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Fixed point sets and the fundamental group I: semi-free actions on G-CW-complexes

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Smith theory says that the fixed point set of a semi-free action of a group G on a contractible space is $\mathbb{Z}_p$-acyclic for any prime factor p of the order of G. Jones proved the converse of Smith theory for the case G is a cyclic group acting semi-freely on contractible, finite CW-complexes. We extend the theory to semi-free group actions on finite CW-complexes of given homotopy types, in various settings. In particular, the converse of Smith theory holds if and only if a certain K-theoretical obstruction vanishes. We also give some examples that show the geometrical effects of different types of K-theoretical obstructions.

Keywords: group actions; algebraic K-theory; Smith theory

1. Introduction

The homological theory of group actions began with the results of P.A. Smith [22] that, if G is a p-group acting on a contractible space, then the fixed set is $\mathbb{Z}_p$-acyclic. While originally the connection between the order of the group and the nature of acyclicity seemed like an artefact of the proof, it was soon realised that this was not the case.

A definitive refutation of this was the result of L. Jones that any $\mathbb{Z}_n$-acyclic, finite CW-complex is the fixed set of semi-free $\mathbb{Z}_n$-action on a finite, contractible CW-complex [12]. Here we recall that a group action is semi-free if all isotropy subgroups are either trivial or the whole group. If one removes the semi-free condition, then R. Oliver's work [16] shows that, when n is not a prime power, the necessary and sufficient condition for a finite CW-complex F to be a fixed set is that the Euler characteristic $\chi(F) = 1$. Incidentally, this is not necessary for general topological actions, but it is for the so called ANR-actions. By an ANR action, we mean a G-action on a space X, which makes X into a finite dimensional G-ANR (equivariant absolute neighbourhood retract)[1].

This paper and a companion one [**8**] study the extensions of the work of Jones and of Oliver, respectively, to non-simply connected spaces. The simply connected theory was interestingly explored by Assadi [**1**] and Oliver-Petrie [**17**], and is largely understood. Both theories depend on a kind of 'equivariant surgery' and involve K_0. Assadi-Vogel [**2**] developed a non-simply connected, semi-free theory for actions on manifolds (therefore only for certain restricted families of groups). Our work extends theirs, in the situation of finite CW-complexes which allows for many more possible finite groups.

The main results of our companion paper [**8**] show that the analogue of Oliver's theorem does not become substantially more subtle in the presence of the fundamental group. The following is a special case:

Theorem. *Suppose X and F are finite CW-complexes, and $G = \mathbb{Z}_n$ is a cyclic group with n not prime power. Then there is a finite G-CW-complex Y with $Y^G = F$ and a G-map $Y \to X$ (giving X the trivial G-action) which is a homotopy equivalence, if and only if $\chi(F) = \chi(X)$.*

Here and below we see, by generalising Oliver's work, including to non-simply connected settings, that the complete analysis of fixed point sets is governed by the Euler characteristics of combinations of the components of putative fixed sets.

The necessity in the theorem is a consequence of the Lefschetz fixed point theorem, and therefore also holds for the G-ANR case. The proof of sufficiency in [**8**] builds on Oliver's work by a series of purely geometric constructions; for our purposes, we remark that the fundamental group of X does not enter.

In this paper, we will see that, even for $G = \mathbb{Z}_p$ there is a rich set of phenomena visible in trying to understand the homotopy types of fixed sets, in contrast to the situation for non-p-groups. In contrast to the generalisation of Oliver's theorem, an analysis of semi-free actions shows a number of interesting phenomena. We will mention some examples before describing our main theorems: theorems 1 and 2.

EXAMPLE 1.1. Let $T(r)$ be the mapping torus of a degree r map from a sphere S^d to itself. Notice that the map $T(r) \to S^1$ is a $\mathbb{Z}_n[\mathbb{Z}]$-homology equivalence if and only if n divides a power of r (i.e., all the primes in n occur in r). The infinite cyclic cover has nontrivial $\mathbb{Q}$-homology, but is also $\mathbb{Z}_n$-acyclic under this divisibility condition.

We will see that there is a semi-free $\mathbb{Z}_n$-action on a finite CW-complex homotopy equivalent to S^1 with fixed set $T(r)$ if and only if n divides r. When n is not square-free, this condition goes beyond Smith theory. It is related to $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}_n])$, and the role of the non-square-free condition is well known to be the condition for the

[1]Recall that being a compact and finite dimensional ANR space is equivalent to the corresponding local condition, and the same is true for G-ANR spaces. Moreover, for G trivial, according to West's celebrated theorem [**30**], any finite dimensional compact ANR is homotopy equivalent to a finite CW-complex. When G is nontrivial, Quinn's examples in [**19**] show this is not true. Moreover, elementary examples show that there is no analogue of Oliver's theorem for general topological actions.

Nil factor to be nontrivial in the Bass–Heller–Swan formula of algebraic K-theory (see Bass–Murthy [**6**]). Concretely, $T(p)$ is fixed under a $\mathbb{Z}_p$-action, but not a semi-free $\mathbb{Z}_{p^2}$-action on some homotopy circle. Its two fold cover $T(p^2)$ is fixed under a semi-free $\mathbb{Z}_{p^2}$-action, but not a semi-free $\mathbb{Z}_{p^3}$-action, etc.

If one studies topological actions on manifolds that are locally smooth, one does not necessarily obtain a finite G-CW-complex [**19**, **20**, **23**, **29**]. The non-uniqueness of such structures, even when they exist, is implicated in the phenomenon of non-linear similarity of linear actions on the sphere [**7**]. In the above examples one can obtain a locally smooth action (or equivalently a G-ANR action) with $T(r)$ as fixed set if and only if one can construct such a G-action on finite CW-complex. The following example shows a difference between these categories.

Example 1.2. Let $T(r_1, r_2)$ be the double mapping torus, obtained by glueing two ends of $S^d \times [0, 1]$ to a copy of S^d by maps of degrees r_1, r_2. Then $T(2, 3)$ is $\mathbb{Z}_6[\mathbb{Z}]$-acyclic. It is the fixed set of a semi-free $\mathbb{Z}_6$-ANR action. On the other hand, it is not the fixed set (up to homotopy) of any finite G-CW-complex homotopy equivalent to the circle S^1.

In this case, the obstructions are nontrivial elements of $K_{-1}(\mathbb{Z}[\mathbb{Z}_6])$ that enter via the Bass–Heller–Swan formula into the obstruction group $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}_6])$; there are similar examples arising for all groups of not prime power order.

One of the reasons to focus so strongly on the case of the circle is the special role that it plays in the Farrell–Jones conjecture in algebraic K-theory [**11**]. The circle is central to this problem because, as we shall soon explain more systematically, the examples on the circle can be promoted to examples on any finite CW-complex whose fundamental group is a torsion free hyperbolic group, or a lattice.

Assuming the Farrell–Jones conjecture, if the fundamental group is torsion free, there are no examples of fixed sets obstructed for $\mathbb{Z}_p$-actions when the Smith condition holds. To give an example where there is an obstruction, we turn to finite fundamental group.

Example 1.3. Let $f\colon L(kp; 1) \to L(p; 1)$ be a degree r map of three dimensional lens spaces, with k, p coprime. There is a $\mathbb{Z}_p$-action on a space of the homotopy type of $L(p; 1)$, such that the inclusion map from the fixed set is homotopic to f, if and only if $r^{p-1} = k^{p-1} \bmod p^2$. The details are in proposition 4.6.

We now state the results from which the above examples follow.

Let G be a group. A G-map between finite G-CW-complexes (or compact G-ANRs) is a pseudo-equivalence if it is a homotopy equivalence after ignoring the group action. Given a G-map $f\colon F \to Y$, we ask whether it is possible to extend F to a bigger finite G-CW-complex (or compact G-ANR) X, and extend f to a pseudo-equivalent G-map $g\colon X \to Y$. We call g a *pseudo-equivalence extension* of f.

In this paper, we concentrate on the following setting. The group G is finite, and all spaces are finite, semi-free G-CW-complexes. Moreover, we only consider $F = X^G$ in the pseudo-equivalence extension. In other words, the extension from F to X is obtained by attaching free G-cells.

The concept of pseudo-equivalence was introduced by Oliver and Petrie [**17**, **18**]. A pseudo-equivalence becomes a homotopy equivalence upon applying the Borel construction. However, a characterisation in terms of Borel equivalence would be inadequate for our present purpose because we require our G-spaces to be finite G-CW-complexes.

For the special case Y is a point, the question of the existence of a pseudo-equivalence extension becomes whether a given space F can be the fixed point set of a semi-free G-action on a contractible, finite CW-complex X. The classical results of Smith [**22**] and of Jones [**12**] give necessary and sufficient condition for semi-free actions by cyclic groups.

THEOREM 1.4 Smith and Jones. *A finite CW-complex F is the fixed set of a finite, contractible, semi-free $\mathbb{Z}_n$-CW-complex if and only if $\widetilde{H}_*(F;\mathbb{Z}_n) = 0$.*

For a general semi-free action of G on contractible X, and any prime factor p of $|G|$, the fixed set $F = X^G$ is the same as the fixed set X^C of a cyclic subgroup C of order p. Then the homological condition in the theorem becomes $\widetilde{H}_*(F;\mathbb{F}_p) = 0$ for all prime factors p of G ($\mathbb{F}_p = \mathbb{Z}_p$ is a field for prime p). This is equivalent to $\widetilde{H}_*(F;\mathbb{Z}_{|G|}) = 0$. We call this the *Smith condition*.

In general, we let Y be a connected G-CW-complex. Let $\widetilde{Y}$ be the universal cover of Y, with action by the fundamental group $\pi = \pi_1(Y)$. Then all actions on $\widetilde{Y}$ covering G-actions on Y form a group Γ that fits into an exact sequence

$$1 \to \pi \to \Gamma \to G \to 1.$$

The definition of Γ here is not always exactly correct, because it ignores the effectiveness of the G-action. See § 2 for the precise definition. In particular, we have $\Gamma = \pi \times G$ if G acts trivially on Y.

Suppose a G-map $g\colon X \to Y$ is a pseudo-equivalence between semi-free G-CW-complexes. Then the mapping cone of g is a contractible semi-free G-CW-complex, and the Smith condition can be applied to the mapping cone to give isomorphisms $H_*(X^G;\mathbb{F}_p) \cong H_*(Y^G;\mathbb{F}_p)$ for all prime factors p of $|G|$. In fact, in § 2, we apply the Smith condition to the universal cover and get isomorphisms $H_*(X^G;\mathbb{F}_p\pi) \cong H_*(Y^G;\mathbb{F}_p\pi)$. This is the necessary Smith condition for constructing pseudo-equivalence extension.

However, it turns out that there is additionally an algebraic K-theoretic obstruction. The following is our first main result, for the case the G-action on Y is trivial.

