix} \mathbf{0} & \mathbf{D}_1 & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_2 & & \\ & & & \ddots & \\ & & & \mathbf{0} & \mathbf{D}_{r-1} \\ \mathbf{UD}_r & \mathbf{0} & & \cdots & \mathbf{0} \end{bmatrix}, \tag{2.6.8}$$

where $\mathbf{D}_i$ is a block matrix each of whose blocks is of the form λI_m ($\lambda \in R$) for $i \leq r$, and is the block diagonal matrix each of whose (diagonal) blocks is equal to U.

We next show that there is a matrix

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & & & & \\ & \mathbf{T}_2 & & & \\ & & \ddots & & \\ & & & \mathbf{T}_{r-1} & \\ & & & & \mathbf{T}_r \end{bmatrix}, \tag{2.6.9}$$

such that the submatrices $\mathbf{T}_i$ are block matrices whose blocks are of the form λI_m ($\lambda \in R$) and such that for $i = 1, 2, \ldots, r-1$, $\mathbf{T}_i \mathbf{D}_i \mathbf{T}_{i+1}^{-1}$ is diagonal, as is $\mathbf{T}_r \mathbf{D}_r \mathbf{T}_1^{-1}$.

To see that such a matrix exists, observe that by taking powers of the matrix (2.6.8), one sees that $(\mathbf{D}_1\mathbf{D}_2 \ldots \mathbf{D}_r)^{\frac{n}{r}} = I_{hm}$, whence, since the characteristic of R does not divide n, $\mathbf{D}_1\mathbf{D}_2 \ldots \mathbf{D}_r$ is diagonalisable by a matrix $\mathbf{T}_1$ of the required form. Then take $\mathbf{T}_2 = \mathbf{T}_1\mathbf{D}_1$, $\mathbf{T}_3 = \mathbf{T}_2\mathbf{D}_2, \ldots, \mathbf{T}_r = \mathbf{T}_{r-1}\mathbf{D}_{r-1}$. It is then easily verified that $\mathbf{T}$ satisfies the stated conditions.

Now the linear transformation $\mathbf{T}$ of M commutes with $\rho(b)$ for each $b \in B$ and $\mathbf{T}\rho(\sigma)\mathbf{T}^{-1}$ is of the form (2.6.8) with each block $\mathbf{D}_i$ diagonal. It follows that if M' is the subspace of M consisting of the sum of the first, $(h+1)$st, $(2h+1)$st, $\ldots, ((r-1)h+1)$st summands of the decomposition (2.6.3) of M, then the subspace $\mathbf{T} \cdot M'$ is invariant under $\rho(B)$ and $\rho(\sigma)$. Since A is generated by B and σ, $\mathbf{T} \cdot M'$ is invariant under $\rho(A)$, whence by irreducibility, $M = M'$, which proves (2.6). □

(2.7) COROLLARY. *Suppose B, A, σ and M are as in the statement of* (2.6). *If the restriction of M from A to B contains a σ-invariant irreducible B submodule M_1, then $M = M_1$. Conversely, every σ-invariant irreducible B module M_1 extends to an irreducible A module.*

Proof. The first statement is just the case $r = 1$ of (2.6). For the converse, observe that since $M_1^\sigma \cong M_1$, there is a transformation $\tau \in \mathrm{End}_R(M_1)$ such that for all elements $b \in B$, $\rho_1(\sigma b \sigma^{-1}) = \tau\rho_1(b)\tau^{-1}$, where ρ_1 is the representation of B on M_1. If we take $\rho(\sigma) = \lambda\tau$ where $\lambda \in R$ is such that $(\lambda\tau)^n = \mathrm{id}_{M_1}$, this defines a representation ρ of A on M_1, which extends ρ_1. □

Using (2.6), the classification of the distinct irreducible $\widetilde{TL_n^a}(q)$ modules is now straightforward.

(2.8) THEOREM. *Assume that R is an algebraically closed field of characteristic not dividing n. The distinct irreducible $\widetilde{TL_n^a}(q)$-modules are classified as follows. For each element $(t, z) \in \widetilde{\Lambda^a(n)}$ (see* (2.2.4)) *and each element $\zeta \in R$ such that $\zeta^n = z^{-t}$ there is an irreducible $\widetilde{TL_n^a}(q)$-module $L_{\widetilde{(t,z)}}(\zeta)$. As a space, $L_{\widetilde{(t,z)}}(\zeta)$ is $L_{t,z}(n)$; the subalgebra $TL_n^a(q)$ acts as in* (2.3), *while $\sigma = \omega \otimes 1$ (see* (1.8)) *acts as $\tau_n\zeta$. The only isomorphisms among the simple $\widetilde{TL_n^a}(q)$ modules $L_{\widetilde{(t,z)}}(\zeta)$ are as follows: $L_{(n,z)}(\zeta) \cong L_{(n,y)}(\zeta z y^{-1})$ for any $z, y \in R^*$, $L_{(t,z)}(\zeta) \cong L_{(t,-z)}(-\zeta)$ for any $(t, z) \in \Lambda^a(n)$, $L_{(0,z)}(\zeta) \cong L_{(0,z^{-1})}(\zeta)$ for any $z \in R^*$, and when $q^2 \neq -1$, $L_{(0,i)}(\zeta) \cong L_{(0,i)}(-\zeta)(\cong \underline{L_{(0,-i)}}(-\zeta))$ for all $\zeta \in R$ satisfying $\zeta^n = 1$.*

Proof. Let M be an irreducible $\widetilde{TL_n^a}(q)$ module. Theorem (2.6) clearly applies, with $A = \widetilde{TL_n^a}(q)$ and $B = TL_n^a(q)$. Suppose first that the restriction of M to $TL_n^a(q)$ contains an irreducible $TL_n^a(q)$-module $M_1 \cong L_{t,z}(n)$ with $(t, z) \neq (0, \pm i)$ as a submodule. Then by (2.5)(i), $M_1^\sigma \cong M_1$ and we may apply (2.7) to deduce that M is an extension of M_1 to A. By Schur's Lemma, $\sigma = \omega \otimes 1$ must act on $L_{t,z}$ as a scalar multiple $\zeta\tau_n$ of τ_n. But τ_n^n acts on $W_{t,z}$ as the scalar z^t, since for any diagram μ: $t \to n$ we have $\tau_n^n \mu \tau_t^{-t} = \mu$ by periodicity. Since $\sigma^n = 1$, we must have $\zeta^n z^t = 1$ as stated. Thus M is the $\widetilde{TL_n^a}(q)$ module $L_{\widetilde{(t,z)}}(\zeta)$ of the statement.

