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p-DIVISIBILITY FOR COHERENT COHOMOLOGY

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Abstract

We prove that the coherent cohomology of a proper morphism of noetherian schemes can be made arbitrarily p-divisible by passage to proper covers (for a fixed prime p). Under some extra conditions, we also show that p-torsion can be killed by passage to proper covers. These results are motivated by the desire to understand rational singularities in mixed characteristic, and have applications in p-adic Hodge theory.

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1. Introduction

Fix a prime p. We will study the following question in mixed and positive characteristic geometry.

QUESTION 1.1. Given a scheme X and a class $\alpha \in H^n(X, \mathcal{O}_X)$ for some $n > 0$, does there exist a 'cover' $\pi : Y \to X$ such that $\pi^*\alpha$ is divisible by p?

Of course, as stated, the answer is trivially yes: take Y to be a disjoint union of opens occurring in a Čech cocyle representing α. However, the question becomes interesting if we impose geometric conditions on the cover π, such as properness. The first obstruction encountered is the potential noncompactness of X: passage to proper covers cannot make cohomology classes p-divisible for the simplest of open varieties (such as $\mathbf{A}^2_{\mathbf{F}_p} - \{0\}$; see Example 2.29). Our main result is that this is the only obstruction. In fact, we affirmatively answer the *relative* version of Question 1.1 for proper maps.

THEOREM 1.2. *Let $f : X \to S$ be a proper morphism of noetherian schemes with S affine. Then there exists an alteration $\pi : Y \to X$ such that $\pi^*(H^i(X, \mathcal{O}_X)) \subset p(H^i(Y, \mathcal{O}_Y))$ for $i > 0$.*

In fact, we prove a more precise derived category statement (see Remark 2.27), an analogous 'global' result (see Corollary 2.28), a stronger result in positive characteristic (see Theorem 3.1), and provide examples to show why the assumptions in Theorem 1.2 are essentially optimal (see Examples 2.29 and 2.30); we refer the reader to the body of the paper for a further discussion of these topics.

Theorem 1.2 is trivially true if p is invertible on S. At the opposite extreme, if S is an $\mathbf{F}_p$-scheme, then Theorem 1.2 says that alterations kill the higher (relative) cohomology of the structure sheaf for proper maps, which is the main theorem of [**Bhab**]. Hence, one may view Theorem 1.2 as a mixed characteristic lift of [**Bhab**]. However, the techniques of [**Bhab**] depend heavily on the availability of the Frobenius endomorphism, and thus do not transfer to the mixed characteristic world. Instead, our proof of Theorem 1.2 is geometric—we crucially use ideas from geometric class field theory and de Jong's work on stable curve fibrations—and can, in fact, be used to reprove [**Bhab**, Theorem 1.5].

The results of [**Bhab**] are quite useful in studying singularities in positive characteristic (see, for example, [**BST**]), and we expect that Theorem 1.2 will be similarly useful in studying singularities in mixed characteristic. Moreover, Theorem 1.2 (together with a derived refinement that is available when $\dim(S) \leqslant 1$; see Remark 2.11) has found surprising applications recently in p-adic Hodge theory: Beilinson's recently discovered h-localization approach to Fontaine's p-adic comparison conjectures (see [**Bei12**, **Bei11**] and also [**Bhac**]) uses Theorem 1.2 as the key geometric ingredient in the proof of the p-adic Poincaré lemma. A generalization of Theorem 1.2 (see Remark 3.3) would help extend these p-adic comparison results to the *relative* setting, and also have purely algebraic applications (see Remark 2.12). Additionally, Theorem 1.2 together with a geometric refinement has also been used by Deninger and Werner in the extension of their theory [**DW05**] of p-adic Higgs bundles to higher-dimensional cases. (This refinement requires an improvement of de Jong's alteration theorems that imposes finer control on the étale locus of the relevant alterations, and will appear elsewhere.)

Outline of the proof of Theorem 1.2. Assume first that f has relative dimension 1. If S was a point, then a natural strategy is the following: replace X with its normalization, identify the group $H^1(X, \mathcal{O}_X)$ with the tangent space to the Picard variety $\mathrm{Pic}^0(X)$ at the origin, and construct maps of curves such that the pullback on Picard varieties is divisible by p, at least at the expense of extending the ground field (these maps can be constructed by pulling back multiplication by p on the

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Picard variety along the Abel–Jacobi map, an old trick from geometric class field theory). For a nontrivial family of curves, the preceding argument can be applied to solve the problem over the generic point. Using the existence of compact moduli spaces of stable curves (or, even better, stable *maps*) and the theory of Neron models, one can then extend the generic solution to one over an alteration of S. This is not quite enough as the alteration is no longer affine, but it reduces Theorem 1.2 for maps of relative dimension $\leqslant 1$ to maps of relative dimension $\leqslant 0$. In general, theorems of de Jong show that an *arbitrary* proper morphism f of relative dimension d can be altered into a sequence of d iterated stable curve fibrations over an alteration of the base. The previous argument then lets us inductively reduce the general problem to that for maps of relative dimension $\leqslant 0$. At this point, we carefully fibre S itself over a lower-dimensional base while preserving certain cohomological properties, and proceed by (nested) induction on $\dim(S)$.

Organization of this paper. Theorem 1.2 is proved in Section 2: we discuss a reduction to relative dimension 0 in Section 2.1, and then prove this case in Section 2.2. Note that, when $\dim(S) \leqslant 1$, the latter step is unnecessary. In Section 3, we explain how to deduce the apparently stronger sounding [**Bhab**, Theorem 1.5] from Theorem 1.2.

Conventions. For any morphism $f : X \to S$ of noetherian schemes of finite Krull dimension, we define the relative dimension of f to be the supremum of the dimensions of the fibres of f over the generic points of S (with the convention that the dimension of the empty set is -1); this nonstandard convention will be useful in inductive arguments. For example, with this convention, any proper morphism of relative dimension 0 between integral noetherian schemes is an alteration. Given S-schemes $f : X \to S$ and $g : Y \to$, as well as an S-map $\pi : Y \to X$, we will write $\pi^*(\mathrm{R}^i f_* \mathcal{O}_X) \subset p(\mathrm{R}^i g_* \mathcal{O}_Y)$ to mean that the image of the pullback $\pi^* : \mathrm{R}^i f_* \mathcal{O}_X \to \mathrm{R}^i g_* \mathcal{O}_Y$ is contained in the subsehaf $p(\mathrm{R}^i g_* \mathcal{O}_Y) \subset \mathrm{R}^i g_* \mathcal{O}_Y$); a similar convention describes the meaning of $\pi^*(H^i(X, \mathcal{O}_X)) \subset p(H^i(Y, \mathcal{O}_Y))$.

2. The main theorem

In order to flesh out the outline from Section 1, we first make the following trivial observation.

LEMMA 2.1. *If Theorem 1.2 is true for excellent schemes S, then it is true in general.*

Proof. This is a standard approximation argument (see [**Gro66**, Section 8] and [**Sta14**, Tag 01YT] for more); for the convenience of the reader, we sketch the

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argument. Let $f : X \to S$ be a proper morphism of noetherian schemes. Then $\bigoplus_{i>0} H^i(X, \mathcal{O}_X)$ is a finite $\mathcal{O}_S$-module, so it suffices to construct alterations of X that make any fixed class $\alpha \in H^n(X, \mathcal{O}_X)$ (for $n > 0$) divisible by p. For a fixed α, the quadruple (X, S, f, α) can be approximated by a quadruple $(X_i, S_i, f_i, \alpha_i)$ with X_i and S_i excellent (see [Sta14, Tag 0A0X] for approximating (X, S, f) and [Sta14, Tag 09RE] for approximating α). By assumption, there is an alteration $\pi_i : Y_i \to X_i$ such that $\pi^*(\alpha_i) \in p(H^n(Y_i, \mathcal{O}_{Y_i}))$. The fibre product $\pi' : Y_i \times_{X_i} X \to X$ is then a proper surjective map such that $\pi'^*(\alpha) = 0$. Pick a closed subscheme $Y \subset Y_i \times_{X_i} X$ such that $Y \to X$ is an alteration; such a scheme Y exists as π' is surjective, and clearly does the job. □

Lemma 2.1 allows us to restrict to excellent schemes in what follows, which will be very convenient: it allows us to normalize our schemes in various constructions without leaving the noetherian world. We now make the following definition, integral to the rest of Section 2.

DEFINITION 2.2. Given a scheme S, we say that Condition $\mathcal{C}_d(S)$ is satisfied if S is excellent, and the following is satisfied by each irreducible component S_i of S: given a proper surjective morphism $f : X \to S_i$ of relative dimension d with X integral, there exists an alteration $\pi : Y \to X$ such that, with $g = f \circ \pi$, we have $\pi^*(\mathrm{R}^i f_* \mathcal{O}_X) \subset p(\mathrm{R}^i g_* \mathcal{O}_Y)$ for $i > 0$.

Note that $\mathcal{C}_{-1}(S)$ is vacuous: there is nothing to prove if S is empty, and, when S is nonempty, we simply observe there are no proper surjective maps between integral noetherian schemes of relative dimension -1. Hence, in what follows, we implicitly assume that $d \geqslant 0$. The main reason to make the preceding definition is the following elementary observation.

LEMMA 2.3. *If $\mathcal{C}_d(S)$ is satisfied for all excellent S and integers $d \geqslant 0$, then Theorem 1.2 is true.*

Proof. Choose $f : X \to S$ as in Theorem 1.2. It is enough to verify the conclusion of Theorem 1.2 when S is excellent thanks to Lemma 2.1. Write $X = \bigcup_i X_i$ as the union of irreducible components, and let $S_i = f(X_i) \subset S$ be the scheme-theoretic image. Then each S_i is an integral excellent scheme, and f induces proper surjective maps $f_i : X_i \to S_i$ of relative dimension d_i for $d_i \geqslant 0$. By assumption, there exist alterations $\pi_i : Y_i \to X_i$ such that, with $g = f_i \circ \pi_i$, we have $\pi_i^*(\mathrm{R}^j f_{i,*} \mathcal{O}_{X_i}) \subset p(\mathrm{R}^j g_i, *\mathcal{O}_{Y_i})$ for $j > 0$. As S is affine by assumption, so is each S_i, and hence the preceding containment translates to $\pi_i^*(H^j(X_i, \mathcal{O}_{X_i})) \subset p(H^j(Y_i, \mathcal{O}_{Y_i}))$ for $j > 0$. The induced map $Y := \bigsqcup_i Y_i \to X$ then does the job. □

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What follows is dedicated to verifying $\mathcal{C}_d(S)$ for excellent S and $d \geqslant 0$. (For the reader's convenience, we note here that this verification is local on the scheme S (see Lemma 2.15 below), so there is no loss of generality in assuming that S is affine.) More precisely, in Section 2.1, we will show that the validity of $\mathcal{C}_0(S)$ for all excellent base schemes S implies the validity of $\mathcal{C}_d(S)$ for all integers d and all excellent schemes S. We then proceed to verify Condition $\mathcal{C}_0(S)$ in Section 2.2.