THEOREM 1.4. *Suppose $f\colon F \to Y$ is a map of finite CW-complexes, with Y connected and $\pi = \pi_1(Y)$. Then F can be the fixed set of a finite, semi-free G-CW-complex X, and f has pseudo-equivalence extension $g\colon X \to Y$, if and only if the following are satisfied:*

1. *The map f induces isomorphisms $H_*(F;\mathbb{F}_p\pi) \cong H_*(Y;\mathbb{F}_p\pi)$ for all prime factors p of $|G|$.*

2. *An obstruction $[C_*(\widetilde{f})] \in \widetilde{K}_0(\mathbb{Z}[\pi \times G])$ vanishes.*

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The first is what we have called the Smith condition. In the second condition, the chain complex $C_*(\widetilde{f})$ of the π-cover $\widetilde{f}\colon \widetilde{F} \to \widetilde{Y}$ of f is a $\mathbb{Z}\pi$-chain complex. Then we regard $C_*(\widetilde{f})$ as a $\mathbb{Z}[\pi \times G]$-chain complex with trivial G-action. We will argue that the Smith condition implies that $C_*(\widetilde{f})$ is chain homotopy equivalent to a finite chain complex of finitely generated projective $\mathbb{Z}[\pi \times G]$-modules, and therefore gives a well-defined element $[C_*(\widetilde{f})] \in \widetilde{K}_0(\mathbb{Z}[\pi \times G])$. Moreover, since the terms in $C_*(\widetilde{f})$ are finitely generated free $\mathbb{Z}[\pi]$-modules, the obstruction lies in the kernel of the homomorphism that forgets the G-action:

$$[C_*(\widetilde{f})] \in \mathrm{Ker}(\ \widetilde{K}_0(\mathbb{Z}[\pi \times G]) \to \widetilde{K}_0(\mathbb{Z}[\pi])\ ).$$

THEOREM 1.5. *Suppose Y is a finite, semi-free, connected G-CW-complex, with $\pi = \pi_1(Y)$. Suppose F is a finite CW-complex and $f\colon F \to Y^G$ is a map. Then F can be the fixed set of a finite, semi-free G-CW-complex X, and f has pseudo-equivalence extension $g\colon X \to Y$, if and only if the following are satisfied*:

1. *The map f induces isomorphisms $H_*(F; \mathbb{F}_p\pi) \cong H_*(Y^G; \mathbb{F}_p\pi)$ for all prime factors p of $|G|$.*
2. *A well-defined obstruction $[C_*(\widetilde{f})] \in \widetilde{K}_0(\mathbb{Z}[\Gamma])$ vanishes.*

The meaning of the two conditions is explained in § 2 and 3. The Smith condition is equivalent to the condition being satisfied on each connected component of Y^G. Then we get a K-theory element on each connected component similar to the first main theorem, and the obstruction $[C_*(\widetilde{f})]$ is the sum of these. Moreover, similar to the remark for theorem 1.4, we know the obstruction lies in the kernel of the forgetful homomorphism $\widetilde{K}_0(\mathbb{Z}[\Gamma]) \to \widetilde{K}_0(\mathbb{Z}[\pi])$.

We also describe how $\widetilde{K}^{top}$ enters to modify the above results in the simple homotopy setting (theorem 3.1) and the G-ANR setting (theorem 3.2).

We remark that Oliver and Petrie [**17**] studied the extension problem in a generally different setting (see also Assadi [**1**], and Morimoto and Iizuka [**15**]). When restricted to our problem, they gave the obstruction such that the extension g induces isomorphism on the integral homology. Therefore they solved our problem for the case Y is simply connected. What is new in our theorem is the non-simply connected case for homotopy equivalences.

Another important paper in this direction was Assadi-Vogel [**2**], that works in a manifold setting. It is quite close to what we do, although their techniques are different (based on ideas of homology propagation rather than G-surgery), formally have less generality (since $\mathbb{Z}_p \times \mathbb{Z}_p$ cannot act semi-freely on a manifold, for example), and their calculations focus on finite fundamental groups. Our focus here is mainly on the phenomena that arise when fundamental groups are torsion free, as this paper is intended to provide foundations for later studies of group actions on aspherical manifolds.

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2. Smith condition

We explain the Smith condition in more detail.

Let G be a finite group. Let X and Y be semi-free G-CW-complexes. Let $g\colon X \to Y$ be a G-map and a (non-equivariant) homotopy equivalence, i.e., g is a pseudo-equivalent G-map. Let $f\colon F = X^G \to Y$ be the restriction of g to the fixed set. Let Y be connected, and let $p\colon \widetilde{Y} \to Y$ be the universal cover, equipped with the free action by the fundamental group $\pi = \pi_1(Y)$.

An element $u \in G$ gives an action $u\colon Y \to Y$. The action lifts to self homeomorphisms $\widetilde{u}\colon \widetilde{Y} \to \widetilde{Y}$ of the universal cover, in the sense that $p\widetilde{u} = up$. If we fix one lifting $\widetilde{u}$ of u, then the other liftings of u are $a\widetilde{u}$, for $a \in \pi$. Let Γ' be the collections of all such liftings. Then Γ' is a group fitting into an exact sequence

$$1 \to \pi \to \Gamma' \to G/G_0 \to 1.$$

We remark that G_0 consists of those $u \in G$ that act trivially on Y, because the liftings $\widetilde{u}$ can only distinguish $u \in G$ through their actions on Y. To further distinguish distinct elements of G that may act the same way on Y, we introduce the group Γ as the pullback of $\Gamma' \to G/G_0 \leftarrow G$:

$$\Gamma = \{(\widetilde{u}, u)\colon \widetilde{u} \in \Gamma' \text{ covers } u \in G\}.$$

Then we get an exact sequence

$$1 \to \pi \to \Gamma \to G \to 1. \tag{2.1}$$

As an extreme case, if G acts trivially on Y, then $\Gamma = \pi \times G$.

As an example, consider $G = \mathbb{Z}_2 = \langle u \rangle$ acting on the real projective space $\mathbb{R}P^2$ by $u([x_0, x_1, x_2]) = [x_0, x_1, -x_2] = [-x_0, -x_1, x_2]$. The universal cover of $\mathbb{R}P^2$ is the sphere S^2, with the covering group π generated by the antipode $a(x_1, x_2, x_3) = (-x_1, -x_2, -x_3)$. The action u lifts to $\widetilde{u}_1(x_0, x_1, x_2) = (x_0, x_1, -x_2)$ and $\widetilde{u}_2(x_0, x_1, x_2) = (-x_0, -x_1, x_2)$. The group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \widetilde{u}_1 \rangle \times \langle \widetilde{u}_2 \rangle$, and π is a subgroup of Γ by $a = \widetilde{u}_1 \widetilde{u}_2$.

We use $\widetilde{\ }$ to denote the lifting/pullback along the universal cover of Y. For example, we have the pullbacks (note that $\widetilde{X}$ is generally not the universal cover of X):

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{g}"] \arrow[d] & \widetilde{Y} \arrow[d, "p"] \\
X \arrow[r, "g"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{F} \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{Y^G} = p^{-1}(Y^G) \arrow[d] \\
F = X^G \arrow[r, "f"] & Y^G
\end{tikzcd}$$

For a connected component C of Y^G, we have the pullback

$$\begin{tikzcd} \widetilde{F}_C = \widetilde{f}^{-1}(\widetilde{C}) \arrow[r] \arrow[d] & \widetilde{C} = p^{-1}(C) \arrow[d] \\ F_C = f^{-1}(C) \arrow[r] & C \end{tikzcd}$$

In our example, we have $(\mathbb{R}P^2)^G = \{[x_0, x_1, 0]\} \sqcup \{[0, 0, 1]\} = \mathbb{R}P^1 \sqcup \{[0, 0, 1]\}$, and $(\widetilde{\mathbb{R}P^2})^G = \{(x_0, x_1, 0)\} \sqcup \{(0, 0, 1)\} \sqcup \{(0, 0, -1)\} = S^1 \sqcup N \sqcup S$ (N and S are the north and south poles).

Let $\widetilde{y} \in \widetilde{Y^G}$ and $y = p(\widetilde{y}) \in Y^G$. Let C and $\widehat{C}$ be the connected components of Y^G and $\widetilde{Y^G}$ containing y and $\widetilde{y}$. Then $\widehat{C}$ covers C. We may use $\widetilde{y}$ to get an isomorphism $\pi_1(Y, y) \cong \pi$. Then the deck transformations $\pi_{\widehat{C}} \subset \pi$ of the covering $\widehat{C} \to C$ is the image of the homomorphism $\pi_1(C, y) \to \pi_1(Y, y)$, and we get $\widetilde{C} = \pi \times_{\pi_{\widehat{C}}} \widehat{C}$.

In general, for $\widetilde{y} \in \widetilde{Y}$ and $y = p(\widetilde{y}) \in Y$, the induced homomorphism $\Gamma_{\widetilde{y}} \to G_y$ of isotropy groups is an isomorphism. In particular, for $y \in Y^G$, the isomorphism gives a splitting $G = G_y \cong \Gamma_{\widetilde{y}} \subset \Gamma$ of (2.1). The splitting depends only upon the connected component $\widehat{C}$ containing $\widetilde{y}$. Therefore we may denote $\Gamma_{\widehat{C}} = \Gamma_{\widetilde{y}}$, and get $\Gamma = \pi \rtimes \Gamma_{\widehat{C}}$.

The other connected components of $\widetilde{C}$ are $a\widehat{C}$, $a \in \pi$. Therefore the connected component C gives a π-conjugation class of isotropy groups (equivalently, a π-conjugation class of splittings of (2.1))

$$\Gamma_C = \{\Gamma_{a\widehat{C}} = a\Gamma_{\widehat{C}}a^{-1} \colon a \in \pi\}.$$

In our example, $(\mathbb{R}P^2)^G$ has two connected components $C_1 = \mathbb{R}P^1$ and $C_2 = [0, 0, 1]$. Their preimages in $(\widetilde{\mathbb{R}P^2})^G$ are respectively $\widetilde{C}_1 = S^1$ and $\widetilde{C}_2 = \{N, S\}$. We may take $\widehat{C}_1 = S^1$, $\widehat{C}_2 = N$, with $a\widehat{C}_1 = \widehat{C}_1$, $a\widehat{C}_2 = S$. Then $\Gamma_{S^1} = \langle \widetilde{u}_1 \rangle$, and $\Gamma_N = \Gamma_S = \langle \widetilde{u}_2 \rangle$. The semi-direct products $\Gamma = \pi \rtimes \Gamma_{S^1} = \langle a \rangle \times \langle \widetilde{u}_1 \rangle$ and $\Gamma = \pi \rtimes \Gamma_N = \pi \rtimes \Gamma_S = \langle a \rangle \times \langle \widetilde{u}_2 \rangle$ are the usual products. Moreover, we have the two-fold cover $\widehat{C}_1 = S^1 \to C_1 = \mathbb{R}P^1$ with the covering group $\pi_{\widehat{C}_1} = \pi$, and $\widetilde{C}_1 = \widehat{C}_1 = \pi \times_\pi \widehat{C}_1$. We also have the one-fold cover $\widehat{C}_2 \to C_2$ with the covering group $\pi_{\widehat{C}_2} = 1$, and $\widetilde{C}_2 = \widehat{C}_2 \sqcup a\widehat{C}_2 = \pi \times_1 \widehat{C}_2$.