To determine the coincidences among these modules, note that the restriction of $L_{\widetilde{(t,z)}}(\zeta)$ to $TL_n^a(q)$ is the irreducible $TL_n^a(q)$-module $L_{t,z}(n)$. By (2.3)(vi), this implies that the equivalence class $\overline{(t, z)}$ of (t, z) in $\widetilde{\Lambda^a(n)}$ is determined by M. If θ is an intertwining map for $L_{(t,z)}(\zeta)$ and $L_{(t',z')}(\zeta')$, then since θ intertwines the $TL_n^a(q)$ actions, we have $\overline{(t, z)} = \overline{(t', z')}$. Moreover, by irreducibility, θ must be a scalar multiple of the map U described in the proof of (2.3). Hence, we may assume, without loss of generality, that $\theta = U$. If $t = n$, τ_n acts on $L_{(n,z)}(n)$ as multiplication by z. Hence, the equation $\theta(\sigma \cdot \mu) = \sigma \cdot \theta(\mu)$ shows that $\zeta z = \zeta' z'$. If $t < n$, then $t = 0$ or $t = t'$ and $z' = -z$. Consider the latter case first. Then

$$\theta(\sigma \cdot \mu) = \theta(\zeta \tau_n \cdot \mu) = \zeta(-1)^{\mathrm{rk}(\mu)+1} \tau_n \cdot \mu = \sigma \cdot \theta(\mu) = \zeta'(-1)^{\mathrm{rk}(\mu)} \tau_n \cdot \mu,$$

whence $\zeta' = -\zeta$. If $t = 0$, $W_{0,z}(n) = W_{0,z^{-1}}(n)$, so $L_{0,z}(\zeta) = L_{0,z^{-1}}(\zeta)$.

The remaining case is when M contains a $TL_n^a(q)$ submodule L^+ isomorphic to $L_{(0,i)^+}$, which occurs only when $q^2 \neq -1$. In this case, M also has a $TL_n^a(q)$ submodule isomorphic to $L_{(0,i)^-}$, viz. $\sigma \cdot L^+$, and $\sigma^2 \cdot L^+ \cong L^+$. Hence we are in the situation of (2.6) with $r = 2$. Application of (2.6) yields that σ acts on $M \cong L_{(0,i)^+} \oplus L_{(0,i)^-}$ via the matrix

$$\begin{bmatrix} 0 & \lambda_1 \tau_n \\ \lambda_2 \tau_n & 0 \end{bmatrix}$$

for some scalars $\lambda_1, \lambda_2 \in R$.

The condition $\sigma^n = 1$ implies that $(\lambda_1 \lambda_2)^{\frac{n}{2}} = 1$. Write $M = L_{(0,i)}(\lambda_1, \lambda_2)$ for this module. We shall show

(2.8.1). *The modules $L_{(0,i)}(\lambda_1, \lambda_2)$ and $L_{(0,i)}(\lambda_1', \lambda_2')$ are isomorphic as $\widetilde{TL_n^a}(q)$ modules if and only if $\lambda_1 \lambda_2 = \lambda_1' \lambda_2'$.*

To see how the theorem follows from (2.8.1), observe that each isomorphism class of modules $L_{(0,i)}(\lambda_1, \lambda_2)$ contains a module $L_{(0,i)}(\zeta, \zeta)$, with $\zeta^n = 1$, because since R is algebraically closed, there is an element $\zeta \in R$ such that $\zeta^2 = \lambda_1 \lambda_2$. In the notation of the statement, $L_{(0,i)}(\zeta, \zeta) = L_{(0,i)}(\zeta)$, and by (2.8.1), $L_{(0,i)}(\zeta) \cong L_{(0,i)}(-\zeta)$ and there are no other coincidences among the $L_{(0,i)}(\zeta)$.

It therefore remains only to prove (2.8.1). First suppose that θ: $L_{(0,i)}(\lambda_1, \lambda_2) \to L_{(0,i)}(\lambda_1', \lambda_2')$ is an intertwining map. Then as above, since $\theta(\sigma^2 \cdot \mu) = \sigma^2 \cdot \theta(\mu)$ for $\mu \in L_{(0,i)}(n)$ and since σ^2 acts on $L_{(0,i)}(\lambda_1, \lambda_2)$ as $\lambda_1 \lambda_2 \tau_n^2$, we have $\lambda_1 \lambda_2 = \lambda_1' \lambda_2'$. Conversely, suppose this condition holds. Write $\kappa = \lambda_1 / \lambda_1' = \lambda_2' / \lambda_2$. Let $\theta : L_{(0,i)}(n) \to L_{(0,i)}(n)$ have matrix description analogous to those above given by

$$\theta = \begin{bmatrix} I & 0 \\ 0 & \kappa I \end{bmatrix},$$

where I denotes the identity matrix. It is then easily checked that θ intertwines $L_{(0,i)}(\lambda_1, \lambda_2)$ and $L_{(0,i)}(\lambda_1', \lambda_2')$. This proves (2.8.1) and completes the proof of the theorem. □

(2.9) DEFINITION. Define the sets $\Delta^a(n)^+$ and $\Delta^a(n)$ by

$$\Delta^a(n)^+ = \{[(t, z), \zeta] \in \Lambda^a(n)^+ \times R^* \mid \zeta^n = z^{-t}\},$$
$$\Delta^a(n) = \{[(t, z), \zeta] \in \Lambda^a(n) \times R^* \mid \zeta^n = z^{-t}\}.$$

Let $\approx$ be the equivalence relation on $\Delta^a(n)^+$ defined by

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(i) $[(n, z), \zeta] \approx [(n, y), \zeta z y^{-1}]$,

(ii) $[(t, z), \zeta] \approx [(t, -z), -\zeta]$,

(iii) $[(0, z), \zeta] \approx [(0, z^{-1}), \zeta]$,

(iv) $[(0, i), \zeta] \approx [(0, i), -\zeta]$.

Define the quotient sets $\Delta^a(n)^{+0}$ and $\Delta^a(n)^0$ by $\Delta^a(n)^{+0} = \Delta^a(n)^+/\approx$ and $\Delta^a(n)^0 = \Delta^a(n)/\approx$.

Notice that in (2.9), the relation (iv) is a consequence of (ii) and (iii), since if $z = i$, $z^{-1} = -z$. Also, if $t = n$, (ii) follows from (i). We shall write (t, z, ζ) for the $\approx$ class of $[(t, z), \zeta]$.

It is clear that in analogy with the irreducible modules $L_{t,z}(\zeta)$ defined in the statement of (2.8), we may define, for any triple $[(t, z), \zeta] \in \Delta^a(n)^+$, the *cell module* (or 'standard module') $W_{t,z}(\zeta)$ for the algebra $\widetilde{TL_n^a}(q)$ by stipulating that as space $W_{t,z}(\zeta) = W_{t,z}$, the cell module for $T^a(n)$, while σ acts as $\zeta\tau_n$.

(2.10) COROLLARY. *The set* $\Delta^a(n)^0$ *defined in* (2.9) *parametrises the isomorphism classes of those cell modules for* $\widetilde{TL_n^a}(q)$ *on which the canonical invariant bilinear form* (cf. [GL2, Definition (2.6) (2), p.188.]) *does not vanish and, hence, parametrises the irreducible modules. Moreover, for elements* $[(t, z), \zeta]$ *and* $[(t', z'), \zeta'] \in \Delta^a(n)^+$,

$$W_{t,z}(\zeta) \cong W_{t',z'}(\zeta') \Longleftrightarrow L_{t,z}(\zeta) \cong L_{t',z'}(\zeta') \Longleftrightarrow [(t, z), \zeta] \approx [(t', z'), \zeta'].$$

Proof. For $[(t, z), \zeta] \in \Delta^a(n)$, $W_{t,z}(\zeta)$ is the $\widetilde{TL_n^a}(q)$ module which is the extension of the $TL_n^a(q)$ module $W_{t,z}(n)$ to $\widetilde{TL_n^a}(q)$, on which σ acts as $\zeta\tau_n$. This is the 'cell module' corresponding to $[(t, z), \zeta]$, and the top quotient $L_{t,z}(\zeta)$ is the corresponding irreducible. It is nonzero if and only if $[(t, z), \zeta] \in \Delta^a(n)$. All the statements of the corollary follow from (2.8). □

3. Multiplicities and Decomposition Numbers

The irreducible $\widetilde{TL_n^a}(q)$-modules classified in (2.8) are all quotients of the spaces $W_{t,z}(n)$ by the radical of the form $\phi_{t,z}$, with $\sigma = \omega \otimes 1$ acting as a scalar multiple of τ_n. In this section we begin our discussion of the composition factors of the cell modules.