2.1. Reduction to the case of relative dimension 0. The objective of the present section is to show that the relative dimension of maps considered in Theorem 1.2 can be brought down to 0 using suitable curve fibrations. The necessary technical help is provided by the following result, essentially borrowed from [**dJ97**], on extending maps between semistable curves.

PROPOSITION 2.4. *Fix an integral excellent base scheme B with generic point η. Assume that we have semistable curves $\phi : C \to B$ and $\phi'_\eta : C'_\eta \to \eta$, and a B-morphism $\pi_\eta : C'_\eta \to C$. If C'_η is geometrically irreducible, then we can alter B to extend π_η to a map of semistable cures over B; that is, there exists an alteration $\tilde{B} \to B$ such that $C'_\eta \times_B \tilde{B}$ extends to a semistable curve over $\tilde{C}' \to \tilde{B}$ with $\tilde{C}'$ integral, and the map $\pi_\eta \times_B \tilde{B}$ extends to a $\tilde{B}$-map $\tilde{\pi} : \tilde{C}' \to C \times_B \tilde{B}$.*

Proof. We may extend C'_η to a proper B-scheme using the Nagata compactification theorem (see [**Con07**, Theorem 4.1]). By taking the closure of the graph of the rational map defined from this compactification to C by π_η, we obtain a proper dominant morphism $\phi' : C' \to B$ of integral schemes whose generic fibre is the geometrically irreducible curve $\phi'_\eta : C'_\eta \to \eta$, and a B-map $\pi : C' \to C$ extending $\pi_\eta : C'_\eta \to C$. The idea, borrowed from [**dJ96**, Section 4.18], is the following: modify B to make the strict transform of $C' \to B$ flat, alter the result to get enough sections which make the resulting datum generically a stable curve, use compactness of the moduli space of stable curves to extend the generically stable curve to a stable curve after further alteration, and then use stability and flatness to get a well-defined morphism from the resulting stable curve to the original one extending the existing one over the generic point. Instead of rewriting the details here, we refer the reader to [**dJ97**, Theorem 5.9], which directly applies to ϕ', to finish the proof (the integrality of $\tilde{C}'$ follows from the irreducibility of the generic fibre $C'_\eta \times_B \tilde{B}$). □

REMARK 2.5. Proposition 2.4, while sufficient for the application we have in mind, is woefully inadequate in terms of the permissible generality. Similar ideas can, in fact, be used to show something much better: for any flat projective morphism $X \to B$, there exists an ind-proper algebraic stack $\overline{\mathcal{M}}_g(X) \to B$ parameterizing B-families of stable maps from genus g curves to X; see [**AO01**].

In addition to constructing maps of semistable curves, we will also need to construct maps that preserve sections. The following lemma says that we can do so at a level of generality sufficient for our purposes.

LEMMA 2.6. *Fix an integral excellent base scheme B, two semistable curves $\phi_1 : C_1 \to B$ and $\phi_2 : C_2 \to B$, and a surjective B-map $\pi : C_2 \to C_1$. Then any section of ϕ_1 extends to a section of ϕ_2 after an alteration of B; that is, given a section $s : B \to C_1$, there exists an alteration $b : \tilde{B} \to B$ such that the induced map $\tilde{B} \to B \to C_1$ factors through a map $\tilde{B} \to C_2$.*

Proof. Let η be the generic point of B, let $s : B \to C_1$ be the section of ϕ_1 under consideration, and let $s_\eta : \eta \to C_1$ denote the restriction of s to the generic point. By the surjectivity of π, the map $\pi_\eta : (C_2)_\eta \to (C_1)_\eta$ is surjective. Thus, there exists a finite surjective morphism $\eta' \to \eta$ such that the induced map $\eta' \to C_1$ factors through some map $s'_\eta : \eta' \to C_2$. If B' denotes the normalization of B in $\eta' \to \eta$, then the map s'_η spreads out to give a rational map $B' \dashrightarrow C_2$. Taking the closure of the graph of this rational map (over B) gives an alteration $b : \tilde{B} \to B$ such that the induced map $\tilde{B} \to C_1$ factors through a map $\tilde{s}_2 : \tilde{B} \to C_2$, proving the claim. $\square$

Proposition 2.4 lets us to construct maps of semistable curves by constructing them generically. We now construct the desired maps generically; the idea of this construction belongs to class field theory.

LEMMA 2.7. *Let X be a proper curve over a field k. Then there exist a field extension k' of k, a proper smooth curve Y over k' with geometrically irreducible connected components, and a finite surjective map $\pi : Y \to X_{k'}$ such that the induced map $\pi^* : \mathrm{Pic}(X_{k'}) \to \mathrm{Pic}(Y)$ of fppf sheaves of abelian groups on $\mathrm{Spec}(k)$ is divisible by p in $\mathrm{Hom}(\mathrm{Pic}(X_{k'}), \mathrm{Pic}(Y))$.*

Proof. We first assume that $k = \overline{k}$ is algebraically closed. To prove the statement, we may replace X with any finite cover, so we can assume that X is normal, and hence smooth as k is perfect. Moreover, it is enough to solve the problem over each connected component of X (by taking a disjoint union), so we may even assume that X is a proper smooth geometrically connected curve. After possibly replacing X with some ramified cover, we may assume that X has genus $\geqslant 1$. Fix a point $x_0 \in X(k)$ (which always exists as k is algebraically closed). The point x_0

defines the Abel–Jacobi map $X \to \mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ via $x \mapsto \mathcal{O}([x]) \otimes \mathcal{O}(-[x_0])$. The Riemann–Roch theorem implies that this map is a closed immersion. We set $\pi : Y \to X$ to be the normalized inverse image of X under the multiplication by p map $[p] : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. It follows that the pullback $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ factors through multiplication by p on $\mathrm{Pic}(X)$ and is therefore divisible by p. Moreover, by construction, Y is a proper smooth k-curve, so this solves the problem over $\overline{k}$. The general case is easily deduced from this case by a limit argument as every proper smooth connected curve over $\overline{k}$ descends to a proper smooth geometrically connected curve over some finite extension k'/k; the details are left to the reader. □

REMARK 2.8. Lemma 2.7 and the discussion below use basic properties of the relative Picard scheme of a proper flat family $f : X \to S$ of curves. A general reference for this object is [**BLR90**, Sections 8–9]. In this paper, we define $\mathrm{Pic}(X/S)$ as the fppf sheaf $\mathrm{R}^1 f_* \mathbf{G}_m$ on the category of all S-schemes. If f has a section, then one can identify $\mathrm{Pic}(X/S)(T) \simeq \mathrm{Pic}(X \times_S T)/\mathrm{Pic}(T)$. Since f has relative dimension 1, deformation theory implies that $\mathrm{Pic}(X/S)$ is smooth (as a functor). Two additional relevant properties are the following: (a) if f has geometrically reduced fibres, then $\mathrm{Pic}(X/S)$ is representable by a smooth group scheme (by Artin's work), and (b) if f is additionally semistable, then the connected component $\mathrm{Pic}^0(X/S)$ is semiabelian.

Lemma 2.7 allows us to construct covers of curves that induce a map divisible by p on Picard schemes. Note that this latter property is meaningful in characteristic 0, and yet implies divisibility by p on cohomology in all characteristics. Using this observation, we globalize the construction in Lemma 2.7 to arrive at one of the primary ingredients of our proof of Theorem 1.2.

PROPOSITION 2.9. *Let $\phi : X \to T$ be a projective family of semistable curves with T integral and excellent. Then there exists a commutative diagram*

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi"] \arrow[d, "\tilde{\phi}"] & X \arrow[d, "\phi"] \\ \tilde{T} \arrow[r, "\psi"] & T \end{tikzcd}$$

satisfying the following.

(1) *The scheme $\tilde{T}$ is integral, and the map ψ is an alteration.*

(2) *$\tilde{\phi}$ is a projective family of semistable curves, and the map π is proper and surjective.*

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(3) *The pullback map* $\psi^*\mathrm{R}^1\phi_*\mathcal{O}_X \to \mathrm{R}^1\tilde{\phi}_*\mathcal{O}_{\tilde{X}}$ *is divisible by* p *in* $\mathrm{Hom}(\psi^*\mathrm{R}^1\phi_*\mathcal{O}_X, \mathrm{R}^1\tilde{\phi}_*\mathcal{O}_{\tilde{X}})$.

Proof. For any family $\phi : X \to T$ of projective semistable curves, there is a natural identification of $\mathrm{R}^1\phi_*\mathcal{O}_X$ with the normal bundle of the zero section of the semiabelian scheme $\mathrm{Pic}^0(X/T) \to T$; see [**BLR90**, Theorem 1, Section 8.4]. Moreover, given another semistable curve $\tilde{\phi} : \tilde{X} \to T$ and a morphism of semistable curves $\pi : \tilde{X} \to X$ over T, the induced map $\mathrm{R}^1\pi : \mathrm{R}^1\phi_*\mathcal{O}_X \to \mathrm{R}^1\tilde{\phi}_*\mathcal{O}_{\tilde{X}}$ can be identified as the map on the corresponding normal bundles at 0 induced by the natural morphism $\mathrm{Pic}^0(\pi) : \mathrm{Pic}^0(X/T) \to \mathrm{Pic}^0(\tilde{X}/T)$. As multiplication by n on smooth commutative T-group schemes induces multiplication by n on the normal bundles at 0, if $\mathrm{Pic}^0(\pi)$ is divisible by p, so is $\mathrm{R}^1\pi$. As the formation of $\mathrm{R}^1\phi_*\mathcal{O}_X$ commutes with arbitrary base change on T, it suffices to show that there exist an alteration $\psi : \tilde{T} \to T$ and a morphism of semistable curves $\pi : \tilde{X} \to X \times_T \tilde{T}$ over $\tilde{T}$ such that the induced map $\mathrm{Pic}^0(\pi)$ is divisible by p. Our strategy will be to construct a solution to this problem generically on T, and then use Proposition 2.4 and properties of semiabelian schemes to globalize.