Denote the homomorphism $G \cong \Gamma_{\widehat{C}} \subset \Gamma$ by $u \to \widetilde{u}$. Then the elements of $\Gamma = \pi \rtimes \Gamma_{\widehat{C}} \cong \pi \rtimes G$ are $a\widetilde{u}$, with $a \in \pi$ and $u \in G$. The multiplication in Γ is given by $a_1\widetilde{u}_1 a_2\widetilde{u}_2 = a_1 u_1(a_2)\widetilde{u_1 u_2}$. Here $u(a)$ is obtained by regarding $a \in \pi_1(Y, y)$ as a loop at y and applying the action of $u \in G$ to the loop. In particular, if $a \in \pi_1(C, y)$ lies in the deck transformation group $\pi_{\widehat{C}}$, then $u(a) = a$. This means $\Gamma_{\widehat{C}}$ acts trivially on $\pi_{\widehat{C}}$, and $\pi_{\widehat{C}} \times \Gamma_{\widehat{C}} \cong \pi_{\widehat{C}} \times G$ is a subgroup of Γ.

We have the $\mathbb{Z}[\Gamma]$-chain complexes

$$C_*(\widetilde{Y^G}) = \oplus_{C \in \pi_0 Y^G} C_*(\widetilde{C}), \quad C_*(\widetilde{C}) = C_*(\pi \times_{\pi_{\widehat{C}}} \widehat{C}) = \mathbb{Z}\pi \otimes_{\mathbb{Z}\pi_{\widehat{C}}} C_*(\widehat{C}).$$

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Since the isotropy group $\Gamma_{\widehat{C}}$ acts trivially on $\widehat{C}$, we may regard $C_*(\widehat{C})$ as a $\mathbb{Z}[\pi_{\widehat{C}} \times \Gamma_{\widehat{C}}]$-chain complex (with trivial action by $\Gamma_{\widehat{C}}$). By $\Gamma = \pi \rtimes \Gamma_{\widehat{C}}$, we get the following interpretation of the $\mathbb{Z}[\Gamma]$-chain complex $C_*(\widetilde{C})$

$$C_*(\widetilde{C}) = \mathbb{Z}[\pi \rtimes \Gamma_{\widehat{C}}] \otimes_{\mathbb{Z}[\pi_{\widehat{C}} \times \Gamma_{\widehat{C}}]} C_*(\widehat{C}) = \mathrm{Ind}_{\mathbb{Z}[\pi_{\widehat{C}} \times G]}^{\mathbb{Z}[\pi \rtimes G]} C_*(\widehat{C}).$$

Similarly, let $\widehat{F}_C \subset \widetilde{F}_C$ correspond to $\widehat{C} \subset \widetilde{C}$. Then we have

$$C_*(\widetilde{F}) = \oplus_{C \in \pi_0 Y^G} C_*(\widetilde{F}_C), \quad C_*(\widetilde{F}_C) = \mathrm{Ind}_{\mathbb{Z}[\pi_Z \times G]}^{\mathbb{Z}[\pi \rtimes G]} C_*(\widehat{F}_C).$$

The pseudo-equivalent G-map $g\colon X \to Y$ between semi-free G-CW-complexes lifts to a pseudo-equivalent Γ-map $\widetilde{g}\colon \widetilde{X} \to \widetilde{Y}$, and $\widetilde{g}$ has the 'fixed part" $\widetilde{f}\colon \widetilde{F} \to \widetilde{Y^G}$. Here the fixed part is not fixed by the whole Γ, but by various isotropy subgroups $\Gamma_{\widehat{C}} \subset \Gamma$. We regard the pseudo-equivalent Γ-map $\widetilde{g}$ as a pseudo-equivalent $\Gamma_{\widehat{C}}$-map. This implies that the mapping cone $C(\widetilde{g})$ is a contractible, semi-free $\Gamma_{\widehat{C}}$-space. The fixed set $C(\widetilde{g})^{\Gamma_{\widehat{C}}}$ is the mapping cone of the map $\widetilde{X}^{\Gamma_{\widehat{C}}} \to \widetilde{Y}^{\Gamma_{\widehat{C}}}$. By the classical Smith theory [**22**], $C(\widetilde{g})^{\Gamma_{\widehat{C}}}$ has trivial $\mathbb{F}_p$-homology for all prime factors p of $|\Gamma_{\widehat{C}}| = |G|$. This implies an isomorphism $H_*(\widetilde{X}^{\Gamma_{\widehat{C}}}; \mathbb{F}_p) \cong H_*(\widetilde{Y}^{\Gamma_{\widehat{C}}}; \mathbb{F}_p)$.

We note that $\Gamma_{\widehat{C}}$ may fix several connected components $\widehat{C}_0, \widehat{C}_1, \ldots, \widehat{C}_k$ of $\widetilde{Y^G}$, in addition to $\widehat{C}_0 = \widehat{C}$. Then the isomorphism $H_*(\widetilde{X}^{\Gamma_{\widehat{C}}}; \mathbb{F}_p) \cong H_*(\widetilde{Y}^{\Gamma_{\widehat{C}}}; \mathbb{F}_p)$ is a direct sum of isomorphisms $H_*(\widehat{F}_{C_i}; \mathbb{F}_p) \cong H_*(\widehat{C}_i; \mathbb{F}_p)$. For $i = 0$, this gives the *local Smith condition*

$$H_*(\widehat{F}_C; \mathbb{F}_p) \cong H_*(\widehat{C}; \mathbb{F}_p), \quad \text{for all prime factors } p \text{ of } |G|. \tag{2.2}$$

For our example, we have the Smith condition on $\widehat{C}_1 = S^1$ with the antipode action by $\pi = \langle a \rangle$, and the Smith condition on $\widehat{C}_2 = N$ with the trivial group action. The first condition is obtained by applying the usual Smith condition to the action of $\Gamma_{S^1} = \langle \widetilde{u}_1 \rangle$ on S^2. The second condition is obtained by applying the usual Smith condition to the action of $\Gamma_N = \Gamma_S = \langle \widetilde{u}_2 \rangle$ on S^2. Although the second condition consists of conditions on N and S, the two conditions are equivalent by the action of $\pi = \langle a \rangle$.

Finally, we combine the local Smith conditions into the global Smith condition in theorems 1.4 and 1.5. We pick one connected component $\widehat{C}$ of $\widetilde{C}$ for each connected component C of Y^G. Then

$$\widetilde{Y^G} = \sqcup_{C \in \pi_0 Y^G} \pi \times_{\pi_{\widehat{C}}} \widehat{C},$$

and

$$\begin{aligned} H_*(Y^G; \mathbb{F}_p\pi) &= H_*(\widetilde{Y^G}; \mathbb{F}_p) \\ &= \oplus_{C \in \pi_0 Y^G} H_*(\pi \times_{\pi_{\widehat{C}}} \widehat{C}; \mathbb{F}_p) \\ &= \oplus_{C \in \pi_0 Y^G} \mathbb{F}_p\pi \otimes_{\mathbb{F}_p\pi_{\widehat{C}}} H_*(\widehat{C}; \mathbb{F}_p), \end{aligned}$$

Similarly, we have

$$H_*(F; \mathbb{F}_p\pi) = H_*(\widetilde{F}; \mathbb{F}_p) = \oplus_{C \in \pi_0 Y^G} \mathbb{F}_p\pi \otimes_{\mathbb{F}_p\pi_{\widehat{C}}} H_*(\widehat{F}_C; \mathbb{F}_p).$$

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Then the direct sum of the local Smith condition (2.2) is the *global Smith condition*

$$H_*(F;\mathbb{F}_p\pi) \cong H_*(Y^G;\mathbb{F}_p\pi), \quad \text{for all prime factors } p \text{ of } |G|. \tag{2.3}$$

For our example, the global Smith condition is the direct sum of the Smith conditions on S^1, N, and S.

3. Proof of main theorems

The proof of the main theorems in the introduction involves an equivariant version of Wall's finiteness obstruction [**28**].

Proof of theorem 1.4. We assume that the first (Smith) condition is satisfied and try to construct a pseudo-equivalence extension g. In the process, we will encounter the obstruction in the second condition, and will see that it is well-defined.

By $H_0(F;\mathbb{F}_p\pi) \cong H_0(Y;\mathbb{F}_p\pi) = \mathbb{F}_p$, we know F is connected. We choose a base point in F and use its image in Y as the base point of Y. For any loop ϵ in Y at the base point, we may attach G copies of loops to F, and equivariantly map these loops to the loop ϵ. Since Y is a finite CW-complex, we may attach finitely many such loops to get a finite, semi-free G-CW-complex X^1 with fixed point set F, and get a G-map $f^1\colon X^1 \to Y$ that is surjective on π_1.

Since X^1 is a finite G-CW-complex, there are finitely many loops ϵ_i generating $\pi_1(X^1)$. Since f^1 is surjective on π_1, the images $f^1(\epsilon_i)$ generate π, which is finitely presented because Y is a finite CW-complex. Therefore π can be presented by $f^1(\epsilon_i)$ as generators, with finitely many words $f^1(w_j)$ of these loops as relations. For each such word $f^1(w_j)$, we glue G-copies of D^2 to X^1 along Gw_j and equivariantly map these 2-cells to Y. We thus get a finite, semi-free G-CW-complex $X^{1.5}$ with fixed point set F, and extend f^1 to a G-map $f^{1.5}\colon X^{1.5} \to Y$ that induces an isomorphism on π_1.

Since $f^{1.5}$ induces an isomorphism on π_1, by the Hurewicz theorem, we have $\pi_2(f^{1.5}) = H_2(f^{1.5};\mathbb{Z}\pi)$. This implies that $\pi_2(f^{1.5})$ is finitely generated as a $\mathbb{Z}\pi$-module. In fact, as G is a finite group, the G-action also makes $\pi_2(f^{1.5})$ into a finitely generated $\mathbb{Z}[\pi \times G]$-module. We represent a finite set of $\mathbb{Z}[\pi \times G]$-generators by maps $S^1 \to X^{1.5}$ and $D^2 \to Y$ compatible with f^2. Then we glue G-copies of D^2 along $G(S^1 \to X^{1.5})$ to $X^{1.5}$ and equivariantly map these 2-cells to Y by $G(D^2 \to Y)$ (for the current case that the G-action on Y is trivial, this is $D^2 \to Y$). We get a finite, semi-free G-CW-complex X^2 with fixed point set F, and extend $f^{1.5}$ to a G-map $f^2\colon X^2 \to Y$, such that f^2 satisfies $\pi_1(f^2) = \pi_2(f^2) = 0$.