As in (2.10), Denote by $W_{t,z}(\zeta)$ the $\widetilde{TL_n^a}(q)$ module in which σ acts as described in (2.8) (for the irreducibles), i.e. as $\zeta\tau_n$. These are the 'cell modules' for $\widetilde{TL_n^a}(q)$. We remark that for convenience, we shall take the triples $[(t, z), \zeta]$ to be in $\Delta^a(n)^+$ rather than in the quotient set $\Delta^a(n)^{+0}$. This means that there are isomorphisms among the modules $W_{t,z}(\zeta)$ which induce the isomorphisms among the irreducible $\widetilde{TL_n^a}(q)$ modules which are referred to in the statement (2.8). In particular, we have isomorphisms for each of the cases of the equivalence relation defined in (2.9).

$$W_{t,z}(\zeta) \cong W_{t,-z}(-\zeta) \quad \text{for all } z \in R^*,$$

$$W_{n,z}(\zeta) \cong W_{n,y}(\zeta z y^{-1}) \quad \text{for all } y, z \in R^*,$$

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$$W_{0,z}(\zeta) \cong W_{0,z^{-1}}(\zeta) \quad \text{for all } z \in R^*,$$
$$W_{0,i}(\zeta) \cong W_{0,i}(-\zeta) \quad \text{for any } \zeta \text{ with } \zeta^n = 1,$$

and similarly for the irreducible heads of these modules.

As above, for any pair $t, z \in \Lambda^a(n)^+$ we denote by $\widetilde{(t,z)}$ its equivalence class in $\widetilde{\Lambda^a}(n)$.

The dimensions of the $W_{t,z}(n)$ are known (see (2.1) above), so the problem of determining the dimensions of the irreducible modules is equivalent to that of determining the multiplicities of the irreducibles in the 'cell modules' $W_{t,z}(\zeta)$. We address this question now. In rough terms, we shall show that with few exceptions, the expression for $L_{t,z}(\zeta)$ as a linear combination of cell modules in the Grothendieck group $\Gamma(\widetilde{TL_n^a}(q))$ is similar to that for $L_{t,z}(n)$ as a linear combination of the $W_{s,y}(n)$ in $\Gamma(T^a(n))$.

As in [GL2], the key is the existence of certain homomorphisms between cell modules.

(3.1) PROPOSITION. *Let* (t, z) *and* (s, y) *be elements of* $\Lambda^a(n)$ *which satisfy the conditions of* [GL2, (3.4)]; *i.e.* $s = t + 2\ell \to (\ell \geqslant 0)$, $z^2 = q^s$, $y = zq^{-\ell}$ *(so that* $y^2 = q^t$*). Assume that* $(t, z) \neq (0, \pm i)$. *Let* $\theta: W_{s,y}(n) \to W_{t,z}(n)$ *be the* $T^a(n)$*-homomorphism defined in* [GL2, loc. cit.]. *Then* θ *is a homomorphism of* $\widetilde{TL_n^a}(q)$ *modules:* $W_{s,y}(\zeta) \to W_{t,z}(\zeta)$ *for any* ζ *which satisfies* $\zeta^n = z^{-t}$ *[note that the assumptions imply that* $z^{-t} = y^{-s}$*].*

Proof. By the construction in [GL2, (3.5)], θ intertwines the $T^a(n)$ actions on $W_{s,y}(n)$ and $W_{t,z}(n)$. Hence for $\mu \in W_{s,y}(n)$, we have

$$\theta(\tau_n \cdot \mu) = \tau_n \cdot \theta(\mu).$$

Multiplying both sides by ζ, we see that θ intertwines the action of σ on $W_{s,y}(\zeta)$ and on $W_{t,z}(\zeta)$. Since $\widetilde{TL_n^a}(q)$ is generated by σ and $TL_n^a(q)$, the result follows. □

(3.2) THEOREM. *Let* $(t, z) \in$. *Suppose that in the Grothendieck group* $\Gamma(T^a(n))$, *we have*

$$W_{t,z}(n) = \sum_{(s,y)\in\Lambda^a(n)^0} m_{t,z}^{s,y} L_{s,y}(n). \tag{3.2.1}$$

Then for any ζ *satisfying* $\zeta^n = z^{-t}$, *we have, in* $\Gamma(\widetilde{TL_n^a}(q))$

$$W_{(t,z)}(\zeta) = \sum_{(s,y)\in\Lambda^a(n)^0} m_{t,z}^{s,y} L_{(s,y)}(\zeta). \tag{3.2.2}$$

Proof. Given (3.2.1), there is a composition series of $W_{t,z}(n)$ as $T^a(n)$-module, with the irreducible $T^a(n)$ module $L_{s,y}(n)$ occurring $m_{t,z}^{s,y}$ times. But by (2.8), the enveloping algebra of $\widetilde{TL_n^a}(q)$ acting on $W_{t,z}(\zeta)$ is precisely that of the $T^a(n)$-action. It follows from this and (3.1) that the composition series of $W_{t,z}(n)$ as a $T^a(n)$-module is also a composition series for $W_{t,z}(\zeta)$ as a $\widetilde{TL_n^a}(q)$-module, for any ζ with $\zeta^n = z^{-t}$; moreover the subquotient $L_{s,y}(n)$ is isomorphic to $L_{s,y}(\zeta)$ as a $\widetilde{TL_n^a}(q)$-module. The result follows. □

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We shall discuss the multiplicities $m_{t,z}^{s,y}$ in the next section, but to make use of our knowledge of these, we prove

(3.3) LEMMA. *Let* $[(t,z),\zeta] \in \Delta^a(n)^+$. *Then* $[(t',z'),\zeta] \in \Delta^a(n)^+$ *satisfies* $[(t,z),\zeta] \approx [(t',z'),\zeta]$ *if and only if* $(t,z) \approx (t',z')$ *in* $\Lambda^a(n)^+$. *That is, for any (appropriate)* ζ, $[(t,z),\zeta] \approx [(t',z'),\zeta]$ *in* $\Delta^a(n)^+$ *if and only if* $(t,z) \approx (t',z')$ *in* $\Lambda^a(n)^+$.