Let η denote the generic point of T. By Lemma 2.7, we can find a finite extension $\eta' \to \eta$, and a proper smooth curve $Y_{\eta'} \to \eta'$ with geometrically irreducible components such that the induced map $\mathrm{Pic}^0(X_{\eta'}) \to \mathrm{Pic}^0(Y_{\eta'})$ is divisible by p. After replacing the map $X \to T$ with its base change along the normalization of T in $\eta' \to \eta$, we may assume that $\eta' = \eta$. The situation so far is summarized in the diagram

$$\begin{tikzcd} \bigsqcup_i Y_{\eta_i} = Y_\eta \arrow[r] \arrow[d] & X \arrow[d] \\ \eta \arrow[r] & T \end{tikzcd}$$

where the Y_{η_i} are the (necessarily) geometrically irreducible components of Y_η. As each of the Y_{η_i} is smooth as well, we may apply Proposition 2.4 to extend each Y_{η_i} to a semistable curve $Y_i \to T_i$, where $T_i \to T$ is some alteration of T, such that the map $Y_{\eta_i} \to X$ extends to a map $Y_i \to X$. Setting $\tilde{T}$ to be a dominating irreducible component of the fibre product of all the T_i over T, and setting $\tilde{X}$ to be the disjoint union of $Y_i \times_{T_i} \tilde{T}$, we find the following: an alteration $\tilde{T} \to T$, a semistable curve $\tilde{X} \to \tilde{T}$ extending $Y_\eta \times_T \tilde{T}$, and a map $\tilde{\pi} : \tilde{X} \to X$ extending the existing one over the generic point. We will now check the required divisibility.

As explained earlier, we must show that the resulting map $\mathrm{Pic}^0(X \times_T \tilde{T}/\tilde{T}) \to \mathrm{Pic}^0(\tilde{X}/\tilde{T})$ is divisible by p. This divisibility holds at the generic point of $\tilde{T}$ by construction, and hence also over a sufficiently small Zariski dense open subset

$W \subset \tilde{T}$ (as the functors involved are finitely presented). Next, note that the relative Pic^0 of any semistable curve is a semiabelian group scheme. The normality of $\tilde{T}$ implies that restriction to W is a fully faithful functor from the category of semiabelian schemes over $\tilde{T}$ to the analogous category over W (see [**FC90**, Proposition I.2.7]). In particular, the divisibility by p over W ensures the global divisibility by p, proving the existence of $\tilde{X}$ with the desired properties. □

Recall that our immediate goal is to reduce Theorem 1.2 to verifying Condition $\mathcal{C}_0(S)$. Proposition 2.9 lets us make the relative cohomology of a curve fibration divisible by p on passage to alterations, while de Jong's theorems let us alter an arbitrary proper dominant morphism into a tower of curve fibrations over an alteration of the base. These two ingredients combine to yield the promised reduction in relative dimension.

PROPOSITION 2.10. *Let S be an excellent scheme such that Condition $\mathcal{C}_0(S)$ is satisfied. Then $\mathcal{C}_d(S)$ is satisfied for all $d \geqslant 0$.*

Proof. As Condition $\mathcal{C}_d(S)$ is defined in terms of the irreducible components of S, we may assume that S is integral itself. Fix integers $d, i > 0$, an integral scheme X, and a proper surjective morphism $f : X \to S$ of relative dimension d. By [**dJ97**, Corollary 5.10], after replacing X by an alteration, we may assume that $f : X \to S$ factors as follows:

$$\begin{tikzcd} X \arrow[d, "\phi"'] \arrow[dr, "f"] & \\ T \arrow[r, "f'"'] & S \end{tikzcd}$$

Here ϕ is a projective semistable curve, and f' is a proper surjective morphism of integral excellent schemes of relative dimension $d-1$. Also, at the expense of altering T further, we may assume that ϕ has a section $s : T \to X$. As ϕ is a semistable curve, we have $\mathcal{O}_T \simeq \phi_* \mathcal{O}_X$. Using the section s and the Leray spectral sequence, we find an exact sequence

$$0 \to \mathrm{R}^i f'_* \mathcal{O}_T \to \mathrm{R}^i f_* \mathcal{O}_X \to \mathrm{R}^{i-1} f'_* \mathrm{R}^1 \phi_* \mathcal{O}_X \to 0$$

that is naturally split by the section s; in fact, this arises by applying $\mathrm{R}^i f'_*$ to the triangle

$$\mathcal{O}_T \to \mathrm{R}\phi_* \mathcal{O}_X \to \mathrm{R}^1 \phi_* \mathcal{O}_X[-1] \stackrel{+1}{\to} \mathcal{O}_T[1],$$

which is split by the choice of s. Our strategy will be to prove divisibility for $\mathrm{R}^i f_* \mathcal{O}_X$ by working with the two edge pieces occurring in the exact sequence

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above. In more detail, we apply the inductive hypothesis to choose an alteration $\pi' : T' \to T$ such that, with $g' = f' \circ \pi'$, we have $\pi'^*(\mathrm{R}^i f'_*\mathcal{O}_T) \subset p(\mathrm{R}^i g'_*\mathcal{O}_{T'})$ for $i > 0$. The base change of ϕ and s along π' define for us a diagram

$$\begin{tikzcd}
X' = X \times_T T' \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\phi'"] & X \arrow[d, "\phi"] \arrow[dr, "f"] & \\
T' \arrow[r, "\pi'"] \arrow[u, bend left, "s'"] & T \arrow[r, "f'"] \arrow[u, bend left, "s"] & S
\end{tikzcd}$$

The commutativity of the preceding diagram gives rise to a morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{R}^i f'_*\mathcal{O}_T \arrow[r] \arrow[d, "\pi'^*"] & \mathrm{R}^i f_*\mathcal{O}_X \arrow[r] \arrow[d, "\mathrm{pr}_1^*"] \arrow[l, bend right, "s^*"'] & \mathrm{R}^{i-1} f'_*\mathrm{R}^1\phi_*\mathcal{O}_X \arrow[r] \arrow[d, "\mathrm{R}^1\,\mathrm{pr}_1^*"] & 0 \\
0 \arrow[r] & \mathrm{R}^i g'_*\mathcal{O}_{T'} \arrow[r] & \mathrm{R}^i(f \circ \mathrm{pr}_1)_*\mathcal{O}_{X'} \arrow[r] \arrow[l, bend right, "s'^*"'] & \mathrm{R}^{i-1} g'_*\mathrm{R}^1\phi'_*\mathcal{O}_{X'} \arrow[r] & 0
\end{tikzcd}$$

compatible with the exhibited splittings. The map ϕ' is a semistable curve with a section s'. Applying Proposition 2.9 and using Lemma 2.6, we can find a commutative diagram

$$\begin{tikzcd}
X'' \arrow[r, "a"] \arrow[d, "\phi''"] & X' = X \times_T T' \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\phi'"] & X \arrow[d, "\phi"] \arrow[dr, "f"] & \\
T'' \arrow[r, "\pi''"] \arrow[u, bend left, "s''"] & T' \arrow[r, "\pi'"] \arrow[u, bend left, "s'"] & T \arrow[r, "f'"] \arrow[u, bend left, "s"] & S
\end{tikzcd}$$

where π'' is an alteration, ϕ'' is a semistable curve, a is an alteration, and s'' is a section of ϕ'' (compatible with s' and s thanks to the commutativity of the picture), such that $a^*\mathrm{R}^1\phi'_*\mathcal{O}_{X'} \to \mathrm{R}^1\phi''_*\mathcal{O}_{X''}$ is divisible by p. Setting $g'' = g' \circ \pi''$ gives a diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{R}^i f'_*\mathcal{O}_T \arrow[r] \arrow[d, "\pi'^*"] & \mathrm{R}^i f_*\mathcal{O}_X \arrow[r] \arrow[d, "\mathrm{pr}_1^*"] \arrow[l, bend right, "s^*"'] & \mathrm{R}^{i-1} f'_*\mathrm{R}^1\phi_*\mathcal{O}_X \arrow[r] \arrow[d, "\mathrm{R}^1\,\mathrm{pr}_1^*"] & 0 \\
0 \arrow[r] & \mathrm{R}^i g'_*\mathcal{O}_{T'} \arrow[r] \arrow[d, "\pi''^*"] & \mathrm{R}^i(f \circ \mathrm{pr}_1)_*\mathcal{O}_{X'} \arrow[r] \arrow[d, "a^*"] \arrow[l, bend right, "s'^*"'] & \mathrm{R}^{i-1} g'_*\mathrm{R}^1\phi'_*\mathcal{O}_{X'} \arrow[r] \arrow[d, "\mathrm{R}^1 a^*"] & 0 \\
0 \arrow[r] & \mathrm{R}^i g''_*\mathcal{O}_{T''} \arrow[r] & \mathrm{R}^i(f \circ \mathrm{pr}_1 \circ a)_*\mathcal{O}_{X''} \arrow[r] \arrow[l, bend right, "s''^*"'] & \mathrm{R}^{i-1} g''_*\mathrm{R}^1\phi''_*\mathcal{O}_{X''} \arrow[r] & 0
\end{tikzcd}$$

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which is compatible with the exhibited splittings of each sequence; here the vertical maps on the right are the evident pullbacks induced by pr_1 and a, respectively. As $\mathrm{R}^1 a^*$ is divisible by p, the image of the right vertical composition is divisible by p. The image of the left vertical composition is divisible by p by construction of π'. By compatibility of the morphism of exact sequences with the exhibited splittings, the image of the middle vertical composition is also divisible by p. Replacing X'' by an irreducible component dominating X then proves the claim. □

REMARK 2.11. Consider the special case of Theorem 1.2 when the base S has dimension $\leqslant 1$; for example, S could be the spectrum of a discrete valuation ring. Any alteration of such an S is a finite cover of S, so $\mathcal{C}_0(S)$ is trivially satisfied. Proposition 2.10 then already implies that Theorem 1.2 is true for such S. In fact, tracing through the proof (and using the strong p-divisibility in Proposition 2.9(3)), one observes that a stronger statement has been shown: for any proper morphism $f : X \to S$, there is a proper surjective morphism $\pi : Y \to X$ such that, with $g = f \circ \pi$, the pullback $\pi^* : \tau_{\geqslant 1} \mathrm{R} f_* \mathcal{O}_X \to \tau_{\geqslant 1} \mathrm{R} g_* \mathcal{O}_Y$ induces the 0 map on $- \otimes^{\mathrm{L}}_{\mathbf{Z}} \mathbf{Z}/p$. Concretely, this means the following: in addition to making higher cohomology classes p-divisible on passage to alterations, one can also kill p-torsion classes by alterations. It is this stronger statement that is used in [**Bei12**, **Bei11**, **Bhac**]. We hope in the future to extend this stronger conclusion to higher-dimensional base schemes S.

