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Long Time Behavior of Leafwise Heat Flow for Riemannian Foliations

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Abstract. For any Riemannian foliation $\mathcal{F}$ on a closed manifold M with an arbitrary bundle-like metric, leafwise heat flow of differential forms is proved to preserve smoothness on M at infinite time. This result and its proof have consequences about the space of bundle-like metrics on M, about the dimension of the space of leafwise harmonic forms, and mainly about the second term of the differentiable spectral sequence of $\mathcal{F}$.

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1. Introduction and Main Results

For a smooth foliation $\mathcal{F}$ on a closed Riemannian manifold M, *leafwise heat flow* means the physical evolution of an initial temperature distribution on M when the leaves are thermally isolated from each other. This evolution is given by the usual heat equation involving the Laplacian on the leaves. A more general leafwise heat flow is given by the usual heat equation involving any leafwise elliptic differential operator⋆⋆ with symmetric leading symbol. In this paper, these operators will be induced by leafwise elliptic differential complexes in the usual way.

Let E be a $\mathbb{Z}$-grade Riemannian vector bundle over M, and d a first-order leafwise elliptic differential complex on $C^\infty(E)$—the space of smooth sections of E. Let δ denote the formal adjoint of d on M—it need not be formal adjoint of d on the leaves. Then $D = d + \delta$ and $\Delta = D^2 = d\delta + \delta d$ are symmetric differential operators on $C^\infty(E)$, and thus essentially selfadjoint in the L^2-completion $L^2(E)$ of $C^\infty(E)$

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⋆⋆ A *leafwise differential operator* is a differential operator on M whose local expressions only contain derivatives along leaf directions, and thus can be restricted to the leaves. If furthermore the restriction to the leaves is elliptic, then it is called a *leafwise elliptic differential operator*. The definition of *leafwise elliptic differential complex* is similar. On the other hand, a *transversely elliptic differential operator* is a differential operator on M whose leading symbol is an isomorphism at nontrivial covectors normal to the leaves—a covector of M is normal to the leaves if it vanishes on vectors tangent to the leaves.

(Theorem 2.2 in [10]). Then the spectral theorem defines the leafwise heat operator $e^{-t\Delta}$ on $L^2(E)$ for each $t \geqslant 0$.

The leafwise heat operator has a nice behavior on smooth sections at finite time: According to the work of J. Roe in [28], $e^{-t\Delta}$ defines a continuous operator on $C^\infty(E)$ which depends continuously on $t \in [0, \infty)$. But the main objective of this paper is to study the behavior of $e^{-t\Delta}$ on $C^\infty(E)$ at the limit when $t \to \infty$.

Let Π be the orthogonal projection onto the kernel of the unbounded operator defined by Δ in $L^2(E)$. Recall that, by the spectral theorem, the heat operator $e^{-t\Delta}$ strongly converges to Π on $L^2(E)$. By setting $e^{-\infty\Delta} = \Pi$, we get a continuous map $[0, \infty] \times L^2(E) \to L^2(E)$ given by $(t, \phi) \to e^{-t\Delta}\phi$, where $[0, \infty]$ is the one point compactification of $[0, \infty]$ (see section 2). Observe that Π need not preserve $C^\infty(E)$ because Δ is not elliptic on M except in the trivial case where $\mathcal{F}$ is of codimension zero.

Suppose there is a transversely elliptic differential operator A on $C^\infty(E)$ that commutes with d and δ. It is well known that Π preserves $C^\infty(E)$ in this case, which can be easily seen as follows. On the one hand, Π obviously commutes with the leafwise elliptic operator Δ, and thus Π preserves leafwise smoothness. On the other hand, Δ commutes with A by our assumption, and thus so does Π, yielding that Π preserves transverse smoothness as well. But the existence of such an A commuting with d and δ is a too strong condition. Fortunately, 'exact' commutation is not needed for Π to preserve $C^\infty(E)$, as shown by the following result, which is proved in section 2.

THEOREM A. *With the above notation, suppose that, on $C^\infty(E)$, there is a transversely elliptic first order differential operator A, and there are morphisms G, H, K and L such that*

$$Ad \pm dA = Gd + dH, \qquad A\delta \pm \delta A = K\delta + \delta L. \tag{1}$$

Then Π defines a continuous operator on $C^\infty(E)$, we have the leafwise Hodge decomposition

$$C^\infty(E) = \ker\Delta \oplus \overline{\operatorname{im}\Delta} = (\ker d \cap \ker\delta) \oplus \overline{\operatorname{im} d} \oplus \overline{\operatorname{im}\delta},$$

and $(t, \phi) \mapsto e^{-t\Delta}\phi$ defines a continuous map $[0, \infty] \times C^\infty(E) \to C^\infty(E)$.

In this paper, Theorem A is applied to *Riemannian foliations*. Such foliations are characterized by having isometric holonomy for some metric on local transversals. This property is equivalent to the existence of a *bundle-like metric* on M—a metric so that the foliation is locally defined by Riemannian submersions. Let us mention that Riemannian foliations were introduced by B. Reinhart [27], and certain description of their structure was given by P. Molino [24, 25].

A leafwise differential complex of $\mathcal{F}$ is constructed as follows. Let $\Omega(M)$ (or simply Ω) be the de Rham algebra of M. Consider the bigrading of Ω given by

$$\Omega^{u,v} = C^\infty\left(\bigwedge^u T\mathcal{F}^{\perp *} \otimes \bigwedge^v T\mathcal{F}^*\right), \quad u, v \in \mathbb{Z}.$$

The de Rham derivative and coderivative decompose as sum of bihomogeneous components,

$$d = d_{0,1} + d_{1,0} + d_{2,-1}, \quad \delta = \delta_{0,-1} + \delta_{1,0} + \delta_{-2,1},$$

where the double subindixes denote the corresponding bidegrees. See [1–3] for the properties of these components; in particular

- each $\delta_{i,j}$ is the formal adjoint of $d_{-i,-j}$ on M,
- $d_{2,-1}$ and $\delta_{-2,1}$ are of order zero,
- $D_0 = d_{0,1} + \delta_{0,-1}$ and $\Delta_0 = D_0^2$ are leafwise elliptic and symmetric, and
- $D_\perp = d_{1,0} + \delta_{-1,0}$ is transversely elliptic and symmetric.

Let $L^2\Omega$ be the L^2-completion of Ω, and Π the orthogonal projection onto the kernel of the unbounded operator defined by Δ_0 in $L^2\Omega$. We prove in section 3 (Proposition 3.1) that, if $\mathcal{F}$ is a Riemannian foliation and the metric bundle-like, $(\Omega, d_{0,1})$ satisfies the conditions of Theorem A with $D_\perp$ as A, obtaining the following result which solves affirmatively a conjecture of the first author and P. Tondeur [6].

THEOREM B. *Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold M with a bundle-like metric. Then Π defines a continuous operator on Ω, we have the leafwise Hodge decomposition*

$$\Omega = \ker\Delta_0 \oplus \overline{\operatorname{im}\Delta_0} = (\ker d_{0,1} \cap \ker\delta_{0,-1}) \oplus \overline{\operatorname{im} d_{0,1}} \oplus \overline{\operatorname{im}\delta_{0,-1}}, \tag{2}$$

and $(t, \alpha) \mapsto e^{-t\Delta_0}\alpha$ defines a continuous map $[0, \infty] \times \Omega \to \Omega$.

The so-called *basic complex* $\Omega(M/\mathcal{F})$ is the part of the kernel of Δ_0 with zero tangential degree, i.e. $\Omega(M/\mathcal{F}) = \Omega^{\cdot,0} \cap \ker\Delta_0$. The closure of $\Omega(M/\mathcal{F})$ in $L^2\Omega$ is $L^2\Omega^{\cdot,0} \cap \ker\Delta_0$, and will be denoted by $L^2\Omega(M/\mathcal{F})$. Observe that Theorem B implies that the orthogonal projection $L^2\Omega \to L^2\Omega(M/\mathcal{F})$ preserves smoothness, i.e. there is an orthogonal projection $\Omega \to \Omega(M/\mathcal{F})$. This projection was explicitly constructed in [3] and [26] by using integration along the leaf closures of $\mathcal{F}$, but such an easy construction does not seem to be possible for the whole kernel of Δ_0.

The above bigrading is useful to understand the so-called *differentiable spectral sequence* (E_i, d_i) of $\mathcal{F}$ (see, e.g., [1, 2]: There are canonical identities

$$(E_0, d_0) \equiv (\Omega, d_{0,1}), \quad (E_1, d_1) \equiv (H(\Omega, d_{0,1}), d_{1,0*}). \tag{3}$$

Moreover, the C^∞ topology induces a topology on each E_i so that d_i is continuous. Such topology on E_1 need not be Hausdorff [15], and so we may define the bigraded

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subcomplex $\bar{0}_1 \subset E_1$ as the closure of the trivial subspace. The quotient bigraded complex $E_1/\bar{0}_1$ will be denoted by $\mathcal{E}_1$. We get

$$\mathcal{E}_1^{u,v} \cong \Omega^{u,v} \cap \ker \Delta_0, \quad u, v \in \mathbb{Z}$$

as a direct consequence of Theorem B, yielding the following dualities where $p = \dim \mathcal{F}$ and $q = \operatorname{codim} \mathcal{F}$: If M is oriented, then $\mathcal{E}_1^{u,v} \cong \mathcal{E}_1^{q-u,p-v}$; if $\mathcal{F}$ is oriented, then $\mathcal{E}_1^{u,v} \cong \mathcal{E}_1^{u,p-v}$; and if $\mathcal{F}$ is transversely oriented, then $\mathcal{E}_1^{u,v} \cong \mathcal{E}_1^{q-u,v}$. These isomorphisms are respectively induced by the Hodge star operators on $\bigwedge TM^*$, $\bigwedge T\mathcal{F}^*$ and $\bigwedge T\mathcal{F}^{\perp *}$