The construction from $f^{1.5}$ to f^2 can be inductively extended to higher dimensions. If we have a G-map $f^i\colon X^i \to Y$ satisfying $\pi_j(f^i) = 0$ for $j \leqslant i$, then we can use a finite set of $\mathbb{Z}[\pi \times G]$-generators of $\pi_{i+1}(f^i) = H_{i+1}(f^i;\mathbb{Z}\pi)$ to equivariantly attach $(i+1)$-dimensional free G-cells to X^i, and extend f^i to a G-map $f^{i+1}\colon X^{i+1} \to Y$ satisfying $\pi_j(f^{i+1}) = 0$ for $j \leqslant i+1$. Inductively, we get $f^n\colon X^n \to Y$ for some $n > \max\{\dim F, \dim Y\}$, such that $\pi_j(f^n) = 0$ for $j \leqslant n$.

Let us consider the effect of one more construction to get $f^{n+1}\colon X^{n+1} \to Y$. The generators used for the construction can be interpreted as a basis of a finitely generated free $\mathbb{Z}[\pi \times G]$-module A in a surjective $\mathbb{Z}[\pi \times G]$-homomorphism $A \to$

$H_{n+1}(f^n;\mathbb{Z}\pi)$. By $n+1>\max\{\dim X^n, \dim Y\}$, we have $H_{n+2}(f^n;\mathbb{Z}\pi)=0$ and an exact sequence

$$\begin{aligned}H_{n+2}(f^n;\mathbb{Z}\pi)=0 &\to H_{n+2}(f^{n+1};\mathbb{Z}\pi)\to H_{n+1}(X^{n+1},X^n;\mathbb{Z}\pi)=A\\ &\to H_{n+1}(f^n;\mathbb{Z}\pi)\to H_{n+1}(f^{n+1};\mathbb{Z}\pi)\to H_n(X^{n+1},X^n;\mathbb{Z}\pi)=0\to\cdots\end{aligned}$$

By the Hurewicz theorem, the exact sequence gives $\pi_j(f^{n+1})=H_j(f^{n+1};\mathbb{Z}\pi)=0$ for $j\leqslant n+1$, and a short exact sequence

$$0\to\pi_{n+2}(f^{n+1})=H_{n+2}(f^{n+1};\mathbb{Z}\pi)\to A\to H_{n+1}(f^n;\mathbb{Z}\pi)\to 0.$$

Note that $\pi_{n+2}(f^{n+1})$ is to be used for the further construction based on f^{n+1}. Therefore, the short exact sequence shows that, if $H_{n+1}(f^n;\mathbb{Z}\pi)$ has a finite resolution of finitely generated free $\mathbb{Z}[\pi\times G]$-modules

$$0\to A_k\to\cdots\to A_2\to A_1\to H_{n+1}(f^n;\mathbb{Z}\pi)\to 0,$$

then the resolution can be used as a recipe for constructing a G-map $f^{n+k}\colon X^{n+k}\to Y$, such that X^{n+k} is a finite, semi-free G-CW-complex with fixed point set F, and f^{n+k} extends f and is a (non-equivariant) homotopy equivalence. This f^{n+k} is the pseudo-equivalence extension in the theorem.

Next we argue that the Smith condition implies that the $\mathbb{Z}[\pi\times G]$-module $H_{n+1}(f^n;\mathbb{Z}\pi)$ has a finite resolution by finitely generated projective $\mathbb{Z}[\pi\times G]$-modules. This induces (by § 3 of [**14**], for example) an element $[H_{n+1}(f^n;\mathbb{Z}\pi)]\in\widetilde{K}_0(\mathbb{Z}[\pi\times G])$, such that the element vanishes if and only if all the projective modules in the resolution can be chosen to be free. Therefore the element is the obstruction for completing our construction.

We identify this obstruction with the K-theory element represented by the $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{f})$, regarded as a $\mathbb{Z}[\pi\times G]$-chain complex with the trivial G-action. This is crucial for detailed calculations. This fact is essentially Lemma 1.7 of [**2**]; but we give a detailed explanation anyway.

There is a (possibly infinite) free $\mathbb{Z}G$-resolution P of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, such that P is finitely generated in each dimension. Then we have

$$C_*(\widetilde{f})=C_*(\widetilde{f})\otimes\mathbb{Z}\simeq C_*(\widetilde{f})\otimes P,$$

where each term in the $\mathbb{Z}[\pi\times G]$-chain complex $C_*(\widetilde{f})\otimes P$ is finitely generated and free.

There is also a (infinitely generated) projective $\mathbb{Z}[1/|G|][G]$-resolution P' of the trivial $\mathbb{Z}[1/|G|][G]$-module $\mathbb{Z}[1/|G|]$, such that P' is nonzero only in dimensions 0 and 1. This implies the second chain homotopy equivalence below

$$C_*(\widetilde{f})\simeq C_*(\widetilde{f})\otimes\mathbb{Z}[\tfrac{1}{|G|}]\simeq C_*(\widetilde{f})\otimes P'.$$

The first chain homotopy equivalence is due to the Smith condition, $H_*(C_*(\widetilde{f});\mathbb{F}_p)=H_*(f;\mathbb{F}_p\pi)=0$, for all prime factors p of $|G|$. Since $C_*(\widetilde{f})$ vanishes above dimension n, and P' vanishes away from dimensions 0 and 1, the chain complex $C_*(\widetilde{f})\otimes P'$ has cohomological dimension $\leqslant n+1$.

By Theorem 3.5 of [14], and the special properties of the two $\mathbb{Z}[\pi \times G]$-chain complexes that are chain homotopy equivalent to $C_*(\widetilde{f})$, we know $C_*(\widetilde{f})$ is chain homotopy equivalent to a finite chain complex of finitely generated projective $\mathbb{Z}[\pi \times G]$-modules. This gives a well-defined element $[C_*(\widetilde{f})] \in \widetilde{K}_0(\mathbb{Z}[\pi \times G])$.

The chain complex $C_*(\widetilde{f})$ fits into an exact sequence of $\mathbb{Z}[\pi \times G]$-chain complexes

$$0 \to C_*(\widetilde{f}) \to C_*(\widetilde{f^n}) \to C_{*-1}(\widetilde{X}^n, \widetilde{F}) \to 0.$$

Since $C_{*-1}(\widetilde{X}^n, \widetilde{F})$ is a finite chain complex of finitely generated free $\mathbb{Z}[\pi \times G]$-modules, $C_*(\widetilde{f^n})$ is also chain homotopy equivalent to a finite chain complex of finitely generated projective $\mathbb{Z}[\pi \times G]$-modules, and $[C_*(\widetilde{f^n})] = [C_*(\widetilde{f})] \in \widetilde{K}_0(\pi \times G)$. On the other hand, we know the homology of $C_*(\widetilde{f^n})$ vanishes in all dimensions except for $H_{n+1}(f^n; \mathbb{Z}\pi)$. Therefore the $\mathbb{Z}[\pi \times G]$-chain complex $\cdots \to 0 \to H_{n+1}(f^n; \mathbb{Z}\pi) \to 0 \to \cdots$ is chain homotopy equivalent to $C_*(\widetilde{f^n})$. This implies that $H_{n+1}(f^n; \mathbb{Z}\pi)$ has a finite resolution by finitely generated projective $\mathbb{Z}[\pi \times G]$-modules, and

$$(-1)^{n+1}[H_{n+1}(f^n; \mathbb{Z}\pi)] = [C_*(\widetilde{f^n})] = [C_*(\widetilde{f})] \in \widetilde{K}_0(\mathbb{Z}[\pi \times G]).$$

The equality to $[C_*(\widetilde{f})]$ shows that the obstruction $(-1)^{n+1}[H_{n+1}(f^n; \mathbb{Z}\pi)]$ is independent of our choice of construction. $\square$

Proof of theorem 1.5. The proof is similar to theorem 1.4. The inductive construction of $f^n \colon X^n \to Y$ is the same, except that the new cells can be mapped to Y instead of just the fixed set, and all the homotopy groups and homology groups (at the universal cover level) are $\mathbb{Z}[\Gamma]$-modules. For sufficiently large n, the obstruction for constructing the pseudo-equivalence is $[H_{n+1}(f^n; \mathbb{Z}\pi)] \in \widetilde{K}_0(\mathbb{Z}[\Gamma])$. We need to argue that $C_*(\widetilde{f})$ represents an element in $\widetilde{K}_0(\mathbb{Z}[\Gamma])$ that is the same as $(-1)^{n+1}[H_{n+1}(f^n; \mathbb{Z}\pi)]$.

In § 2, we saw that the global Smith condition (2.3) in theorem 1.5 is equivalent to the local Smith condition (2.2) for each connected component C of Y^G. Following the same argument for theorem 1.4, we know the $\mathbb{Z}[\pi_{\widehat{C}} \times \Gamma_{\widehat{C}}]$-chain complex $C_*(\widehat{F}_C \to \widehat{C})$ is chain homotopy equivalent to a finite chain complex of finitely generated projective $\mathbb{Z}[\pi_{\widehat{C}} \times \Gamma_{\widehat{C}}]$-modules. This gives a well-defined K-theory element

$$[C_*(\widehat{F}_C \to \widehat{C})] \in \widetilde{K}_0(\mathbb{Z}[\pi_{\widehat{C}} \times \Gamma_{\widehat{C}}]).$$

By $\pi_{\widehat{C}} \times \Gamma_{\widehat{C}} \subset \pi \rtimes \Gamma_{\widehat{C}} = \Gamma$, this inducts to

$$\begin{aligned} [C_*(\widetilde{F}_C \to \widetilde{C})] &= [\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\pi_{\widehat{C}} \times \Gamma_{\widehat{C}}]} C_*(\widehat{F}_C \to \widehat{C})] \\ &= \mathrm{Ind}_{\mathbb{Z}[\pi_{\widehat{C}} \times \Gamma_{\widehat{C}}]}^{\mathbb{Z}[\Gamma]} [C_*(\widehat{F}_C \to \widehat{C})] \in \widetilde{K}_0(\mathbb{Z}[\Gamma]), \end{aligned}$$

which further adds up to

$$[C_*(\widetilde{f})] = \sum_{C \in \pi_0 Y^G} [C_*(\widetilde{F}_C \to \widetilde{C})] \in \widetilde{K}_0(\mathbb{Z}[\Gamma]).$$