REMARK 2.12. The stronger statement discussed at the end of Remark 2.11 has purely algebraic consequences: it lifts [**Bhab**, Theorem 1.5] to p-adically complete noetherian schemes (after a small extra argument). In particular, it implies that splinters over $\mathbf{Z}_p$ have rational singularities after inverting p. Such a statement is interesting from the perspective of the direct summand conjecture (see [**Hoc07**]) as there are *no* known nontrivial restrictions on a splinter in mixed characteristic (to the best of our knowledge).

2.2. The case of relative dimension 0. In this section we will verify Condition $\mathcal{C}_0(S)$ for all excellent schemes S. After unwrapping definitions and some easy reductions, one reduces to showing the following: given an alteration $f : X \to S$ with S affine and a class $\alpha \in H^i(X, \mathcal{O}_X)$ with $i > 0$, there exists an alteration $\pi : Y \to X$ such that $p \mid \pi^*(\alpha)$. If α arose as the pullback of a class under a morphism $X \to \overline{X}$ with $\overline{X}$ proper over an affine base of dimension $\dim(S) - 1$, then we may conclude by induction using Proposition 2.10. The proof below will show that, at the expense of certain technical but manageable modifications, this method can be pushed through; the basic geometric ingredient is Lemma 2.21. The main result is the following.

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PROPOSITION 2.13. *Condition $\mathcal{C}_0(S)$ is satisfied by all excellent schemes S.*

Our proof of Proposition 2.13 will consist of a series of reductions which massage S until it becomes a geometrically accessible object (see Lemma 2.19 for the final outcome of these 'easy' reductions); these reductions are standard, especially in arguments involving the h-topology (see [**Org06**], for example), but they are included here for completeness and clarity.

WARNING 2.14. For conceptual clarity, we often commit the following abuse of mathematics in what follows: when proving a statement of the form that $\mathcal{C}_d(S)$ is satisfied for all integers d and a particular scheme S, we ignore the restrictions on integrality and relative dimension imposed by Condition $\mathcal{C}_d(S)$ while making certain constructions; the reader can check that in each case the statement to be proven follows from our constructions by taking suitable irreducible components (see Lemma 2.15 for an example). We strongly believe that this abuse, while easily fixable, enhances readability.

We first observe that the problem is Zariski local on S.

LEMMA 2.15. *Condition $\mathcal{C}_d(S)$ is local on an excellent scheme S for the Zariski topology; that is, if $\{U_i \hookrightarrow S\}$ is a Zariski open cover of S, then $\mathcal{C}_d(S)$ is satisfied if and only if $\mathcal{C}_d(U_i)$ is satisfied for all i.*

Proof. We will first show that $\mathcal{C}_d(S)$ implies $\mathcal{C}_d(U)$ for any open $j : U \to S$. By Nagata compactification (see [**Con07**, Theorem 4.1]), given any alteration $f : X \to U$, we can find an alteration $\overline{f} : \overline{X} \to S$ extending f over U. As $j : U \to S$ is flat, we have that $j^*\mathrm{R}^i\overline{f}_*\mathcal{O}_{\overline{X}} = \mathrm{R}^i f_*\mathcal{O}_X$. By assumption, we can find an alteration $\overline{\pi} : \overline{Y} \to \overline{X}$ such that, with $\overline{g} = \overline{f} \circ \overline{\pi}$, we have $\overline{\pi}^*\mathrm{R}^i\overline{f}_*\mathcal{O}_{\overline{X}} \subset p(\mathrm{R}^i\overline{g}_*\mathcal{O}_{\overline{Y}})$. Restricting to U and using flat base change for $\overline{g}$ produces the desired result.

Conversely, assume that there exists a cover $\{U_i \hookrightarrow S\}$ such that $\mathcal{C}_d(U_i)$ is true. Given an alteration $f : X \to S$, define $f_i : X_{U_i} \to U_i$ to be the natural map. The assumption implies that we can find alterations $\pi_i : Y_i \to X_{U_i}$ such that, with $g_i = f_i \circ \pi_i$, we have $\pi_i^*(\mathrm{R}^j f_{i*}\mathcal{O}_{X_{U_i}}) \subset p(\mathrm{R}^j g_{i*}\mathcal{O}_{Y_i})$ for each i. By an elementary closure argument (see [**Bhab**, Proposition 4.1]), we can find $\pi : Y \to X$ such that $\pi \times_S U_i$ factors through π_i. As taking higher pushforwards commutes with restricting to open subsets, we see that $\pi^*(\mathrm{R}^j f_*\mathcal{O}_X) \subset \mathrm{R}^j g_*\mathcal{O}_Y$ is a subsheaf that is locally inside $p(\mathrm{R}^j g_*\mathcal{O}_Y)$. As containments between two subsheaves of a given sheaf can be detected locally, the claim follows. $\square$

Next, we note that the problem is insensitive to certain finite covers.

LEMMA 2.16. *Let $g : S' \to S$ be a finite surjective morphism of excellent schemes such that each generic point of S' lies over a generic point of S. Then $\mathcal{C}_d(S)$ is satisfied if and only if $\mathcal{C}_d(S')$ is satisfied.*

Any finite *flat* surjection has the property of this lemma.

Proof. By Lemma 2.15, we may assume that S and S' are affine. Let $\overline{S} := \bigsqcup S_i$ and $\overline{S'} = \bigsqcup S'_j$ be the decomposition of S and S' into irreducible components. The assumption on g ensures that each S'_j dominates some S_i via a finite surjective map, and further that each S_i is dominated by some S'_j via a finite surjective map.

Now assume that $\mathcal{C}_d(S')$ is satisfied. Fix some proper surjective map $f : X \to S_i$ of relative dimension d for some i. Choose some j such that $g(S'_j) = S_i$. Then $X \times_{S_i} S'_j \to S'_j$ is a proper surjective map of relative dimension d. By the assumption, we can find an alteration $Y \to X \times_{S_i} S'_j$ such that the image of $H^k(X \times_{S_i} S'_j, \mathcal{O}_{X \times_{S_i} S'_j})$ in $H^k(Y, \mathcal{O}_Y)$ is divisible by p for $k > 0$. The composite $Y \to X \times_{S_i} S'_j$ then does the job for X.

Conversely, assume that $\mathcal{C}_d(S)$ is satisfied. Fix some proper surjective map $f : X \to S'_j$ of relative dimension d. Choose i such that $g(S'_j) = S_i$. Then the composite $X \to S'_j \to S_i$ is a proper surjective map of relative dimension d, so we can find an alteration $Y \to X$ in the category of S_i-schemes such that the image of $H^k(X, \mathcal{O}_X)$ in $H^k(Y, \mathcal{O}_Y)$ is divisible by p for $k > 0$. Viewing $Y \to X$ as a morphism of S'_j-schemes then solves the problem. $\square$

Finally, we show how to étale localize.

LEMMA 2.17. *Condition $\mathcal{C}_d(S)$ is étale local on S; that is, if $g : S' \to S$ is a surjective étale morphism, then $\mathcal{C}_d(S)$ is satisfied if and only if $\mathcal{C}_d(S')$ is satisfied.*

Proof. Assume first that $\mathcal{C}_d(S)$ is satisfied. By Lemma 2.15, we may assume that S and S' are affine. By Zariski's main theorem [Gro66, Théorème 8.12.6], we can factor g as $S' \xrightarrow{j} \overline{S'} \xrightarrow{\overline{g}} S$ with j a dense open immersion, and $\overline{g}$ finite surjective. By density of j, since g is étale, it follows that $\overline{g}$ carries generic points of $\overline{S'}$ to generic points of S. Lemma 2.16 then shows that $\mathcal{C}_d(\overline{S'})$ is satisfied, whence Lemma 2.15 shows that $\mathcal{C}_d(S')$ is also satisfied.

For the converse direction, assume that $\mathcal{C}_d(S')$ is satisfied. Using Lemma 2.15, we may assume that S and S' are both affine. An observation of Gabber (see

[Bhaa, Lemma 2.1]) lets us find a diagram

$$\begin{tikzcd} \bigsqcup_i U_i \arrow[r] \arrow[d, "h"] & T \arrow[d, "\pi"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

such that π is finite flat and surjective, $\bigsqcup U_i \to T$ forms a Zariski cover, and h is some map of S-schemes. The commutativity of the diagram forces h to be quasifinite, while the flatness of π and g ensures that h carries generic points of $\bigsqcup_i U_i$ to generic points of S' (which, in turn, lie over the generic points of S). We can then factor h as $\bigsqcup_i U_i \xrightarrow{k} \overline{U} \xrightarrow{\overline{h}} S'$ with k a dense open immersion, and $\overline{h}$ a finite morphism that carries generic points of $\overline{U}$ to generic points of S' (but $\overline{h}$ may fail to be surjective). The proof of the second half of Lemma 2.16 then shows that $\mathcal{C}_d(\overline{U})$ is satisfied. Lemma 2.15 then shows that $\mathcal{C}_d(\bigsqcup_i U_i)$ and $\mathcal{C}_d(T)$ are also satisfied. This implies that $\mathcal{C}_d(S)$ is satisfied by Lemma 2.16. □

Having étale localized, we prove an approximation result.