This fits into a more general setting. First obserse that $(\Omega^{0,\cdot}, d_{0,1})$ can be canonically identified with the leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$, where $\Omega(\mathcal{F}) = C^\infty(\bigwedge T\mathcal{F}^*)$ and $d_{\mathcal{F}}$ is defined by the de Rham derivative on the leaves. Thus $E_1^{0,\cdot}$ can be identified with the *leafwise cohomology*, and $\mathcal{E}_1^{0,\cdot}$ with the *leafwise reduced cohomology*. Furthermore the whole $(\Omega, d_{0,1})$ can be considered, up to sign, as the leafwise de Rham complex of $\mathcal{F}$ with coefficients in the vector bundle $\bigwedge(TM/T\mathcal{F})^*$ with the flat $\mathcal{F}$-partial Bott connection; i.e. the partial connection induced by the partial Bott connection on the normal bundle [7, 8]. The condition that $\mathcal{F}$ be Riemannian is equivalent to the existence of a metric on the normal bundle such that the partial Bott connection is Riemannian. Thus, with more generality, we can consider the leafwise de Rham complex $(\Omega(\mathcal{F}, V), d_{\mathcal{F}})$ with coefficients in any Riemannian vector bundle V with a flat Riemannian $\mathcal{F}$-partial connection. The leafwise reduced cohomology with coefficients in V will be denoted by $\mathcal{H}(\mathcal{F}, V)$. The operator $\delta_{\mathcal{F}}$ on $\Omega(\mathcal{F}, V)$, defined by the de Rham coderivative on the leaves, is adjoint to $d_{\mathcal{F}}$. Let $D_{\mathcal{F}} = d_{\mathcal{F}} + \delta_{\mathcal{F}}$ and $\Delta_{\mathcal{F}} = D_{\mathcal{F}}^2$. Let also $L^2\Omega(\mathcal{F}, V)$ be the L^2-completion of $\Omega(\mathcal{F}, V)$, and Π the orthogonal projection of $L^2\Omega(\mathcal{F}, V)$ onto the kernel of $\Delta_{\mathcal{F}}$ in $L^2\Omega(\mathcal{F}, V)$. The following result easily follows from the case with coefficients in $\mathbb{R}$ (section 4).

COROLLARY C. *Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold M, and let V be any Riemannian vector bundle with a flat Riemannian $\mathcal{F}$-partial connection. Fix any Riemannian metric on the leaves, smooth on M. Then Π defines a continuous operator on $\Omega(\mathcal{F}, V)$, we have the leafwise Hodge decomposition*

$$\Omega(\mathcal{F}, V) = \ker \Delta_{\mathcal{F}} \oplus \overline{\operatorname{im} \Delta_{\mathcal{F}}} = (\ker d_{\mathcal{F}} \cap \ker \delta_{\mathcal{F}}) \oplus \overline{\operatorname{im} d_{\mathcal{F}}} \oplus \overline{\operatorname{im} \delta_{\mathcal{F}}},$$

and $(t, \alpha) \mapsto e^{-t\Delta_{\mathcal{F}}}\alpha$ defines a continuous map $[0, \infty] \times \Omega(\mathcal{F}, V) \to \Omega(\mathcal{F}, V)$. Thus $\mathcal{H}(\mathcal{F}, V)$ can be canonically identified with $\ker \Delta_{\mathcal{F}}$, and, if $\mathcal{F}$ is oriented, then the leafwise Hodge star operator on $\ker \Delta_{\mathcal{F}}$ induces on isomorphism $\mathcal{H}^v(\mathcal{F}, V) \cong \mathcal{H}^{p-v}(\mathcal{F}, V^*)$.

If we take coefficients in the normal bundle, then Corollary C helps to understand infinitesimal deformations of [16]. If we take coefficients in the symmetric tensor

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product $S^2((TM/T\mathcal{F})^*)$, then Corollary C has the following consequence (section 5) which solves a problem proposed by E. Macías.*

COROLLARY D. *Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold M. Then, with respect to the C^∞ topology, the space of bundle-like metrics on M is a deformation retract of the space of all metrics on M.*

Corollary C also has the following consequence which partially generalizes results from [4].

COROLLARY E. *Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold M, and let V any Riemannian vector bundle with a flat Riemannian $\mathcal{F}$-partial connection. For any metric on the leaves, smooth on M, if there is a nontrivial integrable harmonic r-form on some leaf with coefficients in V, then $\Omega^r(\mathcal{F}, V) \cap \ker \Delta_{\mathcal{F}}$ is of infinite dimension.*

As suggested in [4], Corollary E may be used to find examples of Riemannian foliations on closed Riemannian manifolds with dense leaves and infinite dimensional space of smooth leafwise harmonic forms.

Theorem B and its proof are also useful to get a better understanding of the term E_2 in the spectral sequence of any Riemannian foliation $\mathcal{F}$ on a closed manifold M. Let $\mathcal{H}_1 = \ker \Delta_0$ and $\tilde{\mathcal{H}}_1 = \overline{\operatorname{im} \Delta_0}$ in Ω, and let $L^2\mathcal{H}_1$ and $L^2\tilde{\mathcal{H}}_1$ be the corresponding closures in $L^2\Omega$. Consider the bigrading of $\mathcal{H}_1$ induced by the one of Ω. If $\mathcal{F}$ is Riemannian and the metric bundle-like, by (3) and Theorem B, the differential map d_1 on $\mathcal{E}_1$ corresponds to the map $\Pi d_{1,0}$ on $\mathcal{H}_1$, which will be also denoted by d_1. Hence, $H^u(\mathcal{E}_1^{\cdot,v}, d_1) \cong H^u(\mathcal{H}_1^{\cdot,v}, d_1)$. Consider also the following operators on $\mathcal{H}_1$: $\delta_1 = \Pi\delta_{-1,0}$, $D_1 = d_1 + \delta_1$ and $\Delta_1 = D_1^2$. Such a δ_1 is adjoint of d_1, and thus D_1 and Δ_1 are symmetric in $L^2\mathcal{H}_1$.

We also define a differential map $\tilde{d}_1$ on $\tilde{\mathcal{H}}_1$ as follows. First we slightly change the bigrading that $\Omega^{\cdot\cdot}$ induces on $\tilde{\mathcal{H}}$: Set

$$\tilde{\mathcal{H}}_1^{u,v} = \overline{d_{0,1}(\Omega^{u,v-1})} \oplus \overline{\delta_{0,-1}(\Omega^{u+1,v})}, \quad u, v \in \mathbb{Z}.$$

Let $\tilde{\Pi}_{\cdot,v}$ be the projection of Ω into $\tilde{\mathcal{H}}_1^{\cdot,v}$ according to (2), and set $\tilde{d}_1 = \tilde{\Pi}_{\cdot,v} d$ on $\tilde{\mathcal{H}}_1^{\cdot,v}$. We shall see that $\tilde{d}_1^2 = 0$ (section 7); indeed $H^u(\bar{0}_1^{\cdot,v}, d_1) \cong H^u(\tilde{\mathcal{H}}_1^{\cdot,v}, \tilde{d}_1)$—see section 7 for a better understanding of this modified bigrading. Then set $\tilde{\delta}_1 = \tilde{\Pi}_{\cdot,v}\delta$ on $\tilde{\mathcal{H}}_1^{\cdot,v}$ for each v, and let $\tilde{D}_1 = \tilde{d}_1 + \tilde{\delta}_1$ and $\tilde{\Delta}_1 = \tilde{D}_1^2$. Such $\tilde{\delta}_1$ is adjoint of $\tilde{d}_1$, and thus $\tilde{D}_1$ and $\tilde{\Delta}_1$ are symmetric in $L^2\tilde{\mathcal{H}}_1$. By using Theorem B and the role played by $D_\perp$ in its proof, we prove in section 7 the following result which generalizes the basic Hodge decompositions of [17] and [18].

* Problem 4.12 of 'Open Problems' in: *Analysis and Geometry in Foliated Manifolds*. X. Masa, E. Macías Virgós and J. A. Álvarez López (Editors). Proceedings of the VII International Colloquium on Differential Geometry, Santiago de Compostela, 26–30 July, 1994. World Scientific, Singapore, 1995.

THEOREM F. *Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold M. For any bundle-like metric on M, the operators Δ_1 and $\tilde{\Delta}_1$ are essentially self-adjoint on $L^2\mathcal{H}_1$ and $L^2\tilde{\mathcal{H}}_1$, respectively. Moreover $L^2\mathcal{H}_1$ and $L^2\tilde{\mathcal{H}}_1$ have complete orthonormal systems, $\{\phi_i : i = 1, 2, \ldots\} \subset \mathcal{H}_1$ and $\{\tilde{\phi}_i : i = 1, 2, \ldots\} \subset \tilde{\mathcal{H}}_1$, consisting of eigenvectors of Δ_1 and $\tilde{\Delta}_1$, respectively, so that the corresponding eigenvalues satisfy $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots$, $0 < \tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \ldots$, ;with $\lambda_i \uparrow \infty$ if $\dim \mathcal{H}_1 = \infty$, and $\tilde{\lambda}_i \uparrow \infty$ if $\dim \tilde{\mathcal{H}}_1 = \infty$; thus all of these eigenvalues have finite multiplicity. In particular we have*

$$\mathcal{H}_1 = \ker \Delta_1 \oplus \operatorname{im} \Delta_1, \quad \tilde{\mathcal{H}}_1 = \operatorname{im} \tilde{\Delta}_1.$$

In Theorem F, the fact that $\ker \Delta_1 = 0$ will follow from the following known result.

THEOREM G (X. Masa [23]; see also [1, 2]). *Under the same conditions we have $H(\bar{0}_1) = 0$ and $E_2 \cong H(\mathcal{E}_1)$, which is of finite dimension and, if M is orientable, satisfies the duality $E_2^{u,v} \cong E_2^{q-u,p-v}$, $u, v \in \mathbb{Z}$, where $p = \dim \mathcal{F}$ and $q = \operatorname{codim} \mathcal{F}$.*

It would be interesting to show that $\ker \tilde{\Delta}_1 = 0$ without using Theorem G, which would be a consequence. In contrast with [23], our approach does not use Molino's description of Riemannian foliations [24, 25]. Theorem G has important implications about tautness of Riemannian foliations [3, 23]; a different approach to the same type of tautness results is also given in [22], where Molino's theory is not used either.

