ON THE CONTINUITY OF PROJECTIONS

M. C. GEMIGNANI

(Received 1 August 1966)

Throughout this note X will be a topological space with geometry G of length $m-1$ with $F^0 = \{\{x\}|x \in x\}$. The terminology will be that of [1].

Let f be an $m-1$-flat, $W \subset X$, and $x \in X-(f \cup W)$ such that $f_1(w, x) \cap f \neq \emptyset$ for each $w \in W$. Then $f_1(w, x) \cap f$ consists of a single point which we denote by $p_x(w)$. p_x then is a function from W into f. Clearly p_x is not necessarily continuous.

If $U \subset f$, define $K(U) = \cup \{f_1(x, u)|u \in U\}$ and $k(U) = K(U)-\{x\}$.

PROPOSITION 1. *If* a) *$K(U)$ is an open subset of X whenever U is open in f, or* b) *if $k(U)$ is an open subset of X whenever U is open in f, then p_x is continuous.*

PROOF. Suppose a) or b) holds. Suppose U is an open subset of f. Then $p_x^{-1}(U) = K(U) \cap W = k(U) \cap W$ is an open subset of W, hence p_x is continuous.

The conditions a) and b) are not exhaustive for p_x to be continuous. For example, if X has the trivial topology, then as a rule neither a) nor b) will hold, even though p_x is then clearly continuous.

PROPOSITION 2. *If X and G form an open m-arrangement, then p_x is continuous.*

PROOF. We show that condition b) holds. Let U be an open subset of f and $x \in U$. Then there is a linearly independent subset $S = \{y_0, \cdots, y_{m-1}\}$ of f such that $x \in \text{Int}\, C(S)$. Set $S_i = (S \cup \{x\})-\{y_i\}$, $i = 0, \cdots, m-1$. Then $f_{m-1}(S_i)$, $i = 0, \cdots, m-1$, disconnects X into two convex, open components A_i (which we assume contains x_i) and B_i. It is readily shown that $k(\text{Int}\, C(S)) \subset k(U)$ and $k(\text{Int}\, C(S)) = (\bigcap_{i=0} A_i) \cup (\bigcap_{i=0} B_i)$. It follows at once that $k(U)$ is open, hence $b)$ is satisfied.

The question of whether p_x is always continuous whenever X and G form an m-arrangement has not as yet been answered. The difficulties in connection with an arbitrary m-arrangement are due to peculiarities which can exist with regard to BdX. Generally, of course, condition a) does not hold in any m-arrangement and condition b) would not hold as a rule in any m-arrangement with $BdX \neq \emptyset$.

If X and G form an m-arrangement and h is any $m-1$-flat of X, then we call $X-h$ a half-space of X (regardless of whether h disconnects X or not). If the collection of half-spaces of X form a subbasis for the topology of X, then p_x can be shown to be continuous in a proof analogous to that of proposition 2. However, the space $j(X)$ with geometry $j(G_X)$ in [2] is an example of an m-arrangement where the half-spaces do not form a subbasis for the topology.

The following propositions give a proof that p_x is continuous for a 2-arrangement as well as some clues to the case for any m.

PROPOSITION 3. *Suppose X and G form an open m-arrangement. Let $\{w_k\}$, $k \in K$, be a net in W, $w_k \to z \in W$. Then the net of flats $\{f_1(x, w_k)\}$, $k \in K$, converges to f_1 (x, z) in topologies* I *and* II *as described in* [3].

PROOF. Let U be any convex, open neighborhood of $u \in f_1(x, z)-\{x\}$, and let h be any $m-1$-flat which contains u. Then there is a linearly independent subset $S = \{y_0, \cdots, y_{m-1}\} \subset h$ such that $u \in \operatorname{Int} C(S) \subset U \cap h$. Letting A_i and B_i be as in the proof of proposition 2, we have $V = (\bigcap_{i=0}^{m-1} A_i) \cup (\bigcap_{i=0}^{m-1} B_i)$ is a neighborhood of z, hence $\{w_k\}$, $k \in K$, is residually in V. It follows then that $\{f_1(x, w_k)\}$, $k \in K$, is residually in $V \cup \{x\}$. Since $\operatorname{Int} C(S) = V \cap C(S)$ and $C(S)$ is the face opposite x of $C(S \cup \{x\})$, if $f_1(x, w_k) \subset V \cup \{x\}$, then $f_1(x, w_k) \cap U \neq \emptyset$. It follows at once that $f_1(w_k, x) \to f_1(z, x)$ in topology I. For if not, then there is either $q \in \overline{\lim}\, f_1(w_k, x) - f_1(w_k, x)$, or $q \in \underline{\lim}\, f_1(w_k, x) - \overline{\lim}\, f_1(w_k, x)$, either case leading to a contradiction of the fact that topology II is T_2. Since $f_1(w_k, x) \to f_1(x, z)$ in topology II, the proposition is proved.