LEMMA 2.18. *Condition $\mathcal{C}_d(S)$ is satisfied by all excellent schemes S if it is satisfied by all affine schemes S of finite type over $\mathbf{Z}$.*

Proof. Assume that $\mathcal{C}_d(S)$ is satisfied for all affine schemes of finite type over $\mathbf{Z}$. By Lemma 2.15, it is enough to check $\mathcal{C}_d(S)$ for a fixed affine excellent S. In fact, by the very definition of $\mathcal{C}_d(S)$, we may assume that S is integral. Fix a proper surjective map $f : X \to S$ of relative dimension d with X integral. Standard approximation results (see [Sta14, Tag 0A0P]) allow us to write $S = \lim S_i$ as an inverse limit of affine schemes of finite type over $\mathbf{Z}$ such that f arises as the inverse limit of a tower $\{f_i : X_i \to S_i\}$ of proper morphisms. By replacing each X_i with the scheme-theoretic closure of the image of X, we may assume that each X_i is integral, and that $X \to X_i$ is dominant. Applying the same procedure to the tower $\{S_i\}$ then allows us to realize $f : X \to S$ as a limit of a tower $\{f_i : X_i \to S_i\}$ of proper surjective maps between integral schemes of finite type over $\mathbf{Z}$ with S_i affine; here the surjectivity of f_i is the consequence of the dominance of $X \to S \to S_i$ and the properness of f_i. If $\eta_i \subset S_i$ denotes the pro-open subset defined by the generic point of each S_i, then base change gives a system $\{X_{\eta_i} \to \eta_i\}$ whose limit realizes the generic fibre $X_\eta \to \eta$ of f. As the category of finitely presented η-schemes is the filtered colimit of the category of finitely presented η_i-schemes (see, for example, [Sta14, Tag 01ZM]), it follows that $d := \dim(X_\eta) = \dim(X_{\eta_i})$ for $i \gg 0$. In other words, after possibly passing to a cofinal index set, each f_i has relative dimension d. Now any cohomology class $\alpha \in H^j(X, \mathcal{O}_X)$ arises as the pullback of some $\alpha_i \in H^j(X_i, \mathcal{O}_{X_i})$ for $i \gg 0$, as in the proof of Lemma 2.1. If $j > 0$, then, by assumption, there exists an alteration $\pi_i : Y_i \to X_i$ such that $\pi_i^* \alpha_i$ is divisible by p. It follows then that any irreducible component of : $Y_i \times_{X_i} X$ dominating X provides the desired alteration. $\square$

Next, we localize at p.

LEMMA 2.19. *Condition $\mathcal{C}_d(S)$ is satisfied by all affine excellent schemes S if it is satisfied by all affine schemes S of finite type over $\mathbf{Z}_{(p)}$.*

Proof. Assume that $\mathcal{C}_d(S)$ is satisfied by all finite type affine $\mathbf{Z}_{(p)}$-schemes. By Lemma 2.18, we must check that $\mathcal{C}_d(S)$ is satisfied for any affine finite type $\mathbf{Z}$-scheme S. Clearly we may assume that S is integral. If p is invertible on S, there is nothing to show. If $p = 0$ on S, then S is itself a finite type $\mathbf{Z}_{(p)}$-scheme, so we know the claim. For the remaining case, we may assume that S is $\mathbf{Z}_{(p)}$-flat. Fix a proper surjective morphism $f : X \to S$ of relative dimension d with X integral, and write $S_{(p)}$ for the localization of S at p, etc. Then $f_{(p)}$ is a proper surjective morphism of relative dimension d between integral schemes as well (as the generic point of S comes from $S_{(p)}$). By assumption, there exists an alteration $\pi_{(p)} : Y_{(p)} \to X_{(p)}$ of integral schemes such that the image of $H^i(X_{(p)}, \mathcal{O}_{X_{(p)}})$ is divisible by p in $H^i(Y_{(p)}, \mathcal{O}_{Y_{(p)}})$ for $i > 0$. Spreading out, there exist an open $U \subset S$ containing $S_{(p)}$ and an alteration $\pi_U : Y_U \to X_U$ of integral U-schemes realizing $\pi_{(p)}$ on restriction to $S_{(p)}$. By Nagata compactification, we may find an alteration $\pi : Y \to X$ of integral S-schemes realizing $\pi_{(p)}$ over $S_{(p)}$. Let $g : Y \to S$ denote the structure map. It remains to check that $\pi^*(\mathrm{R}^i f_* \mathcal{O}_X) \subset p\mathrm{R}^i g_* \mathcal{O}_Y$. This assertion can be checked locally on S and is thus clear: it is trivially true on $S[1/p]$, and true by construction on $S_{(p)}$. $\square$

Finally, we record the following elementary observation for ease of reference later.

LEMMA 2.20. *Fix a noetherian integral scheme S of dimension $\leqslant 1$. Any alteration $f : X \to S$ with X integral is a finite morphism.*

Proof. The fibres of f are forced to be finite (as X has dimension $\leqslant 1$, and f is an alteration), so f is finite by Zariski's main theorem. $\square$

We have reduced the proof of Theorem 1.2 to showing Condition $\mathcal{C}_0(S)$ for affine schemes S of finite type over $\mathbf{Z}_{(p)}$. Given an alteration of such an S, the subset $Z \subset S$ where the alteration is not finite is closed of codimension $\geqslant 2$ by

Lemma 2.20; we will call Z the *centre* of the alteration. Our strategy for proving Theorem 1.2 is to construct, at the expense of localizing a little on S, a partial compactification $S \hookrightarrow \overline{S}$ with $\overline{S}$ proper over a lower-dimensional base such that Z remains closed in $\overline{S}$. This last condition ensures that the alteration in question can be extended to an alteration of $\overline{S}$ without changing the centre. As the centre has not changed, the cohomology of the newly created alteration maps *onto* that of the older alteration, thereby paving the way for an inductive argument via Proposition 2.10. The precise properties needed to carry out the above argument are ensured by the presentation lemma that follows.

LEMMA 2.21. *Let B be the spectrum of a discrete valuation ring with a separably closed residue field. Let $\widehat{S}$ be a local, flat, and essentially finitely presented B-scheme of relative dimension $\geqslant 1$ that is integral. Given a closed subset $\widehat{Z} \subset \widehat{S}$ of codimension $\geqslant 2$, we can find a diagram of B-schemes*

$$\begin{tikzcd} s \in Z \arrow[r, "i_Z"] \arrow[drr] & S \arrow[r, "j"] \arrow[dr] & \overline{S} \arrow[d, "\pi"] & \partial\overline{S} \arrow[l, "i"'] \arrow[dl] \\ & & W & \end{tikzcd}$$

satisfying the following.

(1) *All the schemes in the diagram above are of finite type over B.*

(2) *S is an integral scheme, i_Z is a closed subscheme, s is a closed point, and the germ of i_Z at s agrees with $\widehat{Z} \subset \widehat{S}$.*

(3) *i is the inclusion of a Cartier divisor, and j is the open dense complement of i.*

(4) *W is an integral affine scheme with* $\dim(W) = \dim(S) - 1$.

(5) *π is proper, $\pi\ |_S$ is affine, and both these maps have fibres of equidimension* 1.

(6) *$\pi\ |_Z$ and $\pi\ |_{\partial\overline{S}}$ are finite. In particular, $j(i_Z(Z))$ is closed in $\overline{S}$.*

To avoid confusion, we note that the assumption on $\widehat{S}$, by definition, implies that $\widehat{S} \to B$ is a local morphism of local schemes that is obtained by localizing a finitely presented flat B-scheme at a closed point of the special fibre. In particular, the closed point $s \in \widehat{S}$ maps to the closed point of B.

Proof. The strategy of the proof is to construct the desired data by first compactifying $\widehat{S}$ to a projective B-scheme, constructing a suitable projection

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by using a point not on the closure of $\widehat{Z}$, and then deleting an ample divisor on the base to get affineness. In fact, if B was a point, then this method yields the conclusion of the lemma with the weaker assumption that $\widehat{Z} \subset \widehat{S}$ has codimension $\geqslant 1$; the extra constraint on the codimension is necessary to lift this conclusion from the special fibre in the absence of B-flatness of $\widehat{Z}$.

We begin by choosing an *ad hoc* finite type model of $\widehat{Z} \hookrightarrow \widehat{S}$ over B; that is, we find a map $i_Y : Y \to T$ and a point $y \in Y$ satisfying the following: the map i_Y is a closed immersion of finite type affine B-schemes with codimension $\geqslant 2$ and T integral, and the germ of i_Y at y is the given map $\widehat{Z} \hookrightarrow \widehat{S}$; this is possible since, by assumption, both $\widehat{Z}$ and $\widehat{S}$ are essentially finitely presented over B. Note that, since T is integral, the monomorphism $\widehat{S} \to T$ is scheme-theoretically dense. Next, we choose an *ad hoc* compactification $T \hookrightarrow \overline{T}$ over B; that is, $\overline{T}$ is a projective flat B-scheme containing T as a dense open subscheme. Choose a B-ample B-flat divisor $\partial\overline{T} \subset \overline{T}$ that misses y in the special fibre (and hence in all of $\overline{T}$ by properness); this is possible since k is separably closed. We may then replace T with $\overline{T} - \partial\overline{T}$ to assume that $\overline{T} - T$ is a relatively ample B-flat divisor. Let $\overline{Y}$ be the closure of Y in $\overline{T}$, and let $\partial\overline{Y} = \overline{Y} - Y = \partial\overline{T} \cap \overline{Y}$ be its boundary. As Y has codimension $\geqslant 2$ in T, its closure $\overline{Y}$ also has codimension $\geqslant 2$ in $\overline{T}$, and hence the boundary $\partial\overline{Y}$ has codimension $\geqslant 3$ in $\overline{T}$. We will modify T and $\overline{T}$ to eventually find the required S and $\overline{S}$.