Observe that some parts of Theorem G follow from Theorem F and are not satisfied by arbitrary foliations on closed manifolds [29]. So, if there is a version of Theorem B for more general foliations with the same kind of arguments, then $D_\perp$ should be replaced in its proof by other transversely elliptic operator and perhaps more general conditions should be used in Theorem A (see Remarks 2 and 3). There are related results for non-Riemannian foliations with very different proofs [19, 21].

In possible generalizations of Theorem B, a key role may be played by the fact that our leafwise elliptic operators are symmetric on M instead of being symmetric on the leaves. For Riemannian foliations and bundle-like metrics both points of view are the same. In general, the Laplacian on the leaves acting on functions induces the physical leafwise heat flow, while Δ_0 induces a modification of the physical leafwise heat flow by 'exterior influence'. For non-Riemannian foliations, the physical leafwise heat flow may 'break' continuous functions at infinite time, as can be easily seen for foliations on the two dimensional torus with several Reeb components. We hope the modified leafwise heat flow induced by Δ_0 has a better behavior at infinite time for more general foliations.

Theorems B and F, and the estimates in their proofs, are used in [5] to study relations between spectral sequences of Riemannian foliations and adiabatic limits; concretely, the results in section 5 of [14] are proved without the strong hypothesis that the positive spectrum of Δ_0 is bounded away from zero.

NOTATION. The kth Sobolev completion of $C^\infty(E)$, Ω and $\Omega(\mathcal{F}, V)$ will be respectively denoted by $W^k(E)$, $W^k\Omega$ and $W^k\Omega(\mathcal{F}, V)$. Fix a norm $\|\cdot\|_k$ in any of these kth Sobolev spaces, and let $\|\cdot\|_k$ also denote the corresponding norm of bounded operators on that space. Finally, closure in kth Sobolev spaces will be denoted by cl_k.

2. The General Result

This section is devoted to the proof of Theorem A. To do it, a result of J. Roe in [28] is needed as a first step. We firstly state it in our setting. Let $\mathcal{A}$ be the Fréchet algebra of those functions f on $\mathbb{R}$ that extend to entire functions on $\mathbb{C}$ such that for each compact subset $K \subset \mathbb{R}$ the functions $\{x \mapsto f(x+iy) : y \in K\}$ form a bounded subset of the Schwartz space $\mathcal{S}(\mathbb{R})$. Such an $\mathcal{A}$ is a module over the polynomial ring $\mathbb{C}[z]$, and contains all functions with compactly supported Fourier transform and the Gaussians $x \mapsto e^{tx^2}$. With the notation of Theorem A, without assuming (1), the same arguments as in Propositions 1.4 and 4.1 in [28] give the following (see also [20] for a discussion of the action of functions of tangentially elliptic operators in Sobolev spaces on the ambient manifold).

PROPOSITION 2.1 (J. Roe [28]). *The functional calculus map $f \mapsto f(D)$ given by the spectral theorem restricts to a continuous homomorphism of algebras and $\mathbb{C}[z]$-modules from $\mathcal{A}$ to the space of bounded endomorphisms of each $W^k(E)$, and thus to the space of continuous endomorphisms of $C^\infty(E)$. In particular, $e^{-t\Delta}$ defines a bounded operator on each $W^k(E)$ and a continuous operator on $C^\infty(E)$, which depends smoothly on $t \in [0, \infty)$.*

Now assume (1) is satisfied. Then Theorem A clearly follows from the following six properties, which are proved by induction on $k = 0, 1, 2, \ldots$:

(i) There exists $c_{k,1} > 0$ such that the bounded operator $e^{-t\Delta}$ on $W^k(E)$, defined by Proposition 2.1, satisfies

$$\|e^{-t\Delta}\|_k \leqslant c_{k,1} \qquad \text{for all } t > 0$$

(ii) $De^{-t\Delta}$ defines a bounded operator on $W^k(E)$ and there exists $c_{k,2} > 0$ such that

$$\|De^{-t\Delta}\|_k \leqslant \frac{c_{k,2}}{\sqrt{t}} \qquad \text{for all } t > 0$$

(iii) $\Delta e^{-t\Delta}$ defines a bounded operator on $W^k(E)$ and there exists $c_{k,3} > 0$ such that

$$\|\Delta e^{-t\Delta}\|_k \leqslant \frac{c_{k,3}}{t} \qquad \text{for all } t > 0$$

(iv) The operator $e^{-t\Delta}$ is strongly convergent in $W^k(E)$ as $t \to \infty$. Moreover $(t, \phi) \mapsto e^{-t\Delta}\phi$ defines a continuous map $[0, \infty] \times W^k(E) \to W^k(E)$.

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(v) We have

$$\begin{aligned} W^k(E) &= \ker(\Delta \text{ in } W^k(E)) \oplus \mathrm{cl}_k(\operatorname{im} \Delta) \\ &= \ker(D \text{ in } W^k(E)) \oplus \mathrm{cl}_k(\operatorname{im} D). \end{aligned}$$

The corresponding projection of $W^k(E)$ onto the kernel of Δ in $W^k(E)$ is obviously defined by Π.

(vi) There exists $c_{k,4} > 0$ such that

$$\|d\phi\|_k + \|\delta\phi\|_k \leqslant c_{k,4}\|D\phi\|_k$$

for all $\phi \in C^\infty(E)$. Thus

$$\begin{aligned} \ker(D \text{ in } W^k(E)) &= \ker(d \text{ in } W^k(E)) \cap \ker(\delta \text{ in } W^k(E)) \\ \mathrm{cl}_k(\operatorname{im} D) &= \mathrm{cl}_k(\operatorname{im} d) \oplus \mathrm{cl}_k(\operatorname{im} \delta). \end{aligned}$$

For $k = 0$, properties (i)–(v) follow directly from the spectral theorem.

Proof of property (vi) for $k = 0$. Since the image of d is orthogonal to the image of δ in $L^2(E)$, we get $\|D\phi\|_0^2 = \|d\phi\|_0^2 + \|\delta\phi\|_0^2$ for any $\phi \in C^\infty(E)$. Thus

$$\|d\phi\|_0 + \|\delta\phi\|_0 \leqslant \sqrt{2}\|D\phi\|_0. \qquad \square$$

Now suppose properties (i)–(vi) hold for a given $k = l$ and we shall prove them for $k = l + 1$.

The direct sum decompositions in properties (v) and (vi) for $k = 1$ define projections P and Q of $W^l(E)$ onto $\mathrm{cl}_l(\operatorname{im} \mathrm{d})$ and $\mathrm{cl}_l(\operatorname{im} \delta)$, respectively. Thus $\mathrm{id} = \Pi + P + Q$.

LEMMA 2.2. *There are bounded operators $B_1, \dots, B_5$ on $W^l(E)$ such that*

$$\begin{aligned} Ad \pm dA &= B_1 d + dB_2 \\ A\delta \pm \delta A &= B_1\delta + \delta B_2 \\ [A, \Delta] &= B_3\Delta + \Delta B_4 + DB_5 D. \end{aligned}$$

Proof. We clearly have

$$d\Pi = \Pi d = \delta\Pi = \Pi\delta = dP = P\delta = \delta Q = Qd = 0.$$

So

$$d = dQ = DQ = Pd = PD, \qquad \delta = \delta P = DP = Q\delta = QD.$$

Hence, (1) yields the first two equalities in the statement with $B_1 = GP + KQ$ and $B_2 = QH + PL$. Thus $AD \pm DA = B_1 D + DB_2$, yielding the third equality by using

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the equation

$$[A, \Delta] = (AD \pm DA)D \mp D(AD \pm DA). \qquad \square$$

LEMMA 2.3. *For any bounded operator* $R: W^{l+1}(E) \to W^l(E)$, *we have the following Duhamel type formula*

$$[R, \mathrm{e}^{-t\Delta}] = -\int_0^t \mathrm{e}^{-(t-s)\Delta}[R, \Delta]\mathrm{e}^{-s\Delta}\, ds$$

as a bounded operator $W^{l+1}(E) \to W^l(E)$.

Proof. This follows by arguing as in the proof of the usual Duhamel's formula (see Lemma 12.51 in [11]). $\square$

LEMMA 2.4. $[A, \mathrm{e}^{-t\Delta}]$ *defines a bounded operator on* $W^l(E)$, *and there exists* $\tilde{c}_{l,1} > 0$ *such that* $\|[A, \mathrm{e}^{-t\Delta}]\|_l \leqslant \tilde{c}_{l,1}$ *for all* $t > 0$.

Proof. By Lemmas 2.2 and 2.3 we have $[A, e^{-t\Delta}] = I_2 + I_2$, where

$$I_1 = -\int_0^t \mathrm{e}^{(t-s)\Delta} B_3 \Delta \mathrm{e}^{-s\Delta}\, ds - \int_0^t \mathrm{e}^{(t-s)\Delta} \Delta B_4 \mathrm{e}^{-s\Delta}\, ds,$$
$$I_2 = -\int_0^t \mathrm{e}^{(t-s)\Delta} D B_5 D \mathrm{e}^{-s\Delta}\, ds.$$

On the one hand, property (ii) for $k = l$ yields

$$\|I_2\|_l \leqslant c_{l,2}^2 \int_0^t \frac{ds}{\sqrt{(t-s)s}}$$

which is bounded independently of t since this integral is easily seen to be π.