Proof. Suppose $[(t,z),\zeta] \approx [(t',z'),\zeta]$. Then clearly $t = t'$. If $t = n$, then using the relation (2.9)(i), we see immediately that $z = z'$. If $0 < t < n$, then $z' = \pm z$. But if $z' = -z$, then $\zeta = -\zeta$, i.e. R has characteristic 2. But in that case, $z' = z$. If $t = 0$ and $z^2 \neq -1$, then $z' = z^{\pm 1}$, so that $(t,z) \approx (t',z')$. Finally, if $t = 0$ and $z = \pm i$, we again have $z' = \pm i$, which completes the proof. □

4. Multiplicities, Dimension Formulae and Combinatorics

In this section we take R to be an algebraically closed field of characteristic zero, with q an invertible element of R. We begin by reviewing the results of [GL2] concerning the multiplicities of irreducible $T^a(n)$-modules in the cell modules.

Recall (2.2.1) that $\Lambda^a(n)^{+0} = \Lambda^a(n)^+/\approx$ where $(t,z) \approx (t',z')$ if and only if $t = 0$ and $z' = z^{\pm 1}$, and similarly for $\Lambda^a(n)^0$. Then $\Lambda^a(n)^{+0}$ parametrises the cell modules $W_{t,z}(n)$ and $\Lambda^a(n)^0$ parametrises the distinct (isomorphism classes of) irreducible $T^a(n)$ modules $L_{t,z}(n)$, which correspond to those cell modules on which the canonical form does not vanish.

Let $\preceq$ be the partial order on $\Lambda^a(n)$ which is generated by the preorder $\overset{\circ}{\prec}$ which stipulates that $(t,z) \overset{\circ}{\prec} (s,y)$ if

$$0 \leqslant t \leqslant s \leqslant n, \quad s = t + 2\ell \quad (\ell \in \mathbb{Z},\ \ell > 0) \tag{4.1a}$$

and

$$z^2 = q^{\epsilon(s,z)s} \quad \text{and} \quad y = zq^{-\epsilon(s,z)\ell} \quad \text{for } \epsilon(s,z) = \pm 1. \tag{4.1b}$$

Note that (4.1) implies that

$$y^2 = q^{\epsilon(s,z)t} \quad \text{and} \quad z^t = y^s \tag{4.2a}$$

and

$$(t,z) \preceq (t',z') \Rightarrow z^t = (z')^{t'}. \tag{4.2b}$$

It suffices to verify (4.2b) when $(t,z) \overset{\circ}{\prec} (t',z')$, in which case it follows easily from (4.1).

We note also that

(4.3) LEMMA. *The partial order* $\preceq$ *on* $\Lambda^a(n)$ *induces a partial order, also denoted* $\preceq$, *on the set* $\Lambda^a(n)^0 = \Lambda^a(n)/\approx$.

Proof. We must show that if $(t,z) \approx (t',z')$ in $\Lambda^a(n)$ and $(t,z) \overset{\circ}{\prec} (s,y)$, then $(t',z') \overset{\circ}{\prec} (s,y)$. It clearly suffices to take $(t,z) = (0,z)$ and $(t',z') = (0,z^{-1})$. Then $(s,y) = (2\ell, \pm 1)$ in the notation of (4.1a and 4.1b) and an easy calculation using (4.1a and 4.1b) shows that $\epsilon(s,z^{-1}) = -\epsilon(s,z)$ yields a solution of the equations (4.1). □

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The following result is proved in [GL2, Theorem 5.1].

(4.4) THEOREM. *We have, in the Grothendieck ring* $\Gamma(T^a(n))$*, for any* $(t,z)\in\Lambda^a(n)^+$,

$$W_{t,z}(n)=\sum_{\substack{(s,y)\in\Lambda^a(n)^0\\(t,z)\preceq(s,y)}}L_{s,y}(n). \tag{4.4.1}$$

Thus the matrix expressing the cell modules in terms of the irreducibles in $\Gamma(T^a(n))$ is upper unitriangular, and has entries 0 or 1. Now if (t,z) is confined to $\Lambda^a(n)$, the relation (4.4.1) can clearly be inverted. The next result is a statement about the inverse of the matrix of the partial order on $\Lambda^a(n)^0$. It asserts that the inverse has all entries equal to 0 or ±1. We thank D. Kazhdan for the suggestion that Theorem (4.5) might hold in general.

(4.5) THEOREM. *We have, in the notation of* (4.4),

$$L_{t,z}(n)=\sum_{\substack{(s,y)\in\Lambda^a(n)^0\\(t,z)\preceq(s,y)}}n_{t,z}^{s,y}W_{s,y}(n)$$

where $n_{t,z}^{s,y}=0$ *or* ±1.

Proof. If $L_{t,z}(n)=W_{t,z}(n)$ there is nothing to prove. If not, then by [GL2, Theorem (3.4)], there is a homomorphism of $TL_n^a(q)$-modules

$$\theta\colon W_{s,y}(n)\longrightarrow W_{t,z}(n)$$

for some (s,y) such that $(t,z)\prec(s,y)$ and we may assume (s,y) is minimal with respect to this property, i.e. that $(t,z)\overset{\circ}{\prec}(s,y)$. Then θ is injective (cf. [GL2, p. 214]) and the quotient $Q=W_{t,z}(n)/\mathrm{im}\,\theta$ has head $L_{t,z}(n)$. By the argument given in [GL2, pp214–215], the radical of Q (which is the radical of the form induced by $\phi_{t,z}$ on Q) is either 0 or equal to $L_{s',y'}(n)$, where (s',y') is the unique element of $\Lambda^a(n)^0$ such that $(t,z)\prec(s',y')$ and $(s,y)\not\prec(s',y')$. Then in the Grothendieck ring $\Gamma(T^a(n))$, we have

$$L_{t,z}(n)=W_{t,z}(n)-W_{s,y}(n)-L_{s',y'}(n). \tag{4.5.1}$$

Now arguing by (downward) induction in $\Lambda^a(n)^0$, the result follows for $L_{t,z}(n)$ from the corresponding equation for $L_{s',y'}(n)$, together with the properties of $(s',y')\in\Lambda^a(n)$. □

(4.6) COROLLARY. (i) *Every maximal chain between two elements of the poset* $\Lambda^a(n)^0$ *has the same number of elements.*

(ii) *In the notation of* (4.5)*, we have* $n_{t,z}^{s,y}=(-1)^i$*, where i is the number of links in a maximal chain between* (t,z) *and* $(s,y)\in\Lambda^a(n)$.

This is proved by an easy refinement of the argument in the proof of (4.5) above.

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(4.7) DEFINITION. We say that (t, z) and $(t', z') \in \Lambda^a(n)^0$ are in the same ***block*** if $L_{t',z'}(n)$ is a composition factor of $W_{t,z}(n)$, or vice versa. Define blocks of $\Delta^a(n)^0$ similarly.

Let us now interpret these results as they apply to the representation theory of the algebra $\widetilde{TL_n^a}(q)$.

(4.8) THEOREM. *Let $\widetilde{TL_n^a}(q)$ be the Temperley-Lieb quotient of the affine Hecke algebra of type A_{n-1} (see* (1.8)). *For any element $(t, z, \zeta) \in \Delta^a(n)^0$ (see* (2.9)) *we have*

$$W_{t,z}(\zeta) = \sum_{\substack{(s,y)\in\Lambda^a(n)^0 \\ (t,z)\preceq(s,y)}} L_{s,y}(\zeta) \tag{4.8.1}$$

in the Grothendieck ring $\Gamma(\widetilde{TL_n^a}(q))$ (see (2.8) *and* (2.10) *for the definitions of these modules).*

The irreducibles $L_{s,y}(\zeta)$ occurring on the right side of (4.8.1) *are all distinct (i.e. pairwise nonisomorphic). Thus the multiplicity of any irreducible module in a cell module is* 0 *or* 1.