Let d denote the dimension of a fibre of the flat projective morphism $\overline{T} \to B$. By construction, this is also the relative dimension of the flat local B-scheme $\widehat{S}$. The next step is to find a finite morphism $\phi : \overline{T} \to \mathbf{P}^d_B$ such that $\phi(y) \notin \phi(\partial\overline{T})$. We find such a map by repeatedly projecting. In slightly more detail, say we have a finite morphism $\phi : \overline{T} \to \mathbf{P}^N_B$ for some $N > d$ such that $\phi(y) \notin \phi(\partial\overline{T})$. Then $\phi(\partial\overline{T})$ is a closed subscheme of codimension $\geqslant 2$. Moreover, by the flatness of $\partial\overline{T}$ over B, the same is true in the special fibre $\mathbf{P}^N_k \subset \mathbf{P}^N_B$. By basic facts of projective geometry in the special fibre, we can find a line ℓ through $\phi(y)$ that does not meet $\phi(\partial\overline{T})$. By the ampleness of $\partial\overline{T}$, this line cannot entirely be contained in $\phi(\overline{T})$. Thus, we can find a point on it that is not contained in $\phi(\overline{T})$. By projecting from this point, we see that we can find a finite morphism $\phi' : \overline{T}_k \to \mathbf{P}^{N-1}_k$ such that $\phi'(y) \notin \phi'(\partial\overline{T})$. So far the discussion has been regarding the special fibre. However, by choosing a lift of this point to a B-point (by explicit description of points of projective space) and using the properness of $\partial\overline{T}$ to transfer the nonintersection condition from the special fibre to the total space, this construction can be made over B. Continuing this way, we can find a finite morphism $\phi : \overline{T} \to \mathbf{P}^d_B$ with the same property. As $\phi(\partial\overline{T})$ is now a B-ample effective Cartier divisor, its complement $U \hookrightarrow \mathbf{P}^d_B$ is an open affine containing $\phi(y)$. We may now replace T with $\phi^{-1}(U)$ and Y with $\overline{Y} \cap \phi^{-1}(U)$ (this is permissible as $y \in \overline{Y} \cap \phi^{-1}(U) \subset Y$) to assume that we have produced the following: a finite type B-scheme model $i_Y : Y \to T$ of the germ $\widehat{Z} \to \widehat{S}$ for some point $y \in Y$ with T integral, a compactification $T \hookrightarrow \overline{T}$ with $\overline{T}$ integral and a flat projective B-scheme, and a finite morphism $\phi : \overline{T} \to \mathbf{P}^d_B$ such that $T = \phi^{-1}(U)$ for some open affine $U \subset \mathbf{P}^d_B$ that is the complement of a B-ample B-flat divisor H.

Now we project once more to obtain the desired curve fibration. As explained earlier, the closure $\overline{Y}$ has codimension $\geqslant 2$ in $\overline{T}$. Since we do not know that it is flat over B, the most we can say is that its image $\phi(\overline{Y})$ has codimension $\geqslant 1$ in the special fibre $\mathbf{P}^d_k \subset \mathbf{P}^d_B$. On the other hand, we know that $\partial\overline{T}$ is a B-flat divisor. Thus, its image $\phi(\partial\overline{T})$ also has codimension $\geqslant 1$ in the special fibre $\mathbf{P}^d_k \subset \mathbf{P}^d_B$. It follows that $\phi(\overline{Y} \cup \partial\overline{T})$ has codimension $\geqslant 1$ in the special fibre $\mathbf{P}^d_k \subset \mathbf{P}^d_B$. By choosing a closed point not in this image inside U and lifting to a B-point as above, we find a B-point $p : B \to U \subset \mathbf{P}^d_B$ whose image does not intersect $\phi(\overline{Y} \cup \partial\overline{T})$. Projecting from this point gives rise to the following diagram:

$$\begin{tikzcd}
\mathrm{Bl}_{\phi^{-1}(p)}(T) \arrow[r, "a"] \arrow[d] & \mathrm{Bl}_{\phi^{-1}(p)}(\overline{T}) \arrow[r, "b"] \arrow[d] & \mathrm{Bl}_p(\mathbf{P}^d_B) \arrow[r, "c"] \arrow[d] & \mathbf{P}(T_p(\mathbf{P}^d_B)) \simeq \mathbf{P}^{d-1}_B \\
T \arrow[r] & \overline{T} \arrow[r] & \mathbf{P}^d_B. &
\end{tikzcd}$$

The horizontal maps enjoy the following properties: c is a $\mathbf{P}^1$-fibration (in the Zariski topology), b is a finite surjective morphism, and a is an open immersion. In particular, the composite map cb is a proper morphism with fibres of equidimension 1. As the map $\phi : \overline{T} \to \mathbf{P}^d$ was chosen to ensure that $\phi^{-1}(U) = T$, the composite map cba can be factored as

$$\mathrm{Bl}_{\phi^{-1}(p)}(T) \to \mathrm{Bl}_p(U) \to \mathbf{P}^{d-1}_B.$$

The first map in this composition is finite surjective as ϕ is so, while the second map is an affine morphism with fibres of equidimension 1 thanks to Lemma 2.22 below. It follows that the composite map cba is an affine morphism with fibres of equidimension 1. Lastly, by our choice of p, the map cb restricts to a finite map on $\overline{Y}$ and $\overline{T}$ (here we identify subschemes of $\overline{T}$ not intersecting $\phi^{-1}(p)$ with those of the blowup). As explained earlier, the boundary $\partial\overline{Y}$ has codimension $\geqslant 3$ in $\overline{T}$. This implies that its special fibre has codimension $\geqslant 2$ in the special fibre of $\overline{T}$. Therefore, its image in $\mathbf{P}^{d-1}_B$ has codimension $\geqslant 1$. It follows that we can find an open affine $W \hookrightarrow \mathbf{P}^{d-1}_B$ not meeting the image of $\phi(\partial\overline{Y})$. Restricting the entire picture thus obtained to W, we find a diagram that looks like the following:

$$\begin{tikzcd} y \in Y \arrow[r] \arrow[drr] & \mathrm{Bl}_{\phi^{-1}(p)}(T)_W \arrow[r] \arrow[dr] & \mathrm{Bl}_{\phi^{-1}(p)}(\overline{T})_W \arrow[d, "\pi"] & \partial \overline{T}_W \arrow[l] \arrow[dl] \\ & & W & \end{tikzcd}$$

Setting $s = y$, $Z = Y$, $S = \mathrm{Bl}_{\phi^{-1}(p)}(T)_W$, $\overline{S} = \mathrm{Bl}_{\phi^{-1}(p)}(\overline{T})_W$, and $\partial\overline{S} = \partial\overline{T}_W$ implies the claim. $\square$

The following elementary fact concerning blowups was used in Lemma 2.21.

LEMMA 2.22. *Let B be a scheme, and let $\pi : \mathbf{P} \to B$ be a projective bundle. Let $H \hookrightarrow \mathbf{P}$ be an effective Cartier divisor that is B-flat and B-ample, and let $U = \mathbf{P} - H$. For any point $p \in U(B)$, the blowup map $\mathrm{Bl}_p(U) \to \mathbf{P}(T_p(\mathbf{P}^n))$ is an affine morphism with fibres of equidimension* 1.

Proof. Let $b : \mathrm{Bl}_p(\mathbf{P}) \to \mathbf{P}$ be the blowup map, and let $\overline{\pi} : \mathrm{Bl}_p(\mathbf{P}) \to \mathbf{P}(T_p(\mathbf{P}))$ be the morphism defined by projection. It is easy to see that $\overline{\pi}$ is a $\mathbf{P}^1$-bundle. As H is disjoint from the centre of the blowup, $b^*(H)$ defines an ample divisor on the fibres of π. Using the fibre-wise criterion for ampleness (see [**Laz04**, Theorem 1.7.8]), one concludes that $b^*(H)$ is $\overline{\pi}$-ample, and hence $\mathrm{Bl}_p(U) = \mathrm{Bl}_p(\mathbf{P}) - b^*(H)$ is affine over $\mathbf{P}(T_p(\mathbf{P}))$. The assertion about the fibres is clear. $\square$

REMARK 2.23. The main strategy in the proof of Lemma 2.21 was to first perform the desired construction over the special fibre, and then lift the construction to the total space. In particular, one readily checks that the conclusion of Lemma 2.21 is true *verbatim* if B is assumed to be the spectrum of a field.

Before proceeding to the proof of Theorem 1.2, we record a cohomological consequence of certain geometric hypotheses. The hypotheses in question are the kind ensured by Lemma 2.21, while the consequences are those used in proof of Theorem 1.2.

PROPOSITION 2.24. *Fix a quasicompact quasiseparated scheme $\overline{X}$. Let $j : X \hookrightarrow \overline{X}$ be a dense quasicompact open immersion whose complement $\Delta \subset \overline{X}$ is affine and the support of a Cartier divisor. Then $H^i(\overline{X}, \mathcal{O}_{\overline{X}}) \to H^i(X, \mathcal{O}_X)$ is surjective for all $i > 0$.*

Proof. As $\Delta \subset \overline{X}$ is the support of a Cartier divisor, the complement j is an affine map. This implies that

$$j_*\mathcal{O}_X \simeq \mathrm{R}j_*\mathcal{O}_X.$$

Now consider the exact sequence

$$0 \to \mathcal{O}_{\overline{X}} \to j_*\mathcal{O}_X \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is defined to be the cokernel. As $j_*\mathcal{O}_X \simeq \mathrm{R}j_*\mathcal{O}_X$, the middle term in the preceding sequence computes $H^i(X, \mathcal{O}_X)$. By the associated long exact sequence on cohomology, to show the claim, it suffices to show that $H^i(\overline{X}, \mathcal{Q}) = 0$ for $i > 0$. By construction, we have a presentation

$$j_*\mathcal{O}_X = \mathrm{colim}_n\, \mathcal{O}_{\overline{X}}(n\Delta).$$

Thus, we also have a presentation

$$\mathcal{Q} = \mathrm{colim}_n\, \mathcal{O}_{\overline{X}}(n\Delta)/\mathcal{O}_{\overline{X}}.$$

This presentation defines a natural increasing filtration $F^\bullet(\mathcal{Q})$ with

$$F^n(\mathcal{Q}) = \mathcal{O}_{\overline{X}}(n\Delta)/\mathcal{O}_{\overline{X}}$$

for $n \geqslant 0$. The associated graded pieces of this filtration are

$$\mathrm{gr}^n_F(\mathcal{Q}) = \mathcal{O}_{\overline{X}}(n\Delta) \otimes \mathcal{O}_\Delta.$$

In particular, these pieces are supported on Δ, which is an affine scheme by assumption. Consequently, these pieces have no higher cohomology. By a standard devissage argument, the sheaves $F^n(\mathcal{Q})$ have no higher cohomology for any n. Then $\mathcal{Q}$ has no higher cohomology either (as cohomology commutes with filtered colimits of sheaves on quasicompact quasiseparated schemes; see [**Sta14**, Tag 07TA]), establishing the claim. $\square$

We now have enough tools to finish proving Theorem 1.2.