On the other hand,

$$\begin{aligned} I_1 = {} & \mathrm{e}^{-t\Delta/2}(B_3 - B_4)\mathrm{e}^{-t\Delta/2} + \mathrm{e}^{-t\Delta} B_3 + B_4 \mathrm{e}^{-t\Delta} - \\ & - \int_0^{t/2} \mathrm{e}^{-(t-s)\Delta} \Delta (B_3 + B_4) \mathrm{e}^{-s\Delta}\, ds - \\ & - \int_{t/2}^t \mathrm{e}^{-(t-s)\Delta}(B_3 - B_4)\Delta \mathrm{e}^{-s\Delta}\, ds, \end{aligned}$$

where we have decomposed the integrals in the definition of I_1 as sum of integrals on the intervals $[0, 1/2]$ and $[1/21]$, and we have used integration by parts with the equalities

$$\Delta \mathrm{e}^{-s\Delta} = -\frac{\partial}{\partial s}\mathrm{e}^{-s\Delta}, \qquad \Delta \mathrm{e}^{-(t-s)\Delta} = \frac{\partial}{\partial s}\mathrm{e}^{-(t-s)\Delta}.$$

Hence, by properties (i) and (iii) for $k = l$ we get

$$\begin{aligned}\|I_1\|_l &\leqslant c_{l,1}^2\|B_3 - B_4\|_l + c_{l,1}(\|B_3\|_l + \|B_4\|_l) + \\ &\quad + c_{l,1}c_{l,3}\left(\|B_3 + B_4\|_l \int_0^{t/2} \frac{ds}{t-s} + \|B_3 - B_4\|_l \int_{t/2}^{t} \frac{ds}{s}\right)\end{aligned}$$

which is bounded independently of t since both of these integrals are equal to $\ln 2$. □

LEMMA 2.5. *$AD\mathrm{e}^{-t\Delta} \pm D\mathrm{e}^{-t\Delta}A$ defines a bounded operator on $W^l(E)$ and there exists $\tilde{c}_{l,2} > 0$ such that*

$$\|AD\mathrm{e}^{-t\Delta} \pm D\mathrm{e}^{-t\Delta}A\|_l \leqslant \frac{\tilde{c}_{l,2}}{\sqrt{t}} \qquad \textit{for all } t > 0.$$

Proof. By Lemma 2.2 we have

$$AD \pm DA = B_1D + DB_2$$

yielding

$$\begin{aligned}AD\mathrm{e}^{-t\Delta} \pm D\mathrm{e}^{-t\Delta}A &= A\mathrm{e}^{-t\Delta/2}D\mathrm{e}^{-t\Delta/2} \pm \mathrm{e}^{-t\Delta/2}D\mathrm{e}^{-t\Delta/2}A \\ &= [A, \mathrm{e}^{-t\Delta/2}]D\mathrm{e}^{-t\Delta/2} \mp \mathrm{e}^{-t\Delta/2}D[A, \mathrm{e}^{-t\Delta/2}] + \\ &\quad + \mathrm{e}^{-t\Delta/2}(AD \pm DA)\mathrm{e}^{-t\Delta/2} \\ &= [A, \mathrm{e}^{-t\Delta/2}]D\mathrm{e}^{-t\Delta/2} \mp \mathrm{e}^{-t\Delta/2}D[A, \mathrm{e}^{-t\Delta/2}] + \\ &\quad + \mathrm{e}^{-t\Delta/2}B^1D\mathrm{e}^{-t\Delta/2} + \mathrm{e}^{-t\Delta/2}DB_2\mathrm{e}^{-t\Delta/2}.\end{aligned}$$

Thus the result follows by Lemma 2.4 and properties (i) and (ii) for $k = l$. □

Since D is a leafwise elliptic operator of order one and A is transversely elliptic of order one, there exist $\mathrm{e}_{l,1}, \mathrm{e}_{l,2} > 0$ such that

$$\mathrm{e}_{l,1}\|\phi\|_{l+1} \leqslant \|\phi\|_l + \|D\phi\|_l + \|A\phi\|_l \leqslant \mathrm{e}_{l,2}\|\phi\|_{l+1} \tag{4}$$

for all $\phi \in W^{l+1}(E)$.

Proof of property (i) for $k = l + 1$. For $\phi \in W^{l+1}(E)$, we have the following:

$$\begin{aligned}&\|\mathrm{e}^{-t\Delta}\phi\|_l \leqslant c_{l,1}\|\phi\|_l \leqslant c_{l,1}\mathrm{e}_{l,2}\|\phi\|_{l+1}, \\ &\|D\mathrm{e}^{-t\Delta}\phi\|_l = \|\mathrm{e}^{-t\Delta}D\phi\|_l \leqslant c_{l,1}\|D\phi\|_l \leqslant c_{l,1}\mathrm{e}_{l,2}\|\phi\|_{l+1}, \\ &\|A\mathrm{e}^{-t\Delta}\phi\|_l \leqslant \|[A, \mathrm{e}^{-t\Delta}]\phi\|_l + \|\mathrm{e}^{-t\Delta}A\phi\|_l \\ &\qquad \leqslant \tilde{c}_{l,1}\|\phi\|_l + c_{l,1}\|A\phi\|_l \\ &\qquad \leqslant (\tilde{c}_{l,1} + c_{l,1})e_{l,2}\|\phi\|_{l+1},\end{aligned}$$

where we have used property (i) for $k = l$, Lemma 2.4 and (4). □

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Proof of property (ii) for $k = l + 1$. For $\phi \in W^{l+1}(E)$ we have the following:

$$\begin{aligned}
\|D\mathrm{e}^{-t\Delta}\phi\|_l &\leqslant \frac{c_{l,2}}{\sqrt{t}}\|\phi\|_l \leqslant \frac{c_{l,2}e_{l,2}}{\sqrt{t}}\|\phi\|_{l+1},\\
\|\Delta\mathrm{e}^{-t\Delta}\phi\|_l &= \|D\mathrm{e}^{-t\Delta}D\phi\|_l \leqslant \frac{c_{l,2}}{\sqrt{t}}\|D\phi\|_l \leqslant \frac{c_{l,2}e_{l,2}}{\sqrt{t}}\|\phi\|_{l+1},\\
\|AD\mathrm{e}^{-t\Delta}\phi\| &\leqslant \|(AD\mathrm{e}^{-t\Delta} \pm D\mathrm{e}^{-t\Delta}A)\phi\|_l + \|D\mathrm{e}^{-t\Delta}A\phi\|_l\\
&\leqslant \frac{\tilde{c}_{l,2}}{\sqrt{t}}\|\phi\|_l + \frac{c_{l,2}}{\sqrt{t}}\|A\phi\|_l\\
&\leqslant \frac{(\tilde{c}_{l,2} + c_{l,2})e_{l,2}}{\sqrt{t}}\|\phi\|_{l+1},
\end{aligned}$$

where we have used property (ii) for $k = l$, Lemma 2.5 and (4). □

Proof of property (iii) for $k = l + 1$. This follows directly from property (ii) for $k = l + 1$ since $\Delta\mathrm{e}^{-t\Delta} = D\mathrm{e}^{-t\Delta} = D\mathrm{e}^{-t\Delta/2}D\mathrm{e}^{-t\Delta/2}$ on $C^\infty(E)$. □

Let $\tilde{\Pi} = \mathrm{id} - \Pi$. Set $B = B_4\Pi - B_3\tilde{\Pi}$, which is a bounded operator on $W^l(E)$. For further reference, we point out the following estimates which are similar to (4): There exist $e'_{l,1}, e'_{l,2}, e''_{l,1}, e''_{l,2} > 0$ such that

$$e'_{l,1}\|\phi\|_{l+1} \leqslant \|\phi\|_l + \|D\phi\|_l + \|(A + B)\phi\|_l \leqslant e'_{l,2}\|\phi\|_{l+1}, \tag{5}$$

$$e''_{l,1}\|\phi\|_{l+1} \leqslant \|\phi\|_l + \|D\phi\|_l + \|(A - B_1)\phi\|_l \leqslant e''_{l,2}\|\phi\|_{l+1}. \tag{6}$$

LEMMA 2.6. *The operator* $[A + B, \mathrm{e}^{-t\Delta}]$ *strongly converges to the zero operator on* $W^l(E)$ *as* $t \to \infty$. *Moreover* $(t, \phi) \mapsto [A + B, \mathrm{e}^{-t\Delta}]\phi$ *extends to a continuous map* $[0, \infty] \times W^l(E) \to W^l(E)$ *vanishing on* $\{\infty\} \times W^l(E)$.

Proof. Take any $a > 1$ that will be fixed later. Let us write $[A + B, \mathrm{e}^{-t\Delta}] = I_1 + I_2$, where

$$\begin{aligned}
I_1 &= [A + B, \mathrm{e}^{-(t-t/a)\Delta}]\mathrm{e}^{-t\Delta/a},\\
I_2 &= \mathrm{e}^{-(t-t/a)\Delta}[A + B, \mathrm{e}^{-t\Delta/a}].
\end{aligned}$$

By Lemma 2.2 and since $\Delta\Pi = 0$, we easily get

$$[A + B, \Delta] = (\Delta(B_3 + B_4) + DB_5D)\tilde{\Pi}. \tag{7}$$

Hence, by Lemma 2.3 we have $I_2 = I_{2,1} + I_{2,2}$, where

$$\begin{aligned}
I_{2,1} &= -\mathrm{e}^{-(t-t/a)\Delta}\int_0^{t/a} \mathrm{e}^{-(t-s)\Delta}\Delta(B_3 + B_4)\mathrm{e}^{-s\Delta}\tilde{\Pi}\,ds,\\
I_{2,2} &= -\mathrm{e}^{-(t-t/a)\Delta}\int_0^{t/a} \mathrm{e}^{-(t-s)\Delta}DB_5D\mathrm{e}^{-s\Delta}\tilde{\Pi}\,ds.
\end{aligned}$$

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Properties (i), (iii) and (vi) for $k = l$ yield the following estimate for $\phi \in W^l(E)$:

$$\begin{aligned}\|I_{2,1}\|_l &\leqslant c_{l,1}^2 c_{l,3} \|B_3 + B_4\|_l \|\tilde{\Pi}\phi\|_l \int_0^{t/a} \frac{ds}{t-s} \\ &\leqslant c_{l,1}^2 c_{l,3} \|B_3 + B_4\|_l \|\tilde{\Pi}\phi\|_l \ln \frac{a}{a-1}.\end{aligned}$$

Similarly, properties (i), (ii) and (vi) for k=l yield the following estimate for $\phi \in W^l(E)$:

$$\begin{aligned}\|I_{2,2}\phi\|_l &\leqslant c_{l,1} c_{l,2}^2 \|\tilde{\Pi}\phi\|_l \int_0^{t/a} \frac{ds}{\sqrt{(t-s)s}} \\ &\leqslant c_{l,1} c_{l,2}^2 \|\tilde{\Pi}\phi\|_l \left(\pi + \arctan \frac{2-a}{a}\right).\end{aligned}$$

Therefore I_2 defines a bounded operator on $W^l(E)$ whose norm can be made arbitrarily small uniformly on t by taking a large enough.