Proof. Equation (4.8.1) follows immediately from (3.2) and (4.4). If $L_{s,y}(\zeta) \cong L_{s',y'}(\zeta)$ then by (2.8), $[(s, y), \zeta] \approx [(s', y'), \zeta]$. But by (3.3), this implies that $(s, y) = (s', y')$ in $\Lambda^a(n)^0$, which proves the statement. □

(4.9) COROLLARY. *Two elements $[(s, y), \zeta]$ and $[(s', y'), \zeta']$ of $\Delta^a(n)^0$ are in the same block if and only if there is a triple $[(t, z), \zeta]$ such that $[(s', y'), \zeta'] \approx [(t, z), \zeta]$, and (t, z) is connected to (s, y) in the partial order $\preceq$ on $\Lambda^a(n)^0$.*

Proof. To say that two elements $\delta = [(s, y), \zeta]$ and $\delta' = [(s', y'), \zeta']$ of $\Delta^a(n)^0$ are in the same block is to say that there is a chain $\delta_1 = \delta, \delta_2, \dots, \delta_r = \delta'$ such that for each i, the irreducible module L_{δ_i} is a composition factor of the cell module $W_{\delta_{i+1}}$ or vice versa (cf. [GL1, (3.9)]). By Theorems (3.2) and (4.8), the implied equivalence relation on triples is generated by a relation which preserves the third factor ζ. The result follows. □

Combining (4.5) and (4.6), we also have

(4.10) THEOREM. *Maintain the notation of* (4.8). *For any element $[(t, z), \zeta] \in \Delta^a(n)^0$ (see* (2.9)) *we have*

$$L_{t,z}(\zeta) = \sum_{\substack{(s,y)\in\Lambda^a(n)^0 \\ (t,z)\preceq(s,y)}} n_{t,z}^{s,y} W_{s,y}(\zeta) \tag{4.10.1}$$

in the Grothendieck ring $\Gamma(\widetilde{TL_n^a}(q))$, where $n_{t,z}^{s,y} = (-1)^i$ if i is the length of a maximal chain in $\Lambda^a(n)^0$ from (t, z) to (s, y).

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The last three results may also be formulated in terms of the order relation which $\preceq$ induces on $\Delta^a(n)^0$.

(4.11) THEOREM. *Let $\preceq$ be the partial order on $\Delta^a(n)^0$ induced by the preorder $\overset{\circ}{\preceq}$ defined by*

$$(t, z, \zeta) \overset{\circ}{\prec} (t', z', \zeta) \Longleftrightarrow (t, z) \overset{\circ}{\prec} (t', z') \quad in \quad \Lambda^a(n)^+.$$

If $(t, z, \zeta) \preceq (t', z', \zeta')$, then there is an injective homomorphism of $\widetilde{TL^a_n}(q)$ modules $: W_{t',z'}(\zeta') \longrightarrow W_{t,z}(\zeta)$. The multiplicity of the irreducible module $L_{t',z'}(\zeta')$ in the cell module $W_{t,z}(\zeta)$ is one if $(t, z, \zeta) \preceq (t', z', \zeta')$ and zero otherwise.

This is simply a restatement of (4.8). Of course this and the preceding statements may be rephrased in terms of the representations of the extended affine Hecke algebra, but we leave this to the next section.

5. Description in Terms of Semisimple-Nilpotent Pairs

Let G be the group $SL_N(\mathbb{C})$ and let $\mathfrak{G}$ be the Lie algebra of G. When $R = \mathbb{C}$, the affine Hecke algebra may be realised as a convolution algebra (cf. [KL], [L1], [L2], [Gi], [CG] or [X]), and there is consequently a theory of 'standard modules' for the algebra $\widetilde{TL^a_n}(q)$ which classifies them by G-conjugacy classes of pairs (s, N) such that $\mathrm{Ad}(g) \cdot N = q^2 N$, where s is a semisimple element of G and N is a nilpotent element of $\mathfrak{G}$ (notice that our algebra is a quotient of the affine Hecke algebra which corresponds to the parameter q^2 in the usual notation). These modules 'generically' have top quotients which constitute a complete set of irreducible modules for $\widetilde{TL^a_n}(q)$. When q^2 is a root of unity, the picture is not so well understood (see, however [G] and [A]). We show how our results may be expressed in these terms; in particular our results give complete information on the irreducible modules and decomposition numbers of the standard modules, for pairs (s, N), where N has at most two Jordan blocks. Write $\mathcal{P}^+$ for the set of G-conjugacy classes of pairs (s, N) such that $\mathrm{Ad}(g) \cdot N = q^2 N$ as above.

(5.1) PROPOSITION. *There is a natural bijection between the set $\Delta^a(n)^{+0}$ of (2.9) and the set $\mathcal{P}^+$ of semisimple-nilpotent pairs described above, in which N has at most two Jordan blocks.*

Proof. Let t be any integer satisfying $n = 2k + t$, where $k \in \mathbb{Z}$ and $0 \leqslant k \leqslant n/2$. Let J_k denote the $k \times k$ (Jordan) matrix with zeros everywhere except on the superdiagonal, where all entries are 1. Then each pair (s, N) such that $\mathrm{Ad}(g) \cdot N = q^2 N$ and N has at most two Jordan blocks is G-conjugate to one where

$$N = N_k = \begin{bmatrix} J_{n-k} & \\ & J_k \end{bmatrix} \tag{5.1.1}$$

and

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$$s = \begin{bmatrix} a_1 & & & & & & & & & \\ & a_1 q^{-2} & & & & & & & & \\ & & a_1 q^{-4} & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & a_1 q^{-2(n-k-1)} & & & & & \\ & & & & & a_2 & & & & \\ & & & & & & a_2 q^{-2} & & & \\ & & & & & & & a_2 q^{-4} & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & a_2 q^{-2(k-1)} \end{bmatrix} \tag{5.1.2}$$

where, since det $s = 1$, $a_1, a_2 \in \mathbb{C}$ satisfy

$$a_1^{k+t} a_2^k = q^{\{2k(k-1)+2kt+t(t-1)\}} = q^{\{n(n-1)-2k(n-k)\}}. \tag{5.1.3}$$

Denote the element s of (5.1.2) by $s(a_1, a_2)$. It is easily verified that the ordered pair (a_1, a_2) is uniquely determined by the G conjugacy class of the pair (s, N), except if $n = 2k$, in which case a_1 and a_2 may be reversed; i.e. $(s(a_1, a_2), N_{\frac{n}{2}})$ is G conjugate to $(s(a_2, a_1), N_{\frac{n}{2}})$. Hence

(5.1A). *The set $\mathcal{P}^+$ of the statement is in bijection with the set of pairs $\{(s(a_1, a_2), N_k)\}$, where $0 \leqslant 2k \leqslant n$, and $a_1, a_2 \in \mathbb{C}$ satisfy* (5.1.3), *modulo the equivalence of $(s(a_1, a_2), N_{\frac{n}{2}})$ and $(s(a_2, a_1), N_{\frac{n}{2}})$.*

Define variables A_1, A_2 by

$$A_1 = a_1 a_2 q^{-(n-2)}, \qquad A_2 = a_1 q^{-(n-k-1)}. \tag{5.1.4}$$

With this change of variables, Equation (5.1.3) becomes

$$A_1^k A_2^t = 1. \tag{5.1.5}$$

We now have the following parametrisation of the relevant G-conjugacy classes of pairs (s, N) in terms of solutions of Equation (5.1.5).