Now we know the $\mathbb{Z}[\Gamma]$–chain complex $C_*(\widetilde{f})$ is chain homotopy equivalent to a finite chain complex of finitely generated projective $\mathbb{Z}[\Gamma]$-modules and gives a K-theory element. Then we have short exact sequences of $\mathbb{Z}[\Gamma]$-chain complexes

$$
\begin{aligned}
&0 \to C_*(\widetilde{f}\colon \widetilde{F} \to \widetilde{Y^G}) \to C_*(\widetilde{F} \to \widetilde{Y}) \to C_{*-1}(\widetilde{Y}, \widetilde{Y^G}) \to 0,\\
&0 \to C_*(\widetilde{F} \to \widetilde{Y}) \to C_*(\widetilde{f}^n\colon \widetilde{X}^n \to \widetilde{Y}) \to C_{*-1}(\widetilde{X}^n, \widetilde{F}) \to 0.
\end{aligned}
$$

Since Y is a semi-free G-CW-complex, and X^n is obtained by glueing free G-cells to F, both $C_{*-1}(\widetilde{Y}, \widetilde{Y^G})$ and $C_{*-1}(\widetilde{X}^n, \widetilde{F})$ are finite chain complexes of finitely generated free $\mathbb{Z}[\Gamma]$-modules. Therefore

$$
\begin{aligned}
[C_*(\widetilde{f})] &= [C_*(\widetilde{F} \to \widetilde{Y})]\\
&= [C_*(\widetilde{f}^n\colon \widetilde{X}^n \to \widetilde{Y})]\\
&= (-1)^{n+1}[H_{n+1}(f^n; \mathbb{Z}\pi)] \in \widetilde{K}_0(\mathbb{Z}[\Gamma]).
\end{aligned}
$$

This completes the identification of the K-theory obstruction. □

The main theorems can be modified to get the pseudo-equivalence extension to be a *simple* homotopy equivalence. The only change is the K-theory in which the obstruction lives. The following is the analogue of theorem 1.4 making use of an algebraic K-theory introduced in [2].

THEOREM 3.1. *Suppose $f\colon F \to Y$ is a map of finite CW-complexes, with Y connected and $\pi = \pi_1(Y)$. Then F can be the fixed set of a finite, semi-free G-CW-complex X, and f has simple pseudo-equivalence extension $g\colon X \to Y$, if and only if the following are satisfied:*

1. *The map f induces isomorphisms $H_*(F; \mathbb{F}_p\pi) \cong H_*(Y; \mathbb{F}_p\pi)$ for all prime factors p of $|G|$.*

2. *A well-defined obstruction $[C_*(\widetilde{f})] \in Wh_1^T(\pi \subset \pi \times G)$ vanishes.*

Assadi and Vogel [2] introduced the Grothendick group $Wh_1^T(\pi \subset \pi \times G)$ of the additive category of finitely generated $\mathbb{Z}\pi$-based projective $\mathbb{Z}[\pi \times G]$-modules, such that the $\mathbb{Z}[\pi \times G]$-module $\mathbb{Z}[\pi \times G]$ with the choice of G as $\mathbb{Z}\pi$-basis is trivial in Wh_1^T. They showed that there is an exact sequence ($\widetilde{K}_0(\pi)$ is denoted $Wh_0(\pi)$ in [2])

$$
Wh_1(\pi \times G) \xrightarrow{T} Wh_1(\pi) \xrightarrow{\beta} Wh_1^T(\pi \subset \pi \times G) \xrightarrow{\alpha} \widetilde{K}_0(\mathbb{Z}[\pi \times G]) \xrightarrow{T} \widetilde{K}_0(\mathbb{Z}\pi).
$$

The cells of F and Y give natural $\mathbb{Z}\pi$-bases for the modules in $C_*(\widetilde{f})$. By Lemmas 1.6 and 1.7 of [2], under the Smith condition, the chain complex $C_*(\widetilde{f})$ with the natural $\mathbb{Z}\pi$-bases gives a well-defined element $[C_*(\widetilde{f})] \in Wh_1^T(\pi \subset \pi \times G)$. The image of this element in $\widetilde{K}_0(\mathbb{Z}[\pi \times G])$ is the obstruction in the main theorem.

The proof of theorem 3.1 is the same as the proof of theorem 1.4, with additional tracking of the basis in the construction. The key point is that the free G-cells

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used in the construction give the chain complex $C_{*-1}(\widetilde{X}^n, \widetilde{F})$, where each term is the direct sum of finitely many copies of $\mathbb{Z}[\pi \times G]$ with the $\mathbb{Z}\pi$-basis G. Therefore $[C_{*-1}(\widetilde{X}^n, \widetilde{F})] = 0 \in Wh_1^T(\pi \subset \pi \times G)$.

In the context of theorem 1.5, we may also extend the K-theory to $Wh_1^T(\pi \subset \Gamma)$, by considering the additive category of finitely generated $\mathbb{Z}\pi$-based projective $\mathbb{Z}[\Gamma]$-modules. For the $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}[\Gamma]$, we find a subset $\Gamma_0 \subset \Gamma$, such that the multiplication map $\pi \times \Gamma_0 \to \Gamma$ is a one-to-one correspondence. Then Γ_0 is a $\mathbb{Z}\pi$-basis of $\mathbb{Z}[\Gamma]$, and this represents the trivial element in $Wh_1^T(\pi \subset \Gamma)$. Using this K-theory, we also have the simple homotopy version of theorem 1.5.

The main theorems can also be modified to treat compact ANR-spaces in place of finite CW-complexes. We only present the analogue of theorem 1.4.

Theorem 3.2. *Suppose $f\colon F \to Y$ is a map of compact ANR-spaces, with Y connected and $\pi = \pi_1(Y)$. Then F can be the fixed set of a semi-free compact G-ANR-space X, and f has pseudo-equivalence extension $g\colon X \to Y$, if and only if the following are satisfied:*

1. *The map f induces isomorphisms $H_*(F; \mathbb{F}_p\pi) \cong H_*(Y; \mathbb{F}_p\pi)$ for all prime factors p of $|G|$.*
2. *A well-defined obstruction $[C_*(\widetilde{f})] \in \widetilde{K}_0^{top}(\mathbb{Z}\pi \subset \mathbb{Z}[\pi \times G])$ vanishes.*

The topological K-theory $\widetilde{K}_0^{top}$ was introduced by M. Steinberger and J. West [**23**, **24**], and by F. Quinn [**19**, **20**] as the K-theoretical obstruction for the topological version of the finiteness theorems for G-ANR-spaces. It fits into an exact sequence

$$\begin{aligned} &H_1(\pi; \widetilde{\mathbf{K}}(\mathbb{Z}[G])) \to Wh(\pi \times G) \to Wh^{top}(\pi \subset \pi \times G) \\ &\quad \to H_0(\pi; \widetilde{\mathbf{K}}(\mathbb{Z}[G])) \to \widetilde{K}_0(\mathbb{Z}[\pi \times G]) \to \widetilde{K}_0^{top}(\mathbb{Z}\pi \subset \mathbb{Z}[\pi \times G]) \to \cdots \end{aligned} \tag{3.1}$$

By a theorem of West [**30**], compact ANRs are homotopy equivalent to finite CW-complexes. Therefore we have an obstruction $[C_*(\widetilde{f})] \in \widetilde{K}_0(\mathbb{Z}[\pi \times G])$ in our main pseudo-equivalence extension theorem. Quinn [**19**] showed that controlled finitely dominated CW-complexes over F with free G-actions have controlled Wall finiteness obstructions in $H_0(F; \widetilde{\mathbf{K}}(\mathbb{Z}[G]))$, and all elements of this group can be realised. By glueing on such an element, we can change the obstruction $[C_*(\widetilde{f})] \in \widetilde{K}_0(\mathbb{Z}[\pi \times G])$ by any element in the image of $H_0(F; \widetilde{\mathbf{K}}(\mathbb{Z}[G]))$. We will explain in the next paragraph that the natural map $H_0(F; \widetilde{\mathbf{K}}(\mathbb{Z}[G])) \to H_0(Y; \widetilde{\mathbf{K}}(\mathbb{Z}[G])) \to H_0(\pi; \widetilde{\mathbf{K}}(\mathbb{Z}[G]))$ is surjective. Therefore $H_0(F; \widetilde{\mathbf{K}}(\mathbb{Z}[G]))$ and $H_0(\pi; \widetilde{\mathbf{K}}(\mathbb{Z}[G]))$ have the same image in $\widetilde{K}_0(\mathbb{Z}[\pi \times G])$. Thus we conclude that the obstruction for G-ANR pseudo-equivalence extension problem actually lies in the image of $\widetilde{K}_0(\mathbb{Z}[\pi \times G])$ in $\widetilde{K}_0^{top}(\mathbb{Z}\pi \subset \mathbb{Z}[\pi \times G])$.

Carter's vanishing theorem [**9**] says that $K_{-i}(\mathbb{Z}[G]) = 0$ for all finite groups G and $i > 1$. Therefore the spectral sequence that computes $H_0(F; \widetilde{\mathbf{K}}(\mathbb{Z}[G]))$ consists of only $H_0(F; \widetilde{K}_0(\mathbb{Z}[G]))$ and $H_1(F; \widetilde{K}_{-1}(\mathbb{Z}[G]))$. The same is true for $H_0(Y; \widetilde{\mathbf{K}}(\mathbb{Z}[G]))$ and $H_0(\pi; \widetilde{\mathbf{K}}(\mathbb{Z}[G]))$. To show the surjection, therefore, we only need to show

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that $H_i(F;\widetilde{K}_{-i}(\mathbb{Z}[G])) \to H_i(Y;\widetilde{K}_{-i}(\mathbb{Z}[G])) \to H_i(\pi;\widetilde{K}_{-i}(\mathbb{Z}[G]))$ is surjective for $i = 0, 1$. The Smith condition implies that $\pi_i F \to \pi_i Y$ is surjective for $i = 0, 1$. This implies the surjections on $H_i(*;\widetilde{K}_{-i}(\mathbb{Z}[G]))$ for $i = 0, 1$.

4. Calculations and examples

Let $T(r)$ be the mapping torus of a map $S^d \to S^d$ of degree r. Let

$$f\colon F = T(r) \to Y = S^1.$$

be the projection map. For a finite group G of order n, we try to extend F to be the fixed set of a finite, semi-free G-CW-complex X, and extend f to a pseudo-equivalence $g\colon X \to S^1$.