To study I_1 we shall use the following:

$$[A + B, e^{-s\Delta}]\Pi = 0 \qquad \text{for all } s \geqslant 0. \tag{8}$$

This equation holds because it is obvious for $s = 0$ and moreover

$$\begin{aligned}\frac{d}{ds}([A + B, e^{-s\Delta}]\Pi) &= -[A + B, e^{-s\Delta}\Delta]\Pi \\ &= -[A + B, e^{-s\Delta}]\Delta\Pi - e^{-s\Delta}[A + B, \Delta]\Pi,\end{aligned}$$

which vanishes since $\Delta\Pi$=0 and by (7). Now (8) yields

$$\begin{aligned}I_1 &= [A + B, e^{-(t-t/a)\Delta}]\tilde{\Pi} e^{-t\Delta/a} \\ &= [A + B, e^{-(t-t/a)\Delta}](e^{-t\Delta/a} - \Pi).\end{aligned}$$

Furthermore, by Lemma 2.4 and property (i) for $k = l$, $[A + B, e^{-(t-t/a)\Delta}]$ defines a bounded operator on $W^l(E)$ whose norm is uniformly bounded on t. Therefore, by property (iv) for $k = l$, I_1 strongly converges to zero in $W^l(E)$ as $t \to \infty$ for any $a > 1$, and $(t, \phi) \mapsto I_1\phi$ extends to a continuous map $[0, \infty] \times W^l(E) \to W^l(E)$ vanishing on $\{\infty\} \times W^l(E)$. □

Proof of property (iv) for $k = l + 1$. Consider the following bounded compositions:

$$W^{l+1}(E) \xrightarrow{e^{-t\Delta}} W^{l+1}(E) \hookrightarrow W^l(E), \tag{9}$$

$$W^{l+1}(E) \xrightarrow{e^{-t\Delta}} W^{l+1}(E) \xrightarrow{D} W^l(E), \tag{10}$$

$$W^{l+1}(E) \xrightarrow{e^{-t\Delta}} W^{l+1}(E) \xrightarrow{A+B} W^l(E). \tag{11}$$

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By (5), it is enough to prove that the compositions (9), (10) and (11) are strongly convergent as $t \to \infty$ and to prove the continuous extension to $[0, \infty] \times W^{l+1}(E)$ of the maps $[0, \infty] \times W^{l+1}(E) \to W^l(E)$ defined by these time dependent operators. This holds for (9) and (10) by property (iv) for $k = l$ since these operators are respectively equal to the compositions

$$W^{l+1}(E) \hookrightarrow W^l(E) \xrightarrow{e^{-t\Delta}} W^l(E),$$
$$W^{l+1}(E) \xrightarrow{D} W^l(E) \xrightarrow{e^{-t\Delta}} W^l(E).$$

On the other hand, (11) is the sum of the bounded compositions

$$W^{l+1}(E) \hookrightarrow W^l(E) \xrightarrow{[A+B, e^{-t\Delta}]} W^l(E), \tag{12}$$

$$W^{l+1}(E) \xrightarrow{A+B} W^l(E) \xrightarrow{e^{-t\Delta}} W^l(E). \tag{13}$$

Now Lemma 2.6 and property (iv) for $k = l$ respectively imply the strong convergence of (12) and (13) as $t \to \infty$, as well as the continuous extension to $[0, \infty] \times W^{l+1}(E)$ of the maps $[0, \infty) \times W^{l+1}(E) \to W^l(E)$ defined by both of these time dependent operators. □

COROLLARY 2.7. Π *defines a bounded operator on* $W^{l+1}(E)$.

Proof. This result is a direct consequence of properties (i) and (iv) for $k = l + 1$—observe that, for each $\phi \in W^{l+1}(E)$, the limit of $e^{-t\Delta}\phi$ in $W^{l+1}(E)$ as $t \to \infty$ can only be $\Pi\phi$ since it is so in $L^2(E)$. □

COROLLARY 2.8. $[A + B, \Pi] = 0 : W^{l+1}(E) \to W^l(E)$.

Proof. By Lemma 2.6, $[A+B, e^{-t\Delta}]$ strongly converges to zero on $W^l(E)$ as $t \to \infty$. Hence the result follows because, as $t \to \infty$ and for each $\phi \in W^{l+1}(E)$, $(A + B)e^{-t\Delta}\phi$ and $e^{-t\Delta}(A + B)\phi$ converge in $W^l(E)$ to $(A + B)\Pi\phi$ and $\Pi(A + B)\phi$, respectively, by property (iv) for $k = l + 1, l$. □

Proof of property (v) for $k = l + 1$. By Corollary 2.7 and property (v) for $k = l$, Π defines a projection of $W^{l+1}(E)$ onto the kernel of Δ in $W^{l+1}(E)$.

Now, for $t > 0$ and $s \in \mathbb{R}$ let

$$f_t(s) = \begin{cases} (1 - e^{-ts^2})/s^2 & \text{if } s \neq 0, \\ t & \text{if } s = 0. \end{cases}$$

It is easy to check that f_t is in the algebra $\mathcal{A}$ of Proposition 2.1. Thus $f_t(D)$ defines a bounded operator on $W^{l+1}(E)$ satisfying

$$\mathrm{id} - e^{-t\Delta} = \Delta f_t(D).$$

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So property (iv) for $k = l + 1$ yields that, for any $\phi \in W^{l+1}(E)$, $\tilde{\Pi}\phi$ is the limit of $\Delta f_t(D)\phi$ in $W^{l+1}(E)$ as $t \to \infty$. Hence

$$\tilde{\Pi}(W^{l+1}(E)) \subset \mathrm{cl}_{l+1}(\mathrm{im}\,\Delta).$$

This is really an equality because the reverse inclusion can be easily proved as follows:

$$\begin{aligned}\mathrm{cl}_{l+1}(\mathrm{im}\,\Delta) &\subset W^{l+1}(e) \cap \mathrm{cl}_0(\mathrm{im}\,\Delta)\\ &= W^{l+1}(E) \cap \tilde{\Pi}(L^2(E))\\ &= \tilde{\Pi}(W^{l+1}(E)).\end{aligned}$$

Here we have used property (v) for k=0 and the fact that $\tilde{\Pi}$ is a projection. Therefore we have proved the first direct sum decomposition in property (v) for $k = l + 1$.

The second direct sum decomposition follows similarly by using the functions given by

$$g_t(s) = \begin{cases} (1 - \mathrm{e}^{-ts^2})/s & s \neq 0,\\ t & s = 0\end{cases}$$

instead of the f_t. □

Proof of property (vi) for $k = l + 1$. Take any $\phi \in C^\infty(E)$. Lemma 2.2, property (vi) for $k = l$ and (6) yield

$$\begin{aligned}\|d\phi\|_{l+1} + \|\delta\phi\|_{l+1} &\leqslant \frac{1}{e''_{l,1}}(\|d\|_l + \|\delta\phi\|_l + \|Dd\phi\|_l + \|D\delta\phi\|_l +\\ &\quad + \|(A - B_1)d\phi\|_l + \|(A - B_1)\delta\phi\|_l)\\ &\leqslant \frac{1}{e''_{l,1}}(\|d\phi\|_l + \|\delta\phi\|_l + VdD\phi\|_l + \|\delta D\phi\|_l +\\ &\quad + \|d(\mp A + B_2)\phi|_l + \|\delta(\mp A + B_2)\phi\|_l)\\ &\leqslant \frac{c_{l,4}}{e''_{l,1}}(\|D\phi\|_l + \|\Delta\phi\|_l + \|D(\mp A + B_2)\phi\|_l)\\ &\leqslant \frac{c_{l,4}}{e'_{l,1}}(\|D\phi\|_l + \|\Delta\phi\|_l + \|(A - B_1)D\phi\|_l)\\ &\leqslant \frac{c_{l,4}e''_{l,2}}{e''_{l,1}}\|D\phi\|_{l+1}.\end{aligned}$$ □

Remark 1. If A is symmetric in Theorem A, then one of the two equations in (1) can be removed in its statement because both of them are equivalent by taking adjoints on M.

Remark 2. The following more general condition can be used instead of (1) in Theorem A to get the same long time behavior of leafwise heat flow with a similar proof: There is some first order transversely elliptic differential operator A and zero

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order operators $K_1, \dots, K_{16}$ such that the operators

$$F_1 = Ad \pm dA - K_1 d - dK_2 - K_3\delta - \delta K_4,$$
$$F_2 = A\delta \pm \delta A - K_5\delta - \delta K_6 - K_7 d - dK_8$$

satisfy

$$\begin{aligned} &F_1\delta \mp \delta F^1 + F_2 d \mp dF_2 \\ &\quad = dK_9 d + \delta K_{10} d + dK_{11}\delta + \delta K_{12}\delta + d\delta K_{13} + \delta dK_{14} + K_{15} d\delta + K_{16}\delta d. \end{aligned}$$

Nevertheless, so far we did not find any non-Riemannian foliation with a (non-elliptic) leafwise elliptic differential complex satisfying such a more general condition.

Corollary 2.8 yields $[A, \Pi] = -[B, \Pi]$ on each $W^k(E)$, obtaining the following consequence that will be used later.

COROLLARY 2.9. $[A, \Pi]$ *defines a bounded operator on each* $W^k(E)$, and thus a continuous operator on $C^\infty(E)$.

3. Case of the Leafwise de Rham Complex for Riemannian Foliations

With the notation introduced in section 1, the objective of this section is to prove the following result, which implies Theorem B by using Theorem A and Remark 1.