PROPOSITION 4. *Suppose X and G form an m-arrangement such that each 1-flat in X intersects Int X. Let $\{w_k\}$, $k \in K$, be a net in W, $w_k \to z \in W$. Then the net of flats $\{f_1(x, w_k)\}$, $k \in K$, converges to $f_1(x, z)$ in topologies* I *and* II *as described in* [3].

PROOF. Let U be any convex, open neighborhood of $u \in f_1(x, z)-\{x\}$. If $u \in \operatorname{Int} X$, then since $\operatorname{Int} X$ with geometry $G_{\operatorname{Int} X}$ forms an open m-arrangement, we may use Proposition 3 to show that $u \in \overline{\lim}\, f_1(w_k, x)$. Suppose $u \in BdX$. Choose $p \in \operatorname{Int} \overline{xu} \cap U$. Then $p \in \operatorname{Int} X$. Carrying through a proof entirely analogous to the proof of proposition 3, we obtain that $\{f_1(w_k, x)\}$, $k \in K$, residually intersects U, hence as before the desired conclusion follows.

Note the difficulty even in this highly restricted situation (every 1-flat intersects Int X) in proving the continuity of p_x. p_x would be continuous if given any net $\{w_k\}$, $k \in K$, in W such that $w_k \to z \in W$, $p_x(w_k) \to p(z)$. As is seen from figure 1, it is possible for the 1-flats $f_1(w_k, x)$ to intersect f

in a point outside $U \cap f$, if $u \in BdX$, thus we cannot be assured that $p_x(w_k) \to p(z)$, even though we have shown that $p(z) \in \overline{\lim}\, f_1(w_k, x)$.

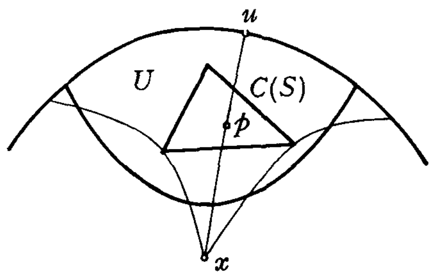

Figure 1

The following example illustrates that if $f_1(x, z) \subset BdX$, $\{f_1(w_k, x)\}$, $k \in K$, may not converge to $f_1(x, z)$ in topology I, even though it does converge in topology II.

EXAMPLE. Let $X = \{(x, y) \mid |x| \leqq 1, y \geqq 0\} \subset R^2$ with the induced topology and geometry. Set $f^n = \{(x, y) \mid y = (1/n)x\} \cap X$, $n = 1, 2, 3, \cdots$. Then $f^n \{(x, y) \mid y = 0\} \cap X$ in topology II, but does not converge in topology I.

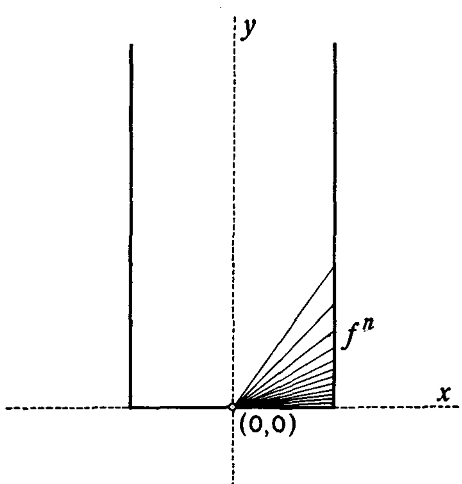

Figure 2

PROPOSITION 5. *If X and G form a 2-arrangement, then p_x is continuous.*

PROOF. Suppose $\{w_k\}$, $k \in K$, is a net in W, $w_k \to z \in W$. We will show that $p_x(w_k) \to p_x(z)$.

https://doi.org/10.1017/S1446788700006108 Published online by Cambridge University Press

CASE 1. $p_x(z) \in \operatorname{Int} f$. There are then points a and b in f such that $p_x(z) \in \operatorname{Int} \overline{ab}$.

Let A_1 be the component of $X - f_1(x, a)$ which contains b and A_2 be the component of $X - f_1(x, b)$ which contains a. Let B_1 and B_2 be the other components (if either is non-empty) of $X - f_1(x, a)$ and $X - f_1(x, b)$, respectively. Then $\operatorname{Int} \overline{ab} \subset A_1 \cap A_2$. If $p_x(w_k) \nrightarrow p_x(z)$, then there is a convex, open neighborhood U of $p_x(z)$, $U \subset A_1 \cap A_2$, and a subnet $\{w_{k_j}\}$, $j \in J$, of $\{w_k\}$, $k \in K$, such that for each $j \in J$, $p_x(w_{k_j}) \notin U \cap f$.