Proof of main theorem. Our goal is to show that Condition $\mathcal{C}_0(\widehat{S})$ is satisfied by an induction on $\dim(\widehat{S})$. By Lemmas 2.19 and 2.15 and a limit argument, we may assume that $\widehat{S}$ is a local integral scheme that is essentially of finite type over the strict henselization B of $\mathbf{Z}_{(p)}$ with a characteristic p residue field at the closed point. We give the argument in the (harder) case that $\widehat{S}$ is flat over B; the remaining case is when $\widehat{S}$ is an $\mathbf{F}_p$-scheme, and this case follows using the same argument below and Remark 2.23 (or simply by invoking [**Bhab**, Theorem 1.5]).

If $\dim(\widehat{S}) = 1$, there is nothing to show thanks to Lemma 2.20, as finite morphisms have no higher cohomology. We may therefore assume that the relative dimension of $\widehat{S}$ over B is at least 1, and that $\mathcal{C}_0(T)$ is satisfied by all schemes T of dimension $< \dim(\widehat{S})$. For such T, Proposition 2.10 then ensures that $\mathcal{C}_d(T)$ is also satisfied for any $d \geqslant 0$, which will be used crucially in the proof below (for B-flat T).

With the assumptions as above, given an alteration $\widehat{f} : \widehat{X} \to \widehat{S}$ with $\widehat{X}$ integral, we want to find an alteration $\widehat{\pi} : \widehat{Y} \to \widehat{X}$ such that $\widehat{\pi}^*(H^i(\widehat{X}, \mathcal{O}_{\widehat{X}})) \subset p(H^i(\widehat{Y}, \mathcal{O}_{\widehat{Y}}))$. After replacing $\widehat{X}$ by a suitable blowup, we may assume that $\widehat{f}$ is projective. As $\widehat{f}$ is an alteration, one has a closed subset $\widehat{Z} \subset \widehat{S}$ of codimension $\geqslant 2$ such that $\widehat{f}$ is finite away from $\widehat{Z}$. Applying the conclusion of Proposition 2.21, we can find a diagram

$$\begin{tikzcd} s \in Z \arrow[r, "i_Z"] \arrow[drr] & S \arrow[r, "j"] \arrow[dr] & \overline{S} \arrow[d, "\pi"] & \partial\overline{S} \arrow[l, "i"'] \arrow[dl] \\ & & W & \end{tikzcd}$$

satisfying the conditions guaranteed by Proposition 2.21. By spreading out $\widehat{f}$, we may choose an open neighbourhood $U \subset \overline{S}$ containing $\widehat{S}$, and a projective alteration $f_U : X_U \to U$ that is finite outside $Z \cap U$, and agrees with $\widehat{f}$ over $\widehat{S}$. Applying Zariski's main theorem (as well as a scheme-theoretic closure trick; see [**Bhab**, Proposition 3.1]) to the restriction of $X_U \to U \to \overline{S}$ over $\overline{S} - Z$, we obtain a finite morphism $f' : X' \to \overline{S} - Z$ that agrees with f_U over $U - U \cap Z$. Set $V = (\overline{S} - Z) \cup U \subset \overline{S}$ to be the displayed open subset. Glueing f_U with f' gives a projective alteration $f'' : X'' \to V$ that is finite over $\overline{S} - Z \subset V$ and extends f_U. Finally, we extend f'' to some projective alteration $\overline{f} : \overline{X} \to \overline{S}$; this is always possible, for example, by taking a closure in a projective embedding. Then $\overline{f}$ is finite outside Z, and agrees with f_U over U, and thus extends $\widehat{f}$. Let $f : X \to S$ denote the restriction of $\overline{f}$ to S. We summarize the preceding constructions by the following diagram:

$$\begin{tikzcd} X \times_S Z \arrow[r] \arrow[d, "f_Z"] & X \arrow[r, "j_X"] \arrow[d, "f"] & \overline{X} \arrow[d, "\overline{f}"] & \Delta = \partial\overline{S} \times_{\overline{S}} \overline{X} \arrow[l, "i_X"'] \arrow[d, "f_{\partial\overline{S}}"] \\ s \in Z \arrow[r, "i_Z"] \arrow[dr] & S \arrow[r, "j"] \arrow[dr] & \overline{S} \arrow[d, "\pi"] & \partial\overline{S} \arrow[l, "i"'] \arrow[dl] \\ & & W & \end{tikzcd},$$

Here the first row is obtained by base change from the second row via $\overline{f}$. In particular, i_X is the inclusion of a Cartier divisor. As $\overline{f}$ is finite away from the closed set Z which does not meet $\partial\overline{S}$, the map $f_{\partial\overline{S}}$ is finite. In particular, the scheme $\partial\overline{S}\times_{\overline{S}}\overline{X}$ is affine. Applying Proposition 2.24 to the map i_X, we find that $H^i(\overline{X},\mathcal{O}_{\overline{X}})\to H^i(X,\mathcal{O}_X)$ is surjective for $i>0$. Since $\dim(W)<\dim(S)$, the inductive hypothesis and Proposition 2.10 ensure that Condition $\mathcal{C}_d(W)$ is true for all d. As $\overline{X}\to W$ is proper surjective, we can find an alteration $\overline{\pi}:\overline{Y}\to\overline{X}$ such that $\overline{\pi}^*(H^i(\overline{X},\mathcal{O}_{\overline{X}}))\subset p(H^i(\overline{Y},\mathcal{O}_{\overline{Y}}))$. It follows that a similar p-divisibility statement holds for the alteration $\pi:Y\to X$ obtained by restricting $\overline{\pi}$ to $X\hookrightarrow\overline{X}$. Lastly, by flat base change, we know that $H^i(X,\mathcal{O}_X)$ generates $H^i(\widehat{X},\mathcal{O}_{\widehat{X}})$ as a module over $\Gamma(\widehat{S},\mathcal{O}_{\widehat{S}})$. Thus, pulling back this alteration along $\widehat{X}\to X$ produces the desired alteration $\widehat{\pi}:\widehat{Y}\to\widehat{X}$. □

REMARK 2.25. One noteworthy feature of the proof of Proposition 2.13 is the following: while trying to show that $\mathcal{C}_0(S)$ is satisfied, we use that $\mathcal{C}_d(S')$ is satisfied for $d>0$ and certain affine schemes S' with $\dim(S')<\dim(S)$. We are allowed to make such arguments thanks to Proposition 2.10 and induction. Moreover, this phenomenon explains why Proposition 2.10 appears before Proposition 2.13 in this paper, despite the relevant statements naturally preferring the opposite order.

REMARK 2.26. Theorem 1.2, while ostensibly being a statement about coherent cohomology, is actually motivic in that it admits obvious analogues for most natural cohomology theories such as de Rham cohomology or étale cohomology. For the former, one can use Theorem 1.2 and the Hodge-to-de Rham spectral sequence to reduce to proving a p-divisibility statement for $H^i(X,\Omega^j_{X/S})$ with $j>0$. Choosing local representatives for differential forms and extracting pth roots out of the relevant functions can then be shown to solve the problem. In étale cohomology, there is an even stronger statement: for any noetherian excellent scheme X, there exist *finite* covers $\pi:Y\to X$ such that $\pi^*(H^i_{\text{ét}}(X,\mathbf{Z}_p))\subset p(H^i_{\text{ét}}(Y,\mathbf{Z}_p))$ for any fixed $i>0$; this statement follows, for example, from [**Bhaa**, Theorem 1.1] using the exact sequences of (continuous p-adic) étale sheaves

$$0\to\mathbf{Z}_p\xrightarrow{p}\mathbf{Z}_p\to\mathbf{Z}/p\to 0.$$

Alternately, one may simply observe that étale cohomology of torsion constructible sheaves is effaceable in the category of torsion constructible sheaves, and that each torsion constructible sheaf is a subsheaf of the pushforward of a constant sheaf along a finite cover (see [**Del77**, Section IV.3, Arcata]). We hope to find finite covers that work for coherent cohomology (see Remark 3.3), but cannot do so yet.

REMARK 2.27. The proof of Theorem 1.2 actually shows the following: given a proper morphism $f : X \to S$ with S excellent, there exists an alteration $\pi : Y \to S$ such that, with $g = \pi \circ f$, we have the following.

(1) $\pi^*(\mathrm{R}^1 f_* \mathcal{O}_X) \subset p(\mathrm{R}^1 g_* \mathcal{O}_Y)$.

(2) The map $\tau_{\geqslant 2}\mathrm{R} f_* \mathcal{O}_X \to \tau_{\geqslant 2}\mathrm{R} g_* \mathcal{O}_Y$ is divisible by p as a morphism in $\mathrm{D}_{\mathrm{coh}}(S)$.

The reason one has to truncate above 2 and not 1 in the second statement above is that divisibility by p in a Hom-group imposes torsion conditions not visible when requiring individual classes to be divisible by p. For instance, the second conclusion above implies that the p-torsion in $\mathrm{R}^i f_* \mathcal{O}_X$ for $i \geqslant 2$ can be killed by alterations. We do not know how to prove this for $i = 1$. The main reason is that the map in Proposition 2.24 is an isomorphism for $i > 1$, but only surjective for $i = 1$. In the notation of the proof of Theorem 1.2 above, this means that p-torsion classes $H^1(X, \mathcal{O}_X)$ need not lift to p-torsion classes in $H^1(\overline{X}, \mathcal{O}_{\overline{X}})$. This last problem, and hence the lacuna discussed in this remark, can be solved by showing that functions in $H^0(X, \mathcal{O}_X/p)$ on the special fibre of X lift to functions on all of X for any scheme X that is proper over an affine, provided we allow passage to alterations.