We have $\pi_1(Y) = \langle t\rangle = \{t^i : i \in \mathbb{Z}\} \cong \mathbb{Z}$, and the only non-trivial $\mathbb{Z}\langle t\rangle$-homology[2] of f is

$$H_d(f;\mathbb{Z}\langle t\rangle) = \mathbb{Z}\langle t\rangle/(rt-1).$$

For a prime p, we have $H_d(f;\mathbb{Z}_p\langle t\rangle) = 0$ if and only if $p|r$. Therefore, the Smith condition is satisfied for G if and only if

$$p|n \implies p|r.$$

This is equivalent to n dividing some power of r. Under this assumption, the condition for the semi-free pseudo-equivalence extension is the vanishing of

$$[\mathbb{Z}\langle t\rangle/(rt-1)] \in \widetilde{K}_0(\mathbb{Z}[G]\langle t\rangle).$$

Proposition 4.1. *Suppose G is a finite group of order n. If r is a multiple of n, then $T(r) \to S^1$ has semi-free pseudo-equivalence extension. Moreover, if G is abelian, then the converse is also true.*

Example 1.1 in the introduction is a direct consequence of proposition 4.1. In fact, $T(p)$ is not only not the fixed set of a semi-free $\mathbb{Z}_{p^2}$-action on homotopy circle, it is also not the fixed set of a semi-free $\mathbb{Z}_p \times \mathbb{Z}_p$-action. The proof given below also shows that $T(p^2)$ is not fixed under a semi-free $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$-action or $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$-action, etc.

Proof. We have the Bass–Heller–Swan decompositions [5]

$$\widetilde{K}_0(\mathbb{Z}[G]\langle t\rangle) = \widetilde{K}_0(\mathbb{Z}[G]) \oplus K_{-1}(\mathbb{Z}[G]) \oplus NK_0(\mathbb{Z}[G]) \oplus NK_0(\mathbb{Z}[G]), \tag{4.1}$$

$$K_1(\mathbb{Z}_n\langle t\rangle) = K_1(\mathbb{Z}_n) \oplus K_0(\mathbb{Z}_n) \oplus NK_1(\mathbb{Z}_n) \oplus NK_1(\mathbb{Z}_n). \tag{4.2}$$

We also have the pullbacks of rings [13] ($\Sigma_G = \sum_{g\in G} g$)

[2]In the literature, $\mathbb{Z}\langle t\rangle$ is usually denoted $\mathbb{Z}[t, t^{-1}]$. We use $\mathbb{Z}\langle t\rangle$ to simplify notation.

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$$\begin{array}{ccc} \mathbb{Z}[G]\langle t\rangle & \longrightarrow & (\mathbb{Z}[G]/\Sigma_G)\langle t\rangle \\ \downarrow & & \downarrow \\ \mathbb{Z}\langle t\rangle & \longrightarrow & \mathbb{Z}_n\langle t\rangle \end{array} \qquad \begin{array}{ccc} \mathbb{Z}[G] & \longrightarrow & \mathbb{Z}[G]/\Sigma_G \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}_n \end{array}$$

that induce the Swan homomorphisms[3] that are compatible with the Bass–Heller–Swan decompositions

$$\begin{aligned} &\partial\colon K_1(\mathbb{Z}_n\langle t\rangle) \to \widetilde{K}_0(\mathbb{Z}[G]\langle t\rangle), \\ &\partial\colon K_1(\mathbb{Z}_n) \to \widetilde{K}_0(\mathbb{Z}[G]), \\ &\partial\colon K_0(\mathbb{Z}_n) \to K_{-1}(\mathbb{Z}[G]), \\ &\partial\colon NK_1(\mathbb{Z}_n) \to NK_0(\mathbb{Z}[G]). \end{aligned}$$

We note that our obstruction is an image of the Swan homomorphism

$$[\mathbb{Z}\langle t\rangle/(rt-1)] = \partial[rt-1], \quad [rt-1] \in K_1(\mathbb{Z}_n\langle t\rangle).$$

Since n divides r, we get $[rt-1] = [-1]$, which comes from $K_1(\mathbb{Z}\langle t\rangle)$. Therefore, $\partial[rt-1] = 0$, and the sufficient part of the proposition.

For the necessary part, we note that the Smith condition requires n dividing some power of r. Now we identify the obstruction in the Bass–Heller–Swan decomposition. By the proof of the decomposition in [**21**, Theorem 3.2.22], we write the automorphism of $R\langle t\rangle$ as an automorphism of R with a nilpotent correction

$$rt-1 = r(t-1) + (r-1) = (r-1)[1 + (r-1)^{-1}r(t-1)] \in \mathbb{Z}_n\langle t\rangle,$$

where $r-1$ is invertible and $(r-1)^{-1}r$ is nilpotent by the Smith condition. This shows that the element $[rt-1]$ becomes $([r-1], 0, 0, [(r-1)^{-1}r])$ in the decomposition, and our obstruction is $\partial[r-1] \in \widetilde{K}_0(\mathbb{Z}[G])$ and $\partial[(r-1)^{-1}r] \in NK_0(\mathbb{Z}[G])$. We will concentrate on the vanishing of $\partial[(r-1)^{-1}r]$, which is the image of $(r-1)^{-1}(rt-1) \in K_1(\mathbb{Z}_n\langle t\rangle)$ under the Swan homomorphism.

Now we assume G is abelian. Then we have the determinant maps from K_1 to the groups of invertible elements. The Swan homomorphism is part of an exact sequence compatible with the determinants

$$\begin{array}{ccccc} K_1(\mathbb{Z}\langle t\rangle) \oplus K_1((\mathbb{Z}[G]/\Sigma_G)\langle t\rangle) & \xrightarrow{\alpha} & K_1(\mathbb{Z}_n\langle t\rangle) & \xrightarrow{\partial} & \widetilde{K}_0(\mathbb{Z}[G]\langle t\rangle) \\ \big\downarrow \mathrm{det} & & \big\downarrow \mathrm{det} & & \\ \mathbb{Z}\langle t\rangle^* \oplus (\mathbb{Z}[G]/\Sigma_G)\langle t\rangle^* & \xrightarrow{\beta} & \mathbb{Z}_n\langle t\rangle^* & & \end{array}$$

The vanishing of $\partial[(r-1)^{-1}r]$ implies that $[(r-1)^{-1}(rt-1)]$ is in the image of α. This further implies that $\det[(r-1)^{-1}(rt-1)] = (r-1)^{-1}(rt-1)$ is in the image

[3] According to the seminal paper [**25**] of Swan where this construction first arose.

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of β. In the appendix of this paper, we prove that $(\mathbb{Z}[G]/\Sigma_G)\langle t\rangle^* = (\mathbb{Z}[G]/\Sigma_G)^*\langle t\rangle$. In other words, the units (i.e., multiplicatively invertible elements) in $(\mathbb{Z}[G]/\Sigma_G)\langle t\rangle$ are monomials. If n does not divide r, then $(r-1)^{-1}(rt-1)$ is not a monomial, and we get a contradiction. This proves the necessary part. $\square$

The necessary part of the proof shows that, if G is abelian and r is not a multiple of n (and n divides a power of r), then the K-theory obstruction $\partial[rt-1] \in \widetilde{K}_0(\mathbb{Z}[G]\langle t\rangle)$ for $T(r) \to S^1$ has nonzero component $\partial[(r-1)^{-1}r]$ in the Bass Nil group $NK_0(\mathbb{Z}[G])$. The Nil group is the K-theory of the additive category of finitely generated R-modules equipped with nilpotent endomorphisms. It is introduced in [**4**] to account for the difference between the K-theory of a ring R and its polynomial extension $R[t]$ or the Laurent extension $R\langle t\rangle$. In case $\pi = \mathbb{Z} = \langle t\rangle$ in the exact sequence (3.1), the map $H_0(\mathbb{Z}; \mathbf{K}(\mathbb{Z}[G])) \to \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}\times G])$ is exactly the comparison (i.e., the inclusion in (4.1)) between $\widetilde{K}_0(\mathbb{Z}[G]) \oplus K_{-1}(\mathbb{Z}[G])$ and $\widetilde{K}_0(\mathbb{Z}[G]\langle t\rangle)$. Therefore, the component $\partial[(r-1)^{-1}r]$ in the $NK_0(\mathbb{Z}[G])$ part of $\widetilde{K}_0(\mathbb{Z}[G]\langle t\rangle)$, being nonzero, implies that the image of the obstruction $\partial[rt-1] \in \widetilde{K}_0(\mathbb{Z}[G]\langle t\rangle)$ in $\widetilde{K}_0^{top}(\mathbb{Z}[\mathbb{Z}] \subset \mathbb{Z}[\mathbb{Z}\times G])$ is nonzero. This image is the K-theory obstruction in theorem 3.2. Therefore, the map $T(r) \to S^1$ does not even have semi-free pseudo-equivalence extension in the G-ANR category.

In short, there are examples of non-existence of semi-free extension in the G-ANR category if and only if the order of G has a square factor.

For the sufficiency part of proposition 4.1, we also give an explicit construction for the special case that G acts freely on a sphere. There is, for example, the case G is a cyclic group acting on the circle S^1 by the standard rotations.

Example 4.2. Suppose r is a multiple of $n = |G|$, and G acts freely on a sphere S^e. By replacing S^e with $S^e * S^e = S^{2e+1}$ and taking the join of G-actions, we may further assume that the action preserves the orientation of S^e. Consider the join $S^{d+e+1} = S^d * S^e$, with the trivial G-action on S^d and the given G-action on S^e. The action is semi-free with fixed set $(S^{d+e+1})^G = S^d$. Let h be a self map of S^{d+e+1} that is the join of the degree r map on S^d and the identity map on S^e. Then h is a G-map of degree r. For any free point $x \in S^{d+e+1} - S^d$, let D be a small disk around x, such that the action of G on D gives disjoint copies. By shrinking the boundary ∂D of D to the point x, we get a map $S^{d+e+1} \to S^{d+e+1} \vee_x (D/\partial D)$. Combining the identity on S^{d+e+1} and a homeomorphism $D/\partial D \to S^{d+e+1}$, we get a map $S^{d+e+1} \to S^{d+e+1} \vee_x S^{d+e+1} \to S^{d+e+1}$. If we do this for all G copies of D, then we get a G-map $h'\colon S^{d+e+1} \to S^{d+e+1} \vee_{Gx} G(D/\partial D) \to S^{d+e+1}$. By choosing a suitable homeomorphism $D/\partial D \to S^{d+e+1}$, the degree of h' is $r+n$ or $r-n$. By repeating the construction for several points in $S^{d+e+1} - S^d$, we get a G-map $h''\colon S^{d+e+1} \to S^{d+e+1}$ of degree $r+an$ for any integer a. Since r is a multiple of n, we take $a = -r/n$ and get a G-map $h''\colon S^{d+e+1} \to S^{d+e+1}$ of degree 0. On the other hand, since the modification happens only on the free part of S^{d+e+1}, the restriction of h'' on the fixed part is still the original degree r map $S^d \to S^d$. The mapping torus $T(h'')$ has semi-free G-action with fixed set $T(r)$, and $T(h'') \to S^1$ extends $T(r) \to S^1$. Moreover, since the degree of h'' is 0, the map $T(h'') \to S^1$ is a homotopy equivalence.