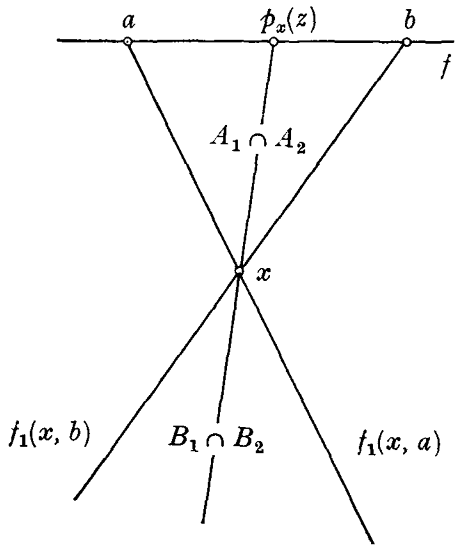

Figure 3

Then $\{p_x(w_{k_j})\}$, $j \in J$, is residually in $\overline{ab}$ since either $A_1 \cap A_2$ or $B_1 \cap B_2$ is a neighborhood of z and each $f_1(w_{k_j}, x)$ cannot intersect $f_1(a, x)$ or $f_1(b, x)$ in two distinct points. Since $\overline{ab}$ is compact, there is a convergent subnet of $\{w_{k_j}\}$, $j \in J$; say this convergent subnet converges to $t \in f$. Then we can find a net of flats which converges to both $f_1(x, p_x(z))$ and $f_1(x, t)$ in F^1 given topology II. But $f_1(x, p_x(z))$ and $f_1(x, t)$ are distinct since no subnet of $\{w_{k_j}\}$, $j \in J$, can converge to $p_x(z)$, a contradiction to the fact that F^1 with topology II is T_2.

CASE 2. $p_x(z) \in Bd f$. Choose $b \in \operatorname{Int} f$. Letting $p_x(z) = a$, let A_1, B_1, A_2 and B_2 be as in Case 1 (whenever these are non-empty). If $z \in B_2$, then $\{w_k\} \subset \operatorname{Cl} B_1 \cap \operatorname{Cl} B_2$ (or else some $f_1(w_k, x)$ does not intersect f). Suppose $z \in A_2$; if $x \in BdX$, it is easily shown that this must be the case. Then $\{w_k\}$ is residually in A_2, hence $\{w_k\}$ is residually in $\operatorname{Cl} A_1 \cap \operatorname{Cl} A_2$. $p_x(w_k)$ is therefore residually in $\overline{bp_x(z)}$ and reasoning similar to that used in Case 1 can be used to complete the proof.

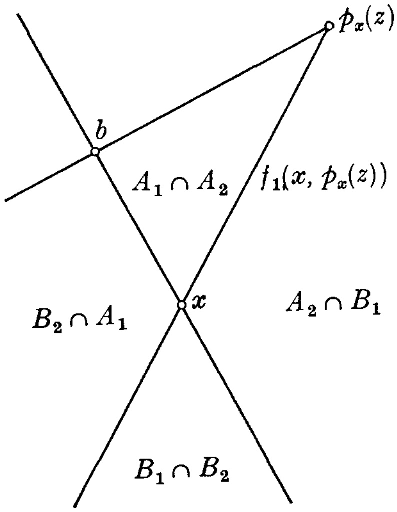

Figure 4

The author has not yet been able to find a valid generalization of this argument to m-arrangements.

We now discuss another type of projection. Let G be an affine geometry. Let f be an $m-1$-flat and g a 1-flat such that $g \cap f$ consists of exactly one point. If g' is any 1-flat parallel to g, then $g' \cap f$ also consists of exactly one point. Let $W \subset X$. If $w \in W$, let g_w be the unique 1-flat which contains w and is parallel to g. Let $p_g(w)$ be the point of intersection of f and g_w. Then p_g is a function from W into f. If $T \subset f$, define $PK(T) = \cup \{g_t | g_t$ is the 1-flat through t which is parallel to $g\}$. Analogous to Proposition 1, we have

PROPOSITION 6. *If $PK(T)$ is open whenever T is open in f, then p_g is continuous.* The proof is that of Proposition 1 with p_g replacing p_x.

Again this condition is sufficient, but not necessary.

PROPOSITION 7. *If X and G form an affine m-arrangement, then p_g is continuous.*

The proof is analogous to the proof of Proposition 2 with 'open boxes' replacing simplices.

References

[1] M. Gemignani, 'Topological geometries and a new characterization of R^m', *Notre Dame Journal of Formal Logic* 7 (1967), 57–100

[2] M. Gemignani, 'A note on *Bd X*', *Notre Dame Journal of Formal Logic* (to appear).

[3] M. Gemignani, 'On topologies for F^i', *Fund. Math.* 54(1966), 153–157.

SUNY at Buffalo

https://doi.org/10.1017/S1446788700006108 Published online by Cambridge University Press