PROPOSITION 3.1. *If $\mathcal{F}$ is a Riemannian foliation and M is endowed with a bundle-like metric, there is a zero order differential operator K on Ω such that*

$$D_\perp \delta_{0,-1} + \delta_{0,-1} D_\perp = K\delta_{0,-1} + \delta_{0,-1} K.$$

To prove Proposition 3.1, choose any open subset $U \subset M$ of triviality for $\mathcal{F}$. Let $n = \dim M$, $p = \dim \mathcal{F}$ and $q = \operatorname{codim} \mathcal{F}$. Fix tangential and transverse orientations for $\mathcal{F}$ in U, obtaining the Hodge star operators $*_\mathcal{F}$ and $*_\perp$ on the restrictions of $\bigwedge T\mathcal{F}^*$ and $\bigwedge T\mathcal{F}^{\perp *}$ to U, respectively. Moreover we get an induced orientation of U so that $*_\perp(1) \wedge *_\mathcal{F}(1)$ is a positive volume form. The following lemma can be easily proved (the statement of Lemma 4.8 in [6] is similar).

LEMMA 3.2. *Over U, the Hodge star operator on $\bigwedge TM^* \equiv \bigwedge T\mathcal{F}^{\perp *} \otimes \bigwedge T\mathcal{F}^*$ is given by*

$$* = (-1)^{(q-u)v} *_\perp \otimes *_\mathcal{F} : \bigwedge^u T\mathcal{F}^{\perp *} \otimes \bigwedge^v T\mathcal{F}^* \to \bigwedge^{q-u} T\mathcal{F}^{\perp *} \otimes \bigwedge^{p-v} T\mathcal{F}^*.$$

Let $\mathcal{X}(\mathcal{F}_\mathcal{U}) \subset \mathcal{X}(\mathcal{U})$ be the Lie subalgebra of vector fields which are tangent to the leaves of $\mathcal{F}_U = \mathcal{F}|_U$, and let $\mathcal{X}(U, \mathcal{F}_U) \subset \mathcal{X}(U)$ be its normalizer—the Lie algebra of infinitesimal transformations of $\mathcal{F}_U$. Let also $\Omega(U/\mathcal{F}_U) \subset \Omega^{\cdot,0}(U)$ denote the basic complex of $\mathcal{F}_U$, and $C^\infty(U/\mathcal{F}_U) = \Omega^0(U/\mathcal{F}_U)$. For $X \in \mathcal{X}(U)$ let $\mathcal{L}_X$ denote the cor-

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responding Lie derivative on $\Omega(U)$ and θ_X its bihomogeneous $(0,0)$-component, which is easily seen to be also a derivation. By comparing bidegrees on Cartan's formula we get that, if X is orthogonal to $\mathcal{F}$, then

$$\theta_X = i_x d_{1,0} \quad \text{on} \quad \Omega^{0,\cdot}(U)$$

yielding

$$\theta_{fX} = f\theta_X \quad \text{on} \quad \Omega^{0,\cdot}(U) \quad \text{for all} \quad f \in C^\infty(U) \tag{14}$$

and

$$d_{1,0}\beta = \sum_{i=1}^{q} \alpha^i \wedge \theta_{Xi}\beta \quad \text{for all} \quad \beta \in \Omega^{0,\cdot}(U) \tag{15}$$

where $X_1, \ldots, X_q$ is any frame of $T\mathcal{F}^\perp$ on U with dual coframe $\alpha_1, \ldots, \alpha_q$ in $\Omega^{1,0}(U)$. Furthermore, if $X \in \mathcal{X}(U, \mathcal{F}_U)$ is orthogonal to the leaves and $Y \in \mathcal{X}(\mathcal{F}_U)$, then

$$i_Y \mathcal{L}_X = \mathcal{L}_X i_Y - i_{[X,Y]} = 0 \quad \text{on} \quad \Omega^{\cdot,0}(U)$$

yielding that the $(-1,1)$-bihomogeneous component of $\mathcal{L}_X$ vanishes on $\Omega^{\cdot,0}$, and thus on $\Omega(U)$. Therefore

$$\theta_X d_{0,1} = d_{0,1}\theta_X \quad \text{on} \quad \Omega(U) \quad \text{for all} \quad X \in \mathcal{X}(U, \mathcal{F}_U) \tag{16}$$

by comparing bidegrees on the formula $\mathcal{L}_X d = d\mathcal{L}_X$.

As was pointed out in section 1, the restriction of $\delta_{0,-1}$ to $\Omega^{0,\cdot} \equiv \Omega(\mathcal{F})$ is defined by the de Rham coderivative on the leaves. This holds whenever the transverse Riemannian volume element is holonomy invariant, and in particular when the metric is bundle-like: On U, $\omega = *_\perp(1)$ satisfies $d\omega = 0$, and thus, by Lemma 3.2, for $\beta \in \Omega^{0,v}(U)$ we have

$$\begin{aligned}
\delta_{0,-1}\beta &= (-1)^{nv+n+1} * d_{0,1} * \beta \\
&= (-1)^{pv+n+1} * d_{0,1}(\omega \wedge *_\mathcal{F}\beta) \\
&= (-1)^{pv+p+1} * (\omega \wedge d_{0,1} *_\mathcal{F} \beta) \\
&= (-1)^{pv+p+1} *_\mathcal{F} d_{0,1} *_\mathcal{F} \beta.
\end{aligned} \tag{17}$$

LEMMA 3.3. *If $X \in \mathcal{X}(U, \mathcal{F}_U)$ is orthogonal to the leaves, then there is a zero order differential operator $R_{U,X}$ on $\Omega^{0,\cdot}(U)$, depending $C^\infty(M/\mathcal{F}_U)$-linearly on X, such that*

$$[\theta_X, \delta_{0,-1}] = [R_{U,X}, \delta_{0,-1}] \quad \textit{on} \quad \Omega^{0,\cdot}.$$

Moreover the assignment $(U, X) \mapsto R_{U,X}$ can be chosen so that the restriction of $R_{U,X}$ to any open subset $U' \subset U$ of triviality for $\mathcal{F}$ is equal to $R_{U',X'}$, where $X' = X|_{U'}$.

Proof. By using (16) and (17), and since $*_{\mathcal{F}}^2 = (-1)^{(p-v)v}$ id, on $\Omega^{0,v}(U)$ we get

$$\begin{aligned}[\theta_X, \delta_{0,-1}] &= (-1)^{pv+p+1}[\theta_X, *_{\mathcal{F}} d_{0,1} *_{\mathcal{F}}] \\ &= (-1)^{pv+p+1}([\theta_X, *_{\mathcal{F}}] d_{01} *_{\mathcal{F}} + *_{\mathcal{F}}\, d_{0,1}[\theta_X, *_{\mathcal{F}}]) \\ &= (-1)^{(p-v+1)(v-1)}[\theta_X, *_{\mathcal{F}}] *_{\mathcal{F}}\, \delta_{0,-1} + \\ &\quad + (-1)^{(p-v)v}\delta_{0,-1} *_{\mathcal{F}} [\theta_X, *_{\mathcal{F}}].\end{aligned}$$

Hence

$$R_{U,X} = (-1)^{(p-v)v}[\theta_X, *_{\mathcal{F}}] *_{\mathcal{F}} \quad \text{on } \Omega^{0,v}$$

satisfies the equality in the statement of this result because

$$0 = [\theta_X, *_{\mathcal{F}}^2] = [\theta_X, *_{\mathcal{F}}] *_{\mathcal{F}} + *_{\mathcal{F}} [\theta_X, *_{\mathcal{F}}].$$

Observe that $R_{U,X}$ is of order zero since θ_X is a derivation on $\Omega^{0,\cdot}$ and $R_{U,X}$ is $C^\infty(M/\mathcal{F}_U)$-linear on X by (14). Finally, the restriction of $R_{U,X}$ to $U' \subset U$ is clearly equal to $R_{U',X'}$ as in the statement of this result. □

Obviously, $d_{0,1}$ and $\delta_{0,-1}$ are $C^\infty(U/\mathcal{F}_U)$-linear on $\Omega(U)$. Indeed we have the following.

LEMMA 3.4. *We have*

$$d_{0,1} \equiv (-1)^u \text{ id} \otimes d_{0,1}, \qquad \delta_{0,-1} \equiv (-1)^u \text{ id} \otimes \delta_{0,-1}$$

with respect to the canonical decomposition

$$\Omega^{u,\cdot}(U) \equiv \Omega^u(U/\mathcal{F}_U) \otimes \Omega^{0,\cdot}(U)$$

as tensor product of $C^\infty(U/\mathcal{F}_U)$-modules.

Proof. The first identity is clear because $d_{0,1}$ vanishes on basic forms. The second identity holds because the metric is bundle-like on U: This is equivalent to

$$*_\perp(\Omega(U/\mathcal{F}_U)) \subset \Omega(U/\mathcal{F}_U)$$

and thus, by Lemma 3.2, for $\alpha \in \Omega^u(U/\mathcal{F}_U)$, $\beta \in \Omega^{0,v}(U)$ and $r = u + v$ we have

$$\begin{aligned}\delta_{0,-1}(\alpha \wedge \beta) &= (-1)^{nr+n+1} * d_{0,1} * (\alpha \wedge \beta) \\ &= (-1)^{nr+n+1+(q-u)v} * d_{0,1}(*_\perp\alpha \wedge *_{\mathcal{F}}\beta) \\ &= (-1)^{nr+n+1+(q-u)(v+1)} * (*_\perp\alpha \wedge d_{0,1} *_{\mathcal{F}} \beta) \\ &= (-1)^u \alpha \wedge \delta_{0,-1}\beta.\end{aligned}$$

□

The proof of Proposition 3.1 can be completed as follows. Let $X_1, \ldots, X_q \in \mathcal{X}(U, \mathcal{F}_U)$ be a frame of $T\mathcal{F}^\perp$ on U, and let $\alpha_1, \ldots, \alpha_q \in \Omega^1(U/\mathcal{F}_U)$ be the dual coframe. Take any $\alpha \in \Omega^u(U/\mathcal{F}_U)$ and any $\beta \in \Omega^{0,\cdot}(U)$. Then Lemmas