(5.1.6) LEMMA. *For any $t = n - 2k \neq 0, n$, the G-conjugacy classes of pairs (s, N) where N has Jordan blocks of size $n - k, k$, are in natural bijection with the solutions (A_1, A_2) of Equation* (5.1.5). *If $t = n$, the classes are in bijection with the solutions A_2 of equation $A_2^n = 1$, while if $t = 0$, the classes are in bijection with the solutions (A_1, A_2) of the Equation* (5.1.5), *modulo the equivalence relation $\approx$, which stipulates that $(A_1, A_2) \approx (A_1, A_1 A_2^{-1})$.*

Proof. Given a solution (A_1, A_2) of (5.1.5), Equations (5.1.4) determine unique values of a_1, a_2 which satisfy (5.1.3) and hence determine a class (s, N) of the required type. Conversely, the class of (s, N) determines (a_1, a_2) uniquely unless $t = 0$, in which case, (s, N) corresponds to both (a_1, a_2) and (a_2, a_1). Hence if $t \neq 0$, both the class of (s, N) and (a_1, a_2) are determined by (A_1, A_2) which is arbitrary, subject to

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(5.1.5). If $t \neq 0, n$, then distinct (a_1, a_2) determine distinct classes, proving the first statement. If $t = n$, a_2 is irrelevant to the class, which is determined by a_1, proving the second statement.

An easy calculation shows that in the bijective correspondence which is defined by (5.1.4) between solutions (a_1, a_2) of (5.1.3) and (A_1, A_2) of (5.1.5), if (a_1, a_2) corresponds to (A_1, A_2), then (a_2, a_1) corresponds to $(A_1, A_1 A_2^{-1} q^{-t})$. Hence, if $t = 0$, the G-conjugacy of the pairs (s, N) corresponding to (a_1, a_2) and (a_2, a_1) is reflected in the equivalence $(A_1, A_2) \approx (A_1, A_1 A_2^{-1})$. □

We shall next give an analogous description of $\Delta^a(n)^{+0}$. Recall (2.9) that $\Delta^a(n)^{+0}$ is a quotient of the set $\Delta^a(n)^+$ of triples (t, z, ζ), where $t = n - 2k$ for some integer k such that $0 \leqslant k \leqslant \frac{n}{2}$ and $z, \zeta \in R$ are such that $\zeta^n z^t = 1$. Equivalently, if we write

$$B_1 = \zeta^2, \qquad B_2 = \zeta z, \tag{5.1.7}$$

then

$$B_1^k B_2^t = 1. \tag{5.1.8}$$

In analogy with (5.1.6) we shall prove

(5.1.9) LEMMA. *If $t \neq 0, n$, the set $\Delta^a(n)^{+0}$ of equivalence classes under $\approx$ of triples (t, z, ζ) are in bijection with the solutions (B_1, B_2) of* (5.1.8). *If $t = n$ the equivalence classes are in bijection with the solutions B_2 of $B_2^n = 1$. If $t = 0$, the equivalence classes of triples are in bijection with the solutions (B_1, B_2) of Equations* (5.1.7), *modulo the equivalence relation $\approx$, which stipulates that $(B_1, B_2) \approx (B_1, B_1 B_2^{-1})$.*

Proof of (5.1.8). If $t \neq 0, n$, the only relation among the relevant triples is (2.9)(ii). It follows that the $\approx$ class of (t, z, ζ) is uniquely determined by $(\zeta^2, \zeta z)$, which may be arbitrary, subject to (5.1.8). The first statement follows. If $t = n$, the $\approx$ class of (n, z, ζ) depends only on $B_2 = \zeta z$, which satisfies $B_2^n = 1$, which is the second statement.

If $t = 0$, the equivalence class of $(0, z, \zeta)$ consists of the four triples

$$\{(0, z, \zeta), (0, z^{-1}, \zeta), (0, -z, -\zeta), (0, -z^{-1}, \zeta)\}.$$

These all have the same value for B_1, while the (two) possible values for B_2 are $B_2 = \zeta z$ and $B_2' = z^{-1}\zeta = B_1 B_2^{-1}$. This proves the third statement. □

The proof of (5.1) is now complete, since Lemmas (5.1.6) and (5.1.9) show that $\mathcal{P}^+$ and $\Delta^a(n)^{+0}$ have the same parameter set.

(5.1B) COROLLARY. *The above correspondence between the set $\mathcal{P}^+$ of G-classes of pairs and the set $\Delta^a(n)^{+0}$ of $\approx$ classes of triples is realized as follows. The class of the pair $(s(a_1, a_2), N_k)$ (see* (5.1.1)) *corresponds to the class of the triple (t, z, ζ)* (see (2.9)) *if*

$$t = n - 2k, \qquad \zeta^2 = a_1 a_2 q^{-(n-2)}, \qquad \zeta z = a_1 q^{-(n-k-1)}. \tag{5.1.10}$$

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Note that the relations (5.1.10) imply

$$z^2 = a_1 a_2^{-1} q^{-t} \tag{5.1.11}$$

Using the identification (5.1B), we may describe the partial order $\preceq$ in terms of the pairs (s, N). First, let us agree to write $M_{s,N} = M_{t,z}(\zeta)$ for the cell module of $\widetilde{TL}_n^a(q)$ if (s, N) and (t, z, ζ) correspond under the map defined in (5.1) or (5.1B). The irreducible $\widetilde{TL}_n^a(q)$ modules are parametrised by the subset $\Delta^a(n)^0$ of $\Delta^a(n)^{+0}$ (the two sets are equal unless $q^2 = -1$). The next result describes the corresponding subset of $\mathcal{P}^+$.

(5.1C) PROPOSITION. *The subset $\mathcal{P}$ of $\mathcal{P}^+$ which corresponds to $\Delta^a(n)^0$ under the bijection* (5.1B) *is given by*

$$\mathcal{P} = \begin{cases} \mathcal{P}^+ & \text{if } q^2 \neq -1, \\ \mathcal{P}^+ \setminus \{(s(\xi, -(-1)^{\frac{n}{2}}\xi), N_{\frac{n}{2}}) \mid \xi^n = (-1)^{\frac{n}{2}}\} & \text{if } q^2 = -1 \text{ and } n \text{ is even.} \end{cases}$$

Proof. The subset $\mathcal{P}$ is obtained from $\mathcal{P}^+$ by excluding the pairs (s, N) which correspond under (5.1B) to the triples $(0, \pm q, \zeta)$ when $q^2 = -1$ and n is even (see (2.2) and (2.2A)). Thus the excluded triples are $\{(0, \pm i, \zeta) \mid \zeta^n = 1\}$. It now remains only to express these triples as pairs (s, N), using the relations in (5.1B). □

If $(s, N) \in \mathcal{P}$ corresponds to $(t, z, \zeta) \in \Delta^a(n)^{+0}$, we write $L_{s,N} = L_{t,z}(\zeta)$ for the corresponding irreducible $\widetilde{TL}_n^a(q)$ module.