We have checked the validity of $\mathcal{C}_d(S)$ for all noetherian S and integers d. This implies the following.

COROLLARY 2.28. *Let $f : X \to S$ be a proper morphism of noetherian schemes. Then there exists an alteration $\pi : Y \to X$ such that, with $g = f \circ \pi$, we have $\pi^*(\mathrm{R}^i f_* \mathcal{O}_X) \subset p(\mathrm{R}^i g_* \mathcal{O}_Y)$ for each $i > 0$.*

Next, we give an example showing that Theorem 1.2 fails as soon as the properness of f is relaxed.

EXAMPLE 2.29. Let k be a characteristic p field, and let $X = \mathbf{P}^n_k - \{x\}$ for some $x \in \mathbf{P}^n(k)$ and $n \geqslant 2$. Then $H^{n-1}(X, \mathcal{O}_X) \simeq H^n_x(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k})$ is nonzero. Moreover, for any proper surjective morphism $\pi : Y \to X$, the pullback $\mathcal{O}_X \to \mathrm{R}\pi_* \mathcal{O}_Y$ is a direct summand (by [**Bhab**, Corollary 8.10]), so $H^{n-1}(X, \mathcal{O}_X) \to H^{n-1}(Y, \mathcal{O}_Y)$ is also a direct summand. In particular, nonzero classes in $H^{n-1}(X, \mathcal{O}_X)$ cannot be killed by proper covers. Replacing X with the obvious mixed characteristic variant X' gives an example of a $\mathbf{Z}_p$-flat scheme X' with nonzero higher coherent cohomology that cannot be made divisible by p on passage to proper covers: the annihilation result for X' implies that the result for X as $H^{n-1}(X', \mathcal{O}_{X'}) \to H^{n-1}(X, \mathcal{O}_X)$ is surjective by explicit calculation.

We end with an example showing that one cannot replace 'alteration' with 'modification' in Theorem 1.2, even when f is itself a modification.

EXAMPLE 2.30. Let $S \subset \mathbf{A}^3$ be the affine cone over an elliptic curve $E \subset \mathbf{P}^2_k$ over some perfect field k of characteristic p. Let $f : X \to S$ be the blowup of S at the origin $s \in S$, so X is smooth, and f is an isomorphism over $U := S - \{s\}$. One can compute easily that $H^1(X, \mathcal{O}_X) \simeq H^1(E, \mathcal{O}_E)$ is a one-dimensional k-vector space. We will show the following: for any modification $\pi : Y \to X$, the pullback $H^1(X, \mathcal{O}_X) \to H^1(Y, \mathcal{O}_Y)$ is injective. This shows that one must allow genuine alterations in Theorem 1.2. To see the previous claim, it is enough to show that $\mathcal{O}_X \to \mathrm{R}\pi_*\mathcal{O}_Y$ is a split monomorphism. By resolution of singularities for surfaces (see [**Lip78**]), there exists a further modification $\pi' : Z \to Y$ such that the composite $\psi : Z \to X$ is a blowup along a smooth centre. In particular, one computes $\mathcal{O}_X \simeq \mathrm{R}\psi_*\mathcal{O}_Z$ via the natural pullback. The claim now follows by factoring this pullback as $\mathcal{O}_X \to \mathrm{R}\pi_*\mathcal{O}_Y \to \mathrm{R}\psi_*\mathcal{O}_Z$.

3. A stronger result in positive characteristic

Our goal in this section is to explain an alternative proof of the [**Bhab**, Theorem 1.5] (the main result of [**Bhab**]) using Theorem 1.2. We first recall the following statement.

THEOREM 3.1. *Let $f : X \to S$ be a proper morphism of noetherian $\mathbf{F}_p$-schemes. Then there exists a finite surjective map $\pi : Y \to X$ such that, with $g = f \circ \pi$, the pullback $\pi^* : \tau_{\geqslant 1}\mathrm{R}f_*\mathcal{O}_X \to \tau_{\geqslant 1}\mathrm{R}g_*\mathcal{O}_Y$ is* 0.

Applying Theorem 1.2 in positive characteristic, *a priori*, only allows us to kill cohomology on passage to proper covers. The point of the proof below, therefore, is that annihilation by proper covers implies annihilation by finite covers for coherent cohomology; see [**Bhaa**, Section 6] for an example with étale cohomology with coefficients in an abelian variety where such an implication fails.

Proof of Theorem 3.1. We first explain the idea informally. Using Corollary 2.28, one finds proper surjective maps $Y' \to X$ and $Y'' \to Y'$ annihilating the higher coherent cohomology of $X \to S$ and $Y' \to X$, respectively; then one simply checks that the Stein factorization of $Y'' \to X$ does the job.

In more detail, by repeatedly applying Corollary 2.28 and using elementary facts about derived categories (see [**Bhab**, Lemma 3.2]), we may find a proper surjective map $\pi' : Y' \to X$ such that, with $g' = f \circ \pi'$, the pullback $\tau_{\geqslant 1}\mathrm{R}f_*\mathcal{O}_X \to$

$\tau_{\geqslant 1}\mathrm{R}g'_*\mathcal{O}'_Y$ is 0. Applying the same reasoning now to the map $\pi' : Y' \to X$, we find a map $\pi^\flat : Y'' \to Y'$ such that, with $\pi'' = \pi^\flat \circ \pi'$, we have that $\tau_{\geqslant 1}\mathrm{R}\pi'_*\mathcal{O}_Y \to \tau_{\geqslant 1}\mathrm{R}\pi''_*\mathcal{O}_{Y''}$ is 0. The picture obtained thus far is

$$\begin{tikzcd}
Y'' \arrow[r, "\pi^\flat"] \arrow[dr, "\pi''"'] & Y' \arrow[d, "\pi'"] \arrow[dr, "g'"] & \\
& X \arrow[r, "f"] & S
\end{tikzcd}$$

The diagram restricted to X gives rise to the following commutative diagram of exact triangles in $\mathrm{D}_{\mathrm{coh}}(X)$:

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, equal] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathcal{O}_X[1] \arrow[d] \\
\pi'_*\mathcal{O}_{Y'} \arrow[r] \arrow[d, "a"] & \mathrm{R}\pi'_*\mathcal{O}_{Y'} \arrow[r] \arrow[d, "b"] \arrow[dl, dotted, "s"'] & \tau_{\geqslant 1}\mathrm{R}\pi'_*\mathcal{O}_{Y'} \arrow[r] \arrow[d, "c=0"] & \pi'_*\mathcal{O}_{Y'}[1] \arrow[d, "a{[1]}"] \\
\pi''_*\mathcal{O}_{Y''} \arrow[r] & \mathrm{R}\pi''_*\mathcal{O}_{Y''} \arrow[r] & \tau_{\geqslant 1}\mathrm{R}\pi''_*\mathcal{O}_{Y''} \arrow[r] & \pi''_*\mathcal{O}_{Y''}[1]
\end{tikzcd}$$

Here the vertical arrows are the natural pullback maps, and the dotted arrow s is a chosen lifting of b guaranteed by the condition $c = 0$ (which is true by construction). Applying $\mathrm{R}f_*$ to the above diagram, we find a factorization:

$$\begin{tikzcd}
\mathrm{R}f_*\mathcal{O}_X \arrow[rr, "h"] \arrow[dr, "d"] & & \mathrm{R}f_*(\pi''_*\mathcal{O}_{Y''}) \\
& \mathrm{R}(f \circ \pi')_*\mathcal{O}_{Y'} = \mathrm{R}g'_*\mathcal{O}_{Y'} \arrow[ur, "e"] &
\end{tikzcd}$$

The map d induces the 0 map on $\tau_{\geqslant 1}$ by construction. It follows that the same is true for the map h. On the other hand, the sheaf $\pi''_*\mathcal{O}_{Y''}$ is a coherent sheaf of algebras on X. Hence, it corresponds to a finite morphism $\pi : Y \to X$. In fact, π is simply the Stein factorization of π''. In particular, π is surjective. It then follows that $\pi : Y \to X$ is a finite surjective morphism such that, with $g = f \circ \pi$, the induced map $\tau_{\geqslant 1}\mathrm{R}f_*\mathcal{O}_X \to \tau_{\geqslant 1}\mathrm{R}g_*\mathcal{O}_Y$ is 0, as desired. $\square$

REMARK 3.2. There is an alternative and more conceptual explanation of the preceding reduction from proper covers to finite covers in the case of H^1. Namely, let $\alpha \in H^1(X, \mathcal{O}_X)$ be a cohomology class, and let $f : Y \to X$ be a proper surjective map such that $f^*\alpha = 0$. We may represent α as a $\mathbf{G}_a$-torsor $T \to X$.

The assumption on Y then says that there is an X-map $Y \to T$. By the defining property of the Stein factorization $Y \to Y'' \to X$, the map $Y \to T$ factors as a map $Y'' \to T$; that is, the pullback of T (or, equivalently α) along the finite surjective map $Y'' \to X$ is the trivial torsor, as wanted. The key cohomological idea underlying this argument is that the pullback $H^1(Y'', \mathcal{O}_{Y''}) \to H^1(Y, \mathcal{O}_Y)$ is injective. This injectivity fails for higher cohomological degree, so one cannot argue similarly in all degrees.

REMARK 3.3. Assume for a moment that the conclusion of Theorem 1.2 can be lifted to the derived category as discussed in Remark 2.11; that is, we can kill p-torsion in higher coherent cohomology by passage to alterations. Then the argument given in the proof of Theorem 3.1 applies directly to show that, in fact, one can make cohomology p-divisible (in the derived sense) by passage to finite covers. In particular, we can then replace 'alteration' with 'finite surjective map' in the statement of Theorem 1.2. We have checked this consequence in a few nontrivial examples (like the blowup of an elliptic two-dimensional singularity over $\mathbf{Z}_p$), and we hope that it is a reasonable expectation in general.

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