3.3 and 3.4, and (15) yield

$$\begin{aligned}(d_{1,0}\delta_{0,-1}+\delta_{0,-1}d_{1,0})(\alpha\wedge\beta) &= (-1)^u d_{1,0}\alpha\wedge\delta_{0,-1}\beta+\alpha\wedge d_{1,0}\delta_{0,-1}\beta+\\ &\quad +(-1)^{u+1}d_{1,0}\alpha\wedge z\delta_{0,-1}\beta+\alpha\wedge\delta_{0,-1}d_{1,0}\beta\\ &= \sum_{i=1}^{q}\alpha\wedge\alpha_i\wedge[\theta_{X_i},\delta_{0,-1}]\beta\\ &= \sum_{i=1}^{q}\alpha\wedge\alpha_i\wedge[R_{U,X_i},\delta_{0,-1}]\beta\\ &= (K_U\delta_{0,-1}+\delta_{0,-1}K_U)(\alpha\wedge\beta),\end{aligned}$$

where

$$K_U(\alpha\wedge\beta)=(-1)^u\sum_{i=1}^{q}\alpha\wedge\alpha_i\wedge R_{U,X_i}\beta$$

for α and β as above. By the properties of the R_{U,X_i}, it is clear that K_U is independent of the choices of the X_i, and moreover the restriction of K_U to any open subset $U'\subset U$ of triviality for $\mathcal{F}$ is equal to $K_{U'}$. Thus the K_U defines a global operator K on Ω satisfying

$$d_{1,0}\delta_{0,-1}+\delta_{0,-1}d_{1,0}=K\delta_{0,-1}+\delta_{0,-1}K$$

which finishes the proof of Proposition 3.1 since (see e.g. [2])

$$d_{1,0}d_{0,1}+d_{0,1}d_{1,0}=0.$$

Remark 3. Proposition 3.1 is slightly stronger than (1) in Theorem A. In this case $K=L$, and thus

$$D_\perp D_0+D_0D_\perp=B_1D_0+D_0B_2$$

with

$$B_1=K^*P+KQ,\qquad B_2=QK^*+PK$$

in Lemma 2.2. In particular $B_2=B_1^*$. Observe that, if K is symmetric, then $D_\perp-K$ commutes with Δ_0 and thus with $\mathrm{e}^{-t\Delta_0}$. Therefore the proof of Theorem B would follow with a much simpler induction argument. Of course K is not symmetric in general.

Remark 4. The above proof of Proposition 3.1 gives explicit local descriptions of K and L. But an alternative proof can be made by using Molino's description of Riemannian foliations: It allows to reduce the proof to the case of transversely parallelizable foliations, where our local arguments can be made globally.

Remark 5. In the setting of this section, Corollary 2.9 states that $[D_\perp,\Pi]$ defines a bounded operator on each $W^k\Omega$.

4. Case of Leafwise Differential Forms with Appropriate Coefficients

Corollary C is proved in this section. Let thus $\mathcal{F}$, M and V be as in the statement of that result. Since any metric on the leaves, smooth on M, can be extended to a bundle-like metric on M, then Corollary C follows directly from Theorem B when V is any trivial Riemannian vector bundle with the trivial $\mathcal{F}$-partial connection.

In the general case we follow Molino's idea to describe Riemannian foliations [24, 25]. Let $\pi : F \to M$ be the principal $O(k)$-bundle of orthonormal frames of V, where k is the rank of V; observe that such an F is a closed manifold. The metric $\mathcal{F}$-partial connection on V can be understood as an $O(k)$-invariant vector subbundle $H \subset TF$ so that $\pi_* : H_f \to T_{\pi(f)}\mathcal{F}$ is an isomorphism for every frame $f \in F$. Moreover the flatness of the connection means that H defines a completely integrable distribution, and let thus $\hat{\mathcal{F}}$ be the corresponding foliation on F. It is clear that $\hat{\mathcal{F}}$ is also a Riemannian foliation, $\pi^* V$ has a canonical trivialization as Riemannian vector bundle, the pull-back of the partial connection on V is the trivial $\hat{\mathcal{F}}$-partial connection on $\pi^* V$, and π^* defines an injection $\Omega(\mathcal{F}, V) \hookrightarrow \Omega(\hat{\mathcal{F}}, \pi^* V)$. Moreover it is easy to check that, for the lift of any given metric on the leaves of $\mathcal{F}$ to the leaves of $\hat{\mathcal{F}}$, $\Delta_{\mathcal{F}}$ is the restriction of $\Delta_{\hat{\mathcal{F}}}$ by the above injection. Therefore Corollary C for $\mathcal{F}$ and V follows from the case of $\hat{\mathcal{F}}$ and $\pi^* V$.

5. The Space of Bundle-like Metrics

We prove Corollary D in this section. First we recall some technicalities from [9]. For a given foliation $\mathcal{F}$ on a manifold M, let $\nu = TM/T\mathcal{F}$, $Q = S^2(\nu^*)$, and $Q^+ \subset Q$ the subbundle given by the positive definite elements in Q. Hence $C^\infty(Q^+)$ is the space of metrics on the normal bundle ν. Such metrics are the key point to prove Corollary D because any metric g on M is uniquely determined by fixing three objects: A metric $g_{\mathcal{F}}$ on $T\mathcal{F}$, a subbundle $N \subset TM$ which is complementary of $T\mathcal{F}$, and a metric $\bar{g}$ on ν. In fact, $\bar{g}$ determines a metric g_N on N by the canonical isomorphism $N \cong \nu$, and g is determined as the orthogonal sum of $g_{\mathcal{F}}$ and g_N. Conversely, g determines $g_{\mathcal{F}} = g|_{T\mathcal{F}}$, $N = T\mathcal{F}^\perp$ and $\bar{g}$ as the only metric that corresponds to $g|_N$ by the above isomorphism. According to this notation, the metric g is bundle-like if and only if the corresponding metric $\bar{g}$ is parallel with respect to the $\mathcal{F}$-partial Bott connection on $S^2(\nu^*)$; i.e. $d_{\mathcal{F}}(\bar{g}) = 0$. Thus, by modifying only $\bar{g}$ for each metric g, it is clear that Corollary D follows from the following result by using Corollary C with $V = Q$.

LEMMA 5.1. *Suppose $\mathcal{F}$ is Riemannian and M is closed with a fixed bundle-like metric. Then the corresponding leafwise heat operator* $\mathrm{e}^{-t\Delta_{\mathcal{F}}}$ on $\Omega(\mathcal{F}, Q)$ preserves $C^\infty(Q^+)$ for each $t \in [0, \infty]$.

Proof. Consider the metric $\bar{g}$ on ν determined as above by the bundle-like metric on M. Let ν^1 be the sphere bundle over M given by the normal vectors of $\bar{g}$-norm

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one. Then the result follows by checking that, for any $g \in C^\infty(Q^+)$, we have

$$\min_{v \in v^1}(\mathrm{e}^{-t\Delta_{\mathcal{F}}}\bar{g})(v, v) \geqslant \min_{v \in v^1} \bar{g}(v, v) \quad \text{for all } t \in [0, \infty).$$

This in turn follows by checking that, for any given $T \in [0, \infty)$, if the map $v \in v^1 \mapsto (\mathrm{e}^{-T\Delta_{\mathcal{F}}}\bar{g})(v, v) \in \mathbb{R}$ reaches the minimum at some $v_m \in v^1_x$ for some $x \in M$, then

$$\frac{\partial(\mathrm{e}^{-t\Delta_{\mathcal{F}}}\bar{g})}{\partial t}(v_m, v_m)\bigg|_{t=T} \geqslant 0.$$

This property can be proved as follows. Extend v_m to a local field V of normal vectors of $\bar{g}$-norm one satisfying $d_{\mathcal{F}}V = 0$; this is always possible on some open subset $U \subset M$ of triviality for $\mathcal{F}$ since $d_{\mathcal{F}}\bar{g} = 0$. Then, if P is the plaque in U containing x, the restriction f_t of $(\mathrm{e}^{-t\Delta_{\mathcal{F}}}\bar{g})(V, V)$ to P satisfies the parabolic equation $\partial f_t/\partial t + \Delta_P f_t = 0$ and f_T reaches the minimum at x; here Δ_P is the Laplacian on P. Hence

$$\frac{\partial f_t}{\partial t}(x)\bigg|_{t=T} \geqslant 0$$

by the maximum principle and the proof is completed. □

6. Dimension of the Space of Leafwise Harmonic Forms

Corollary E is proved in this section. With the notation of that corollary, let ϕ be a nontrivial integrable harmonic r-form on some leaf L with coefficients on V. Such ϕ determines a continuous linear functional $\tilde{\phi}$ on $\Omega^{p-r}(\mathcal{F}, V^*)$, $p = \dim \mathcal{F}$, given by

$$\tilde{\phi}(\psi) = \int_L \phi \wedge \psi|_L$$

Thus $\tilde{\phi}$ is a singular element in $W^k\Omega^r(\mathcal{F}, V)$ for some negative $k \in \mathbb{Z}$. Take any sequence ϕ_i in $\Omega^r(\mathcal{F}, V)$ converging to ϕ in $W^k\Omega^r(\mathcal{F}, V)$. By Corollary C, $\Pi\phi_i$ is a sequence in $\Omega^r(\mathcal{F}, V) \cap \ker \Delta_{\mathcal{F}}$ converging to the singular $\tilde{\phi} = \Pi\tilde{\phi}$ in $W^k\Omega^r(\mathcal{F}, V)$, and so the $\Pi\phi_i$ generate a space of infinite dimension.

7. Application to the Second Term of the Spectral Sequence of Riemannian Foliations

The objective of this section is to prove Theorem F. Consider thus the notation and conditions introduced to state this result.

LEMMA 7.1. *D_1 and $\tilde{D}_1$ are essentially self-adjoint in $L^2\mathcal{H}_1$ and $L^2\tilde{\mathcal{H}}$, respectively*

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Proof. By Theorem 2.2 in [10], $D_\perp$ is essentially self-adjoint in $L^2\Omega$. Then, by using e.g. Lemma XII.1.6-(c) in [13], so is $\Pi D_\perp \Pi$ because Π is a bounded self-adjoint operator on $L^2\Omega$. But $\Pi D_\perp \Pi$ is equal to D_1 in $L^2\mathcal{H}_1$ and vanishes in its orthogonal complement. Hence D_1 is essentially self-adjoint.