(5.2) PROPOSITION. *Suppose $(s(a_1, a_2), N_k)$ is the semisimple-nilpotent pair defined in* (5.1) ($\in \mathcal{P}^+$). *The corresponding cell module, $M_{s(a_1,a_2),N_k}$, is irreducible unless there is a solution (ϵ, t') of the equations*

$$a_1 a_2^{-1} = q^{(t+\epsilon t')}, \qquad \epsilon = \pm 1, \qquad t < t' \leq n, \qquad t \equiv t' (\text{mod } 2) \tag{5.2.1}$$

If Equations (5.2.1) *have a solution, then there is an injective homomorphism*

$$M_{s(a'_1,a'_2),N_{k'}} \longrightarrow M_{s(a_1,a_2),N_k},$$

where

$$k' = \frac{1}{2}(n - t'), \qquad a'_1 = a_1 q^{(1-\epsilon)(k-k')}, \qquad a'_2 = a_2 q^{-(1-\epsilon)(k-k')}. \tag{5.2.2}$$

Proof. The statement is a translation into the language of semisimple-nilpotent pairs of the fact (cf. (4.11) above) that if $(t, z, \zeta) \preceq (t', z', \zeta')$ in $\Delta^a(n)^0$, then there is an injective homomorphism between the corresponding cell modules of $\widetilde{TL}_n^a(q)$, while if (t, z, ζ) is maximal, then the corresponding cell module is irreducible. But (5.1B) shows that in the above correspondence between pairs and triples,

$$z^2 = a_1 a_2^{-1} q^{-t}. \tag{5.2.3}$$

By (4.1), (t, z, ζ) is maximal unless there is a solution to the equation $z^2 = q^{\epsilon t'}$ with ϵ, t' as in (5.2.1). If (t, z, ζ) is a triple corresponding to $(s(a_1, a_2), N_k)$, then translating this using (5.2.3) yields the first statement.

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Given a solution (ϵ, t') of (5.2.1) one uses (4.1) to determine the corresponding triple (t', z', ζ) such that $(t, z, \zeta) \stackrel{\circ}{\precsim} (t', z', \zeta)$. The corresponding semisimple-nilpotent pair $(s(a_1', a_2'), N_{k'})$ may then be determined using Equations (5.1.10) and the Equations (5.2.2) are the result. □

The partial order $\preceq$ may now be expressed in terms of the set $\mathcal{P}^+$ of semisimple-nilpotent pairs.

(5.3) DEFINITION. Let $\mathcal{P}^+ = \{(s(a_1, a_2), N_k)\}$ be the set of semisimple-nilpotent pairs as described in (5.1). Define the partial order $\preceq$ on $\mathcal{P}^+$ as that which is generated by the preorder $\stackrel{\circ}{\precsim}$ which asserts that $(s(a_1, a_2), N_k) \stackrel{\circ}{\precsim} (s(a_1', a_2'), N_{k'})$ if there exists $\epsilon = \pm 1$ such that if $t = n - 2k$, $t' = n - 2k'$,

$$k' < k, \qquad a_1 a_2^{-1} = q^{(t+\epsilon t')} \qquad a_1' = a_1 q^{(1-\epsilon)(k-k')} \qquad a_2' = a_2 q^{-(1-\epsilon)(k-k')}.$$

We may now express (4.11) in the language of pairs. As well as doing this, the next result gives some properties of the ordered set $\mathcal{P}^+$. The statement (iv) of Proposition (5.4) is related to a result of Zelevinsky [Z1].

(5.4) PROPOSITION. (i) *Let (s, N) and (s', N') be two elements of $\mathcal{P}^+$. The irreducible module $L_{s',N'}$ is a composition factor of multiplicity one in the cell module $M_{s,N}$ if $(s, N) \preceq (s', N')$ in $\mathcal{P}^+$. Otherwise its multiplicity is zero.*

(ii) *In Equations* (5.3)*, if $\epsilon = +1$, then $a_1' = a_1$ and $a_2' = a_2$. If $\epsilon = -1$, then $a_1' = a_2$ and $a_2' = a_1$. Thus in all cases, we have*

$$a_1'(a_2')^{-1} = q^{(t'+\epsilon t)} = (a_1 a_2^{-1})^{\epsilon}.$$

(iii) *If Equations* (5.3) *have a solution (ϵ, k'), then $a_1^n = q^{n(n-1)+k(\epsilon t'-n)}$.*

(iv) *If q is not a root of unity, then each cell module has at most two composition factors.*

Proof. The statement (i) is clear from (4.11). The assertions in (ii) are obtained by elementary manipulations of Equations in (5.3), while (iii) is a consequence of Equations (5.3), together with (5.1.3).

Now suppose q is not a root of unity, and that $(s(a_1, a_2), N_k) \stackrel{\circ}{\precsim} (s(a_1', a_2'), N_{k'})$ as in (5.3). Then by (ii) above, $a_1'(a_2')^{-1} = q^{(t'+\epsilon t)} = (a_1 a_2^{-1})^{\epsilon}$. Therefore if $(s(a_1', a_2'), N_{k'}) \stackrel{\circ}{\precsim} (s(a_1'', a_2''), N_{k''})$, we require $k'' < k' < k$ and $a_1'(a_2')^{-1} = q^{(t'+\epsilon' t')}$. Since q is not a root of unity, this entails $t' + \epsilon' t'' = t' + \epsilon t$, whence $t'' = t \not> t'$, so that no solution exists. The result now follows from (i). □

We now summarise our results as they apply to the affine Hecke algebra.

(5.5) THEOREM. *Let $\widetilde{H_n^a}(q)$ be the extended affine Hecke algebra of type $\tilde{A}_{n-1}$* (see (1.3)) *over an algebraically closed field R of characteristic zero. Let $G = SL_n(R)$ and let $\mathcal{P}^+$ be the set of G-conjugacy classes of pairs (s, N) with $s \in G$ semisimple,*

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$N \in Lie(G)$ nilpotent with at most two Jordan blocks, and $Ad(s) \cdot N = q^2 N$. Let $\mathcal{P}$ be the subset of $\mathcal{P}^+$ defined in Proposition (5.1C). Then $\widetilde{H_n^a}(q)$ has a set $\{M_{s,N} \mid (s, N) \in \mathcal{P}^+\}$ of 'cell modules' which have the following properties.

(i) *For each cell module $M_{s(a_1,a_2),N_k}$ there is a canonical $\widetilde{H_n^a}(q)$ invariant bilinear pairing*

$$\phi_{s(a_1,a_2),N_k} : M_{s(a_1,a_2),N_k} \times M_{s(a'_1,a'_2),N_k} \longrightarrow R,$$

where $a'_1 = a_1^{-1} q^{2(n-k-1)}$ and $a'_2 = a_2^{-1} q^{2(k-1)}$, such that $\phi_{s(a_1,a_2),N_k} \neq 0$ if and only if $(s(a_1, a_2), N_k) \in \mathcal{P}$.