Now we turn to another application showing other phenomena. Suppose n is not a prime power. Then $n = n_1 n_2$, with $n_1, n_2 > 1$ and coprime. Let a satisfy $a = 1 \bmod n_1$ and $a = 0 \bmod n_2$. Then $b = 1 - a$ satisfies $b = 0 \bmod n_1$ and $b = 1 \bmod n_2$. Let $T(a, b)$ be the double torus of two maps of S^d to itself of respective degrees a, b. Let $f\colon T(a, b) \to S^1$ be the projection map. We consider the pseudo-equivalence extension of f for the action by the cyclic group $G = \mathbb{Z}_n$.

Similar to the mapping torus in the earlier example, the only non-trivial $\mathbb{Z}\langle t\rangle$-homology of f is

$$H_d(f; \mathbb{Z}\langle t\rangle) = \mathbb{Z}\langle t\rangle/(at - b).$$

By $(at - b)(at^{-1} - b) = a^2 = 1 \bmod n_1$, and $(at - b)(at^{-1} - b) = b^2 = 1 \bmod n_2$, and n_1, n_2 coprime, we know $at - b$ is invertible in $\mathbb{Z}_n\langle t\rangle$. This verifies the Smith condition for f.

PROPOSITION 4.3. *Suppose $G = \mathbb{Z}_n$ is the cyclic group of not prime power order n. Then for suitable a, b, the map $T(a, b) \to S^1$ satisfies the Smith condition for G, but has no semi-free pseudo-equivalence extension.*

Example 1.2 in the introduction is a direct consequence of proposition 4.3. The proof shows that the obstruction effectively lies in the direct summand $K_{-1}(\mathbb{Z}[\mathbb{Z}_n]) \subset \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_n]\langle t\rangle)$ according to (4.1). By theorem 3.2 and the explanation after the proof of proposition 4.1, this implies that, although $T(a, b) \to S^1$ has no semi-free pseudo-equivalence extension in the G-CW-complex category, the map does have semi-free pseudo-equivalence extension in the G-ANR category.

Proof. The obstruction for pseudo-equivalence extension is

$$[\mathbb{Z}\langle t\rangle/(at - b)] \in \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_n]\langle t\rangle).$$

This is the image of

$$[at - b] \in \mathbb{Z}_n\langle t\rangle^* \subset K_1(\mathbb{Z}_n\langle t\rangle)$$

under the Swan homomorphism. We carry out the argument similarly to proposition 4.1.

We have $a(1 - a) = ab = 0 \bmod n$. This means $a^2 = a \bmod n$, or a is an idempotent mod n. In particular, $a\mathbb{Z}_n$ is a projective $\mathbb{Z}_n$-module. By the proof of [**21**, Theorem 3.2.22], the obstruction $[at - b]$ on the left of (4.2) corresponds to $(0, [a\mathbb{Z}_n], 0, 0)$ on the right, with $[a\mathbb{Z}_n] \in K_0(\mathbb{Z}_n)$. Therefore our obstruction is the image $\partial[a\mathbb{Z}_n] \in K_{-1}(\mathbb{Z}[\mathbb{Z}_n])$ under the Swan homomorphism. By the calculation of [**6**], this element is a non-divisible element of $K_{-1}(\mathbb{Z}[\mathbb{Z}_n])$.

The Swan homomorphism fits into an exact sequence

$$\begin{aligned}\widetilde{K}_0(\mathbb{Z}) \oplus \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_n]/\Sigma_{\mathbb{Z}_n}) \to \widetilde{K}_0(\mathbb{Z}_n) &\xrightarrow{\partial} K_{-1}(\mathbb{Z}[\mathbb{Z}_n]) \\ \to K_{-1}(\mathbb{Z}) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}_n]/\Sigma_{\mathbb{Z}_n}) &\to K_{-1}(\mathbb{Z}_n).\end{aligned}$$

Here the image of $\widetilde{K}_0(\mathbb{Z}_n)$ is the same as the image of $K_0(\mathbb{Z}_n)$. By [**6**, **26**], we have $\widetilde{K}_0(\mathbb{Z}) = K_{-1}(\mathbb{Z}) = 0$, $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_n]/\Sigma_{\mathbb{Z}_n})$ is finite, and $\widetilde{K}_0(\mathbb{Z}_n)$ and $K_{-1}(\mathbb{Z}[\mathbb{Z}_n]/\Sigma_{\mathbb{Z}_n})$ are free abelian. Therefore $\widetilde{K}_0(\mathbb{Z}_n)$ embeds as a direct summand of $K_{-1}(\mathbb{Z}[\mathbb{Z}_n])$.

If $n = p_1^{m_1} \dots p_l^{m_l}$ is the decomposition into distinct primes, then

$$K_0(\mathbb{Z}_n) = \oplus_{i=1}^l K_0(\mathbb{Z}_{p_i^{m_i}}) = \oplus_{i=1}^l \mathbb{Z}.$$

We note that the projective $\mathbb{Z}_n$-module $a\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{n_1}$. Under the isomorphism $K_0(\mathbb{Z}_n) = K_0(\mathbb{Z}_{n_1}) \oplus K_0(\mathbb{Z}_{n_2})$, $[a\mathbb{Z}_n] \in K_0(\mathbb{Z}_n)$ corresponds to $([\mathbb{Z}_{n_1}], 0) \in K_0(\mathbb{Z}_{n_1}) \oplus K_0(\mathbb{Z}_{n_2})$. If we start by choosing $n_1 = p_1^{m_1}$ and $n_2 = n/n_1$, then the obstruction for $f\colon T(a, b) \to S^1$ is the image of $(1, 0, \dots, 0) \in K_0(\mathbb{Z}_n)$ under the injective Swan homomorphism. Similarly, we can make other choices of n_1, n_2, such that the obstructions for the corresponding $f\colon T(a, b) \to S^1$ are the images of other unit vectors, in $K_0(\mathbb{Z}_n)$. The upshot is that, if n is not a prime power, then we can construct $f\colon F \to S^1$ satisfying the Smith condition, and the obstruction is a nonzero element in $K_{-1}(\mathbb{Z}[\mathbb{Z}_n])$. □

The examples in propositions 4.1 and 4.3 can fit into other spaces.

Theorem 4.4. *Suppose Y is a finite CW-complex with torsion free $\pi = \pi_1(Y)$. Suppose the Farrell–Jones conjecture holds for π. Suppose G is finite cyclic, and $|G|$ is not a prime and has no square factor. Then there is a map $F \to Y$ satisfying the Smith condition but has no semi-free pseudo-equivalence extension, if and only if $H_1\pi \neq 0$.*

The condition on the order of G is that $|G|$ is a product of more than one distinct primes.

Proof. The hypothesis that $|G|$ has no square factor implies that $NK_0(\mathbb{Z}[G]) = 0$. Then for torsion free π, the Farrell–Jones conjecture asserts that

$$K_0(\mathbb{Z}[\pi \times G]) = K_0(\mathbb{Z}[G]) \oplus H_1(\pi; K_{-1}(\mathbb{Z}[G])).$$

If $H_1\pi = 0$, then $H_1(\pi; K_{-1}(\mathbb{Z}[G])) = 0$, and the Swan homomorphism relevant to constructing the G-action lies in the $K_0(\mathbb{Z}[G])$ part. Thus we are reduced to the classical Swan homomorphism. By [**25**, Corollary 6.1], the Swan homomorphism vanishes for cyclic G.

If $H_1\pi \neq 0$, then a generator of $H_1\pi = H_1 Y$ can be represented by a loop $S^1 \to Y$. By proposition 4.3, there is a map $f\colon F \to S^1$ satisfying the Smith condition but has non-vanishing pseudo-equivalence extension obstruction $[C(\widetilde{f})] \in \widetilde{K}_{-1}(\mathbb{Z}[G]) = H_1(S^1, \widetilde{K}_{-1}(\mathbb{Z}[G])) \subset \widetilde{K}_0(\mathbb{Z}[G]\langle t\rangle)$, where t is the generator of $\pi_1(S^1)$. We use the loop $S^1 \to Y$ to extend f to $f'\colon F \cup_{S_1} Y \to Y$. Then f' also satisfies the Smith condition, and the pseudo-equivalence extension obstruction $[C(\widetilde{f'})]$ for f' is the image of $[C(\widetilde{f})]$ under the homomorphism

$$\widetilde{K}_{-1}(\mathbb{Z}[G]) = H_1(S^1, \widetilde{K}_{-1}(\mathbb{Z}[G])) \to H_1(\pi, \widetilde{K}_{-1}(\mathbb{Z}[G])).$$

Since the circle represents a generator of $H_1\pi$, the image obstruction is still nonzero. □

Theorem 4.5. *Suppose Y is a finite CW-complex with torsion free $\pi = \pi_1(Y)$, the Farrell–Jones conjecture holds for π, and π has maximal infinite cyclic subgroup*

C, such that the normaliser of C is C itself. Suppose G is a finite abelian group, such that $|G|$ has square factor. Then there is a map $F \to Y$ satisfying the Smith condition, but has no semi-free pseudo-equivalence extension.

The theorem implies that, if every map $F \to Y$ satisfying the Smith condition has semi-free pseudo-equivalence extension for $\mathbb{Z}_n$-action, then n is a product of at least two distinct primes.

Proof. The proof is similar to theorem 4.4, except that the Farrell–Jones conjecture is more complicated because $NK_0(\mathbb{Z}[G]) \neq 0$. In this case, by [3], the formula for $K_0(\mathbb{Z}[\pi \times G])$ has another factor (i.e., a direct summand), namely $H_*^{\pi\times G}(E_{\mathcal{VC}}(\pi \times G), E_{\mathcal{FIN}}(\pi \times G); \mathbf{K})$. Here $H_*^{\pi\times G}$ is the homology over the category of $\pi \times G$-orbits (by Davis and Lück [10]), $E_{\mathcal{VC}}$ is the classifying space for the family of virtually cyclic subgroups, $E_{\mathcal{FIN}}$ is the classifying space for finite subgroups, and $\mathbf{K}$ is the non-connective K-theory spectrum of the isotropy groups of points. Note that this relative homology is concentrated on points with infinite isotropy. Under the condition of C being normalised only by itself, by [11], the set of points with isotropy C contributes two copies of $NK_0(\mathbb{Z}[G])$, and the glueing trick using our example from proposition 4.1 constructs an obstructed example.

Since this element is nonzero in $\widetilde{K}_0^{G,top}$ in the sense of [20, 23], it even obstructs the existence of an ANR-action. □

Finally, we study the pseudo-equivalence extension of a map between 3-dimensional lens spaces.