The proof that $\tilde{D}_1$ is essentially self-adjoint is similar. □

LEMMA 7.2. *We have the following properties*:

(i) $D\Pi - \Pi D_\perp \Pi$ *defines a bounded operator on* $L^2\Omega$.
(ii) *For each* $v \in \mathbb{Z}$, $D\tilde{\Pi}_{\cdot,v} - \tilde{\Pi}_{\cdot,v} D\tilde{\Pi}_{\cdot,v}$ *defines a bounded operator on* $L^2\Omega$.

Proof. Property (i) can be proved as follows. Because $D_0\Pi = 0$, we get

$$D\Pi - \Pi D\Pi = (\mathrm{id} - \Pi)D_\perp \Pi + (d_{2,-1} + \delta_{-2,1})\Pi.$$

which is bounded on $L^2\Omega$ by Remark 5 and the formula

$$(\mathrm{id} - \Pi)D_\perp \Pi = [D_\perp, \Pi]\Pi.$$

The proof of property (ii) is slightly more complicated. If $\phi = \phi_1 + \phi_2 \in \tilde{\mathcal{H}}_1^{\cdot,v}$ with

$$\phi_1 \in \overline{d_{0,1}(\Omega^{\cdot,v-1})} \quad \text{and} \quad \phi_2 \in \overline{\delta_{0,-1}(\Omega^{\cdot,v})},$$

then

$$d_{0,1}\phi_1 = \delta_{0,-1}\phi_2 = 0, \quad \delta_{0,-1}\phi_1, d_{0,1}\phi_2 \in \tilde{\mathcal{H}}_1^{\cdot,v}$$

yielding

$$(D_0\tilde{\Pi}_{\cdot,v} - \tilde{\Pi}_{\cdot,v}D_0\tilde{\Pi}_{\cdot,v})\phi = 0.$$

Hence, since $d_{2,-1} + \delta_{-2,1}$ is of order zero, it is enough to prove that

$$D_\perp\tilde{\Pi}_{\cdot,v} - \tilde{\Pi}_{\cdot,v}D_\perp\tilde{\Pi}_{\cdot,v}$$

defines a bounded operator on $L^2\Omega$. But this operator is clearly equal to

$$\Pi D_\perp\tilde{\Pi}_{\cdot,v} + \tilde{\Pi}_{\cdot,v-1}D_\perp\tilde{\Pi}_{\cdot,v} + \tilde{\Pi}_{\cdot,v+1}D_\perp\tilde{\Pi}_{\cdot,v}. \tag{18}$$

Moreover

$$\Pi D_\perp\tilde{\Pi}_{\cdot,v} = [\Pi, D_\perp]\tilde{\Pi}_{\cdot,v}$$

which is bounded on $L^2\Omega$ by Remark 5. Therefore the result follows once we have proved that the last two terms of (18) define bounded operators on $L^2\Omega$. In fact, by taking adjoints, it is enough to prove that one of them defines a bounded operator for arbitrary v.

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Let $\phi = \phi_1 + \phi_2$ as above. Then obviously $\tilde{\Pi}_{\cdot,v-1} D_\perp \phi_1 = 0$. On the other hand, ϕ_2 is the C^∞ limit of $\delta_{0,-1}\psi_i$ for some sequence ψ_i in $\Omega^{\cdot,v}$. So

$$D_\perp \phi_2 = \lim_i (-\delta_{0,-1} D_\perp + K\delta_{0,-1} + \delta_{0,-1} K)\psi_i$$

by Proposition 3.1, yielding

$$\tilde{\Pi}_{\cdot,v-1} D_\perp \phi = \tilde{\Pi}_{\cdot,v-1} K\phi$$

because $\tilde{\Pi}_{\cdot,v-1}\delta_{0,-1}\Omega^{\cdot,v} = 0$ and $\tilde{\Pi}_{\cdot,v-1}$ is continuous on Ω. Thus $\tilde{\Pi}_{\cdot,v-1} D_\perp \tilde{\Pi}_{\cdot,v}$ is bounded on $L^2\Omega$. □

Define the norms $\|\cdot\|_{D_1,k}$ on $\mathcal{H}_1$ and $\|\cdot\|_{\tilde{D}_1,k}$ on $\tilde{\mathcal{H}}_1$ by setting

$$\|\phi\|_{D_1,k} = \|(\mathrm{id} + D_1)^k \phi\|_0, \quad \|\psi\|_{\tilde{D}_1,k} = \|(\mathrm{id} + \tilde{D}_1)^k \psi\|_0,$$

and let $W^k\mathcal{H}_1$ and $W^k\tilde{\mathcal{H}}_1$ be the corresponding completions of $\mathcal{H}$ and $\tilde{\mathcal{H}}_1$. Then the following result follows directly from Lemma 7.2.

COROLLARY 7.3. *The restrictions of the* kth *Sobolev norm* $\|\cdot\|_k$ *to* $\mathcal{H}_1$ *and* $\tilde{\mathcal{H}}_1$ *are respectively equivalent to the norms* $\|\cdot\|_{D_1,k}$ *and* $\|\cdot\|_{\tilde{D}_1,k}$. *Thus* $W^k\mathcal{H}_1$ *and* $W^k\tilde{\mathcal{H}}_1$ *are the closures of* $\mathcal{H}_1$ *and* $\tilde{\mathcal{H}}_1$ *in* $W^k\Omega$, *respectively*.

The inclusions $W^{k+1}\mathcal{H}_1 \hookrightarrow W^k\mathcal{H}_1$ and $W^{k+1}\tilde{\mathcal{H}}_1 \hookrightarrow W^k\tilde{\mathcal{H}}_1$ are compact operators by Corollary 7.3. Then, by Proposition 2.44 in [6] and Lemma 7.1, the Hilbert spaces $L^2\mathcal{H}_1$ and $L^2\tilde{\mathcal{H}}_1$ have complete orthonormal systems, $\{\phi_i : i = 1, 2, \ldots\} \subset \mathcal{H}_1$ and $\{\tilde{\phi}_i : i = 1, 2, \ldots\} \subset \tilde{\mathcal{H}}_1$, consisting of eigenvectors of Δ_1 and $\tilde{\Delta}_1$, respectively, so that the corresponding eigenvalues satisfy $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots$, $0 \leqslant \tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \ldots$, with $\lambda_i \uparrow \infty$ if $\dim \mathcal{H}_1 = \infty$ and $\tilde{\lambda}_i \uparrow \infty$ if $\dim \tilde{\mathcal{H}}_1 = \infty$. Thus it only remains to check that $\tilde{\lambda}_1 > 0$ to complete the proof of Theorem F; i.e. to check that $\ker \tilde{\Delta}_1 = 0$

For each $v \in \mathbb{Z}$ let

$$\mathcal{Z}_v = \bigoplus_{w<v} \Omega^{\cdot,w} \oplus \ker(d_{0,1} : \Omega^{\cdot,v} \to \Omega^{\cdot,v+1}),$$
$$\mathcal{B}_v = \bigoplus_{w<v} \Omega^{\cdot,w} \oplus d_{0,1}(\Omega^{\cdot,v-1}).$$

The bigrading of Ω is used to define these spaces only for the sake of simplicity, indeed they depend only on $\mathcal{F}$ [1, 30]. The de Rham derivative preserves the above spaces, and the quotient topological complex $\overline{\mathcal{B}_v}/\mathcal{B}_v$ is canonically isomorphic to $(\bar{0}_1, d_1)$. Moreover we have the following known result whose proof is easy and does not require that $\mathcal{F}$ be Riemannian and M closed.

LEMMA 7.4 (V. Sergiescu [30]; see also [1]). *The cohomology of the quotient complex* $\mathcal{B}_v/\mathcal{Z}_{v-1}$ *is trivial*.

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On the other hand, the decomposition (2) in Theorem B implies that the inclusion $\tilde{\mathcal{H}}_1^{\cdot,v} \hookrightarrow \overline{\mathcal{B}_v}$ induces an isomorphism $\tilde{\mathcal{H}}_1^{\cdot,v} \cong \overline{\mathcal{B}_v}/\mathcal{Z}_{v-1}$ of topological vector spaces whose inverse is induced by $\tilde{\Pi}_{\cdot,v}$. Moreover $\tilde{d}_1$ corresponds to differential map of the quotient complex $\overline{\mathcal{B}_v}/\mathcal{Z}_{v-1}$. Hence $\tilde{d}_1^2 = 0$, and

$$\ker \tilde{\Delta}_1 \cong H\left(\tilde{\mathcal{H}}_1, \tilde{d}_1\right) \cong H\left(\overline{\mathcal{B}_v}/\mathcal{Z}_{v-1}\right)$$

as topological vector spaces. Furthermore

$$H\left(\overline{\mathcal{B}_v}/\mathcal{Z}_{v-1}\right) \cong H\left(\overline{\mathcal{B}_v}/\mathcal{B}_v\right) \cong H\left(\bar{0}_1^{\cdot,v}, d_1\right) = 0$$

as vector spaces by Lemma 7.4, and

$$H(\overline{\mathcal{B}_v}/\mathcal{B}_v) \cong H\left(\bar{0}_1^{\cdot,v}, d_1\right) = 0$$

by Theorem G. This completes the proof of Theorem F.

Remark 6. Observe that the duality stated in Theorem G can be realized as follows. When M is oriented, the corresponding Hodge star operator commutes with Π and Δ_1, and thus induces duality in $\ker \Delta_1$.

Remark 7. Let $E(\lambda)$ and $\tilde{E}(\tilde{\lambda})$ be the eigenspaces corresponding to eigenvalues λ of Δ_1 and $\tilde{\lambda}$ of $\tilde{\Delta}_1$. This eigenspaces have bigradings induced by the bigradings of $\mathcal{H}_1$ and $\tilde{\mathcal{H}}_1$. Then, if M is oriented, the Hodge star operator also induces duality $E(\lambda)^{u,v} \cong E(\lambda)^{q-u,p-v}$, and skew duality $\tilde{E}(\tilde{\lambda})^{u,v} \cong \tilde{E}(\tilde{\lambda})^{q-u-1,p-v+1}$.

Remark 8. If we use appropriate zero order modifications of $D_\perp$, then we get the Hodge theoretic version of the results of [12], which have important implications about tenseness.

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