(ii) *If $(s, N) \in \mathcal{P}$, $M_{s,N}$ has a unique simple quotient $L_{s,N}$. This is the quotient of $M_{s,N}$ by the radical of the form $\phi_{s,N}$.*

(iii) *The simple modules $\{L_{s,N} \mid (s, N) \in \mathcal{P}\}$ are pairwise non-isomorphic and form a complete set of irreducible $\widetilde{H_n^a}(q)$ modules which factor through the Temperley–Lieb quotient $\widetilde{TL_n^a}(q)$ (see* (1.7)*,* (1.8) *for the definition).*

(iv) *Each cell module has all its composition factors among the simple modules in* (ii).

(v) *Using the description of $\mathcal{P}$ in* (5.1) *and* (5.1.6)*, the elements of $\mathcal{P}$ are represented by pairs $(s(a_1, a_2), N_k)$. Then $\dim M_{s(a_1,a_2),N_k} = \binom{n}{k}$ and the multiplicity $[M_{s(a_1,a_2),N_k} : L_{s(a'_1,a'_2),N_{k'}}]$ is one if $(s(a_1, a_2), N_k) \preceq (s(a'_1, a'_2), N_{k'})$ in the partial order defined in* (5.3)*, and is zero otherwise.*

(vi) *In particular, $M_{s(a_1,a_2),N_k}$ is irreducible unless a_1^n and a_2^n are powers of q.*

(vii) *In the* (*unique*) *expression of the irreducible module $L_{s,N}$ as a linear combination of the cell modules $M_{s',N'}$ in the Grothendieck group $\Gamma(\widetilde{H_n^a}(q))$, the coefficients occurring are all* 0 *or* ± 1.

The statement (i) is a reformulation of the properties of the pairing $\phi_{t,z}$: $W_{t,z}(n) \times W_{t,z^{-1}}(n)R$ described just before (2.2). The relationship between (a_1, a_2) and (a'_1, a'_2) follows from the relations (5.1.10) applied to the triples (t, z, ζ) and (t, z^{-1}, ζ^{-1}). The last statement (vii) is a reformulation of Theorem (4.5) above. All the other statements are clear from the foregoing discussion, and the results of [GL2].

To illustrate our results, we give firstly an explicit description of the subregular case (cf. [L2] for a K-theoretic description of this case for all classical groups) and then two examples, one where q is not a root of unity, but the cell module has two composition factors, the other where q is a root of unity and the cell module has $n/4$ composition factors.

(5.6) EXAMPLE. Suppose N is subregular and that s corresponds to (a_1, a_2). Then $t = n - 2$ and $k = 1$ above, and if there were a solution to the equations in (5.3), we would have $k' = 0$, so that $t' = n$. A short computation shows that $M_{s,N}$ is irreducible unless

$$a_1^n = q^{n(n-2+\epsilon)} \tag{5.6.1}$$

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for $\epsilon = \pm 1$. If this equation has a solution, then $M_{s,N}$ has a composition factor $L_{s',N'}$ where N' is regular nilpotent and s' is semisimple with $a_1' = a_1 q^{(1-\epsilon)}$. Equation (5.6.1) has a solution for both $\epsilon = 1$ and $\epsilon = -1$ if and only if $q^{2n} = 1$. Hence if $q^{2n} \neq 1$, then for the n values of a_1 which satisfy (5.6.1), $M_{s,N}$ has two distinct composition factors, of dimension $1, n-1$. If $q^{2n} = 1$, then $M_{s,N}$ has three distinct composition factors, of dimension $1, 1, n-2$, except if $q^2 = 1$ in which case there are again two.

(5.7) EXAMPLES. (i) Suppose that q is not a root of unity. Take $n = 2m$ to be even with $m > 4$ and let a_1 be one of the n solutions of $a_1^n = q^{2m^2+10m-16}$. Let $a_2 = a_1 q^{-12}$. Then $(s(a_1, a_2), N_{m-2}) \in \mathcal{P}$ and a short computation using Equations in (5.3) shows that $M_{s(a_1,a_2),N_{m-2}}$ has two composition factors, viz. $L_{s(a_1,a_2),N_{m-2}}$ and $L_{s(a_1,a_2),N_{m-4}}$.

Similarly, if a_1 is one of the n solutions of $a_1^n = q^{2m^2-6m+16}$, and $a_2 = a_1 q^4$, then $M_{s(a_1,a_2),N_{m-2}}$ again has two composition factors, viz. $L_{s(a_1,a_2),N_{m-2}}$ and $L_{s(a_1 q^4, a_2 q^{-4}),N_{m-4}}$ (this is the case $\epsilon = -1$). In both cases, the dimension of the irreducible module $L_{s(a_1,a_2),N_{m-2}}$ is

$$\binom{2m}{m-4} \frac{12m+6}{(m-2)(m-3)},$$

and the second composition factor coincides with its cell module.

(ii) Suppose that $q^2 = -1$ and that $n = 4m$ is divisible by 4. Take $a_1 = a_2 = a$, with $a^n = 1$. Then the pair $(s(a_1, a_2), N_{\frac{n}{2}})$ lies in $\mathcal{P}$. Consider the cell module $M_{s(a,a),N_{2m}}$. It is straightforward to show that in this case, $(s(a,a), N_{2m}) \prec (s, N)$ if and only if $(s(a,a), N_{2m}) \overset{\circ}{\prec} (s, N)$, and that the set of pairs (s, N) for which this is true is $\{(s(a,a), N_{2m-2i}) \mid i = 0, 1, 2, \ldots, m\}$. Thus $M_{s(a,a),N_{2m}}$ has $m+1$ composition factors.

We conclude with some remarks concerning the connection between our results and the more general ones of Grojnowski [G]. According to [G], the irreducible modules L_ϕ, as well as the 'cell' or standard modules M_ϕ for $\widetilde{H_n^a}(q))$ correspond (in our case) to elements ϕ of a certain subset Φ_q° of the set of G-conjugacy classes of pairs (s, N) with $\mathrm{Ad}(s){\cdot}N = q^2 N$. It may be verified, using [G, Theorem 2], that our $\mathcal{P}$ is a subset of Φ_q°.

Moreover, although it speaks only of dimensions, [G, Theorem 1] implies that the multiplicity of the irreducible L_ψ $(\psi \sim (s', N'))$ in the cell module M_ϕ $(\phi \sim (s, N))$ is given by a 'Kazhdan–Lusztig' type coefficient $a_{\phi\psi}$, which is the multiplicity of the constant local system on $Z_G(s) \cdot N$ in the perverse extension of the corresponding local system on $Z_G(s') \cdot N'$ to its closure in the Lie algebra of G. Assuming this result, if we also assume★ that our cell modules coincide with those in [G], our results show that if N is a two-step (or regular) nilpotent element, the coefficients $a_{\phi\psi}$ are 0 or 1. Moreover, the coefficients $e_{\phi\psi}$ of the inverse matrix are $0, \pm 1$.

★**Note added in Proof.** This has now been proved and will appear in a subsequent work of the authors.

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Similar remarks apply to the connection between our results and those concerning standard modules and their decomposition in [Z2] and [R].

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