Proposition 4.6. *Suppose $f\colon L(kp;1) \to L(p;1)$ is a degree d map, where p is a prime not dividing d and k. Then f has a pseudo-equivalence extension for $\mathbb{Z}_p$-action if and only if $d^{p-1} = k^{p-1} \bmod p^2$.*

Proof. The obstruction lies in $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_p \times \mathbb{Z}_p])$, where the first $\mathbb{Z}_p = \pi_1(L(p;1))$, and the second $\mathbb{Z}_p$ is the action group. The obstruction is given by the chain complex of the map $\widetilde{f}\colon \widetilde{L}(kp;1) \to \widetilde{L}(p;1)$ obtained by pulling back along the universal cover $\widetilde{L}(p;1) = S^3 \to L(p;1)$. We have the long exact sequence

$$\begin{aligned} &H_3(L(kp;1);\mathbb{Z}[\mathbb{Z}_p]) = \mathbb{Z} \to H_3(L(p;1);\mathbb{Z}[\mathbb{Z}_p]) = \mathbb{Z} \to H_3(f;\mathbb{Z}[\mathbb{Z}_p]) = \mathbb{Z}_d \\ &\quad \to H_2(L(kp;1);\mathbb{Z}[\mathbb{Z}_p]) = 0 \to H_2(L(p;1);\mathbb{Z}[\mathbb{Z}_p]) = 0 \to H_2(f;\mathbb{Z}[\mathbb{Z}_p]) = \mathbb{Z}_k \\ &\quad \to H_1(L(kp;1);\mathbb{Z}[\mathbb{Z}_p]) = \mathbb{Z}_{kp} \to H_1(L(p;1);\mathbb{Z}[\mathbb{Z}_p]) = \mathbb{Z}_p \to H_1(f;\mathbb{Z}[\mathbb{Z}_p]) = 0. \end{aligned}$$

We note that $H_*(f;\mathbb{Z}[\mathbb{Z}_p]) = 0,\ 0,\ \mathbb{Z}_k,\ \mathbb{Z}_d,\ 0,\ \dots$ are trivial $\mathbb{Z}[\mathbb{Z}_p \times \mathbb{Z}_p]$-modules, and have $\mathbb{Z}[\mathbb{Z}_p \times \mathbb{Z}_p]$-projective resolutions. Therefore, the Euler characteristic of $H_*(f;\mathbb{Z}[\mathbb{Z}_p])$ gives an element of $K_1(\mathbb{Z}_{p^2})$ (p^2 is the order of the group $\mathbb{Z}_p \times \mathbb{Z}_p$), and the obstruction is the image of this element under the Swan homomorphism for the group $\mathbb{Z}_p \times \mathbb{Z}_p$

$$K_1(\mathbb{Z}_{p^2}) = (\mathbb{Z}_{p^2})^* \to \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_p \times \mathbb{Z}_p]).$$

In the multiplicative group $(\mathbb{Z}_{p^2})^*$, the Euler characteristic of $H_*(f;\mathbb{Z}[\mathbb{Z}_p])$ is k/d. The group $(\mathbb{Z}_{p^2})^*$ is additively isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_{p-1}$. By [27, Proposition 3],

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the image of the Swan homomorphism is an additive group $\mathbb{Z}_p \subset \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_p \times \mathbb{Z}_p])$. Therefore an element r is in the kernel of the Swan homomorphism $(\mathbb{Z}_{p^2})^* \cong \mathbb{Z}_p \oplus \mathbb{Z}_{p-1} \to \mathbb{Z}_p$ if and only if $r^{p-1} = 1$. In particular, the pseudo-equivalence extension obstruction $\frac{k}{d} \in (\mathbb{Z}_{p^2})^*$ for f vanishes if and only if $(k/d)^{p-1} = 1$ in $(\mathbb{Z}_{p^2})^*$. This gives the condition $d^{p-1} = k^{p-1} \bmod p^2$ in the proposition. □

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Appendix A. Units of $(\mathbb{Z}[G]/\Sigma_G)\langle t\rangle$

LEMMA A.1. *If G is a finite abelian group, then the units of $(\mathbb{Z}[G]/\Sigma_G)\langle t\rangle$ are monomials.*

Proof. The proof is based on the fact that $R\langle t\rangle^* = R^*\langle t\rangle$ for an integral domain R. In other words, the units in the polynomial ring $R\langle t\rangle$ are monomials. In general, $\mathbb{Z}[G]/\Sigma_G$ is not an integral domain, but can be detected by sufficiently many homomorphisms to integral domains. Specifically, a character $\lambda\colon G \to \langle \xi_n\rangle \subset \mathbb{C}$ ($\xi_n = e^{2\pi i/n}$ is the n-th root of unity and $\langle \xi\rangle$ is all powers of ξ) induces a ring homomorphism $\lambda\colon \mathbb{Z}[G] \to \mathbb{Z}[\xi_n] = \mathbb{Z}[s]/\varphi_n(s)$, where φ_n is the minimal polynomial of ξ_n. This further induces a homomorphism $\lambda\colon \mathbb{Z}[G]/\Sigma_G \to \mathbb{Z}[\xi_n]$ unless λ is trivial. Then we get a homomorphism of units

$$\lambda\colon (\mathbb{Z}[G]/\Sigma_G)\langle t\rangle^* \to \mathbb{Z}[\xi_n]\langle t\rangle^* = \mathbb{Z}[\xi_n]^*\langle t\rangle.$$

The equality is due to the fact that $\mathbb{Z}[\xi_n]$ is an integral domain. Then for a unit $x = \sum x_i t^i$, $x_i \in \mathbb{Z}[G]$, on the left, we know $\lambda(x) = \sum \lambda(x_i)t^i \in \mathbb{Z}[\xi_n]^*\langle t\rangle$ means that there is $i(\lambda)$, such that $\lambda(x) = \lambda(x_{i(\lambda)})t^{i(\lambda)}$ (i.e., $\lambda(x_i) = 0$ for all $i \neq i(\lambda)$).

If we take λ to be all non-trivial characters, with the corresponding $n = p^k$ being prime powers, then we get an embedding

$$\mathbb{Z}[G]/\Sigma_G \subset \prod_\lambda \mathbb{Z}[\xi_{p^k}].$$

This induces an embedding of the units (for the specific selections of λ)

$$(\mathbb{Z}[G]/\Sigma_G)\langle t\rangle^* \subset \prod_\lambda \mathbb{Z}[\xi_{p^k}]\langle t\rangle^* = \prod_\lambda \mathbb{Z}[\xi_{p^k}]^*\langle t\rangle.$$

The embedding of the units shows that, if we prove that $i(\lambda)$ is independent of the choice of λ, then the original unit $x = \sum x_i t^i$ on the left is also a monomial. This proves the lemma.

First, by $\varphi_{p^k}(s) = (1 - s^{p^k})/(1 - s^{p^{k-1}}) = 1 + s^{p^{k-1}} + s^{2p^{k-1}} + \cdots + s^{(p-1)p^{k-1}}$, we have a homomorphism by sending s to 1

$$\mu\colon \mathbb{Z}[\xi_{p^k}] = \mathbb{Z}[s]/\varphi_{p^k}(s) \to \mathbb{F}_p.$$

Then we have the composition $\mu \circ \lambda\colon \mathbb{Z}[G]/\Sigma_G \to \mathbb{F}_p$ that sends every group element to 1. In particular, the induced map $\mu \circ \lambda\colon (\mathbb{Z}[G]/\Sigma_G)\langle t\rangle \to \mathbb{F}_p\langle t\rangle$ depends only on p.

Then $\mu \circ \lambda(x) = \mu(x_{i(\lambda)})t^{i(\lambda)}$ depends only on p. This implies that, if two characters λ and λ' correspond to the same prime, then $i(\lambda) = i(\lambda')$.

It remains to show that, if λ and λ' correspond to ξ_p and ξ_q, where p, q are distinct primes, then $i(\lambda) = i(\lambda')$. By what we proved above, we only need to verify this for any one pair λ and λ' corresponding to ξ_p and ξ_q. Consider a character $\Lambda \colon G \to C = \langle \xi_{pq} \rangle \subset \mathbb{C}$. Then we have

$$\begin{array}{ccc}
\mathbb{Z}[G]/\Sigma_G & & \\
\downarrow & & \\
\mathbb{Z}[C]/\Sigma_C = \mathbb{Z}[s]/(\frac{1-s^{pq}}{1-s}) & \longrightarrow & \mathbb{Z}[\xi_{pq}] = \mathbb{Z}[s]/\varphi_{pq}(s) \\
\downarrow & & \downarrow \\
\mathbb{Z}[\xi_p] = \mathbb{Z}[s]/\varphi_p(s) & \longrightarrow & \mathbb{Z}[\xi_{pq}]/\varphi_p(\xi_{pq}) = \mathbb{Z}[s]/(\varphi_{pq}(s), \varphi_p(s))
\end{array}$$

The compositions to $\mathbb{Z}[\xi_p]$ and $\mathbb{Z}[\xi_{pq}]$ are respectively λ and Λ. Therefore, the images of a unit $x = \sum x_i t^i \in (\mathbb{Z}[G]/\Sigma_G)\langle t \rangle^*$ are respectively $\lambda(x_{i(\lambda)})t^{i(\lambda)} \in \mathbb{Z}[\xi_p]^*\langle t \rangle$ and $\lambda(x_{i(\Lambda)})t^{i(\Lambda)} \in \mathbb{Z}[\xi_{pq}]^*\langle t \rangle$. Both are further mapped to $(\mathbb{Z}[\xi_{pq}]/\varphi_p(\xi_{pq}))^*\langle t \rangle$. To show $i(\Lambda) = i(\lambda)$, therefore, we only need to show that $\mathbb{Z}[\xi_{pq}]/\varphi_p(\xi_{pq}) \neq 0$.

By $\mathbb{Z}[\xi_{pq}]/\varphi_p(\xi_{pq}) = \mathbb{Z}[s]/(\varphi_{pq}(s), \varphi_p(s))$, the ring is 0 if and only if $1 = \varphi_{pq}(s)u(s) + \varphi_p(s)v(s)$ for some polynomials $u(s)$, $v(s) \in \mathbb{Z}[s]$. Taking $s = \xi_p$, we get $1 = \varphi_{pq}(\xi_p)v(\xi_p)$. By

$$\varphi_{pq}(s) = \frac{(s^{pq}-1)(s-1)}{(s^p-1)(s^q-1)} = \frac{1 + s^p + s^{2p} + \cdots + s^{(q-1)p}}{1 + s + s^2 + \cdots + s^{q-1}},$$

we have

$$\varphi_{pq}(\xi_p) = \frac{q}{1 + \xi_p + \xi_p^2 + \cdots + \xi_p^{q-1}}.$$

Then $1 = \varphi_{pq}(\xi_p)v(\xi_p)$ means $1 + \xi_p + \xi_p^2 + \cdots + \xi_p^{q-1} = qv(\xi_p)$ is a multiple of q in $\mathbb{Z}[\xi_p]$. Since p, q are coprime, this is not true.

We conclude that the ring is non-zero. This implies $i(\Lambda) = i(\lambda)$. Switching p and q, we also get $i(\Lambda) = i(\lambda')$. This completes the proof of $i(\lambda) = i(\lambda')$. □

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