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ON LOWER BOUNDS FOR THE RADICAL OF A BLOCK IDEAL IN A FINITE p-SOLVABLE GROUP

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Dedicated to Professor Hirosi Nagao on his 60th birthday

Let F be any field of characteristic $p>0, G$ a finite p-solvable group, p^a the order of Sylow p-subgroups of G, FG the group algebra of G over F, and $J(FG)$ the Jacobson radical of FG. Following Wallace **[11]** we write $t(G)$ for the least integer $t \geqq 1$ such that $J(FG)^t = 0$.

D. A. R. Wallace **[11]** proved that

$$t(G) \geqq a(p-1)+1.$$

The purpose of the present paper is to generalize the above result as follows: Let B be a block ideal of FG with defect d, and let $t(B)$ be the least integer $t \geqq 1$ such that $J(B)^t = 0$ where $J(B)$ is the Jacobson radical of B. Then

$$t(B) \geqq d(p-1)+1.$$

Since the defect groups of the principal block ideal of FG are Sylow p-subgroups of G, our result is a generalization of that of Wallace.

We use the following notation and terminology. Throughout this paper we fix a field F of characteristic $p>0$ and a finite group G, all modules are finitely generated right modules, and all groups are finite. For an Artinian ring R and an integer $n \geqq 1$ let us denote by $\mathrm{Mat}(n, R)$ the full matrix ring of degree n over R, by $Z(R)$ the centre of R, by $J(R)$ the Jacobson radical of R, and by $t(R)$ the least integer $t \geqq 1$ such that $J(R)^t = 0$. In particular, we write $t(G)$ for $t(FG)$. Following **[8,** §2] we call $B \leftrightarrow e$ a block of FG if e is a centrally primitive idempotent of FG such that $B = FGe$, and in this case we call B a block ideal of FG. When B is a block ideal of FG, we write $\delta(B)$ for a defect group of B and $d(B)$ for the defect of B, i.e. $|\delta(B)| = p^{d(B)}$ (cf. **[9,** p. 211] and **[8,** Definition 3.9]), and we say that B has full defect if $\delta(B)$ is a Sylow p-subgroup of G. When $H \lhd G$ and $b \leftrightarrow f$ is a block of FH, we write $T_G(b)$ or $T_G(f)$ for the inertia group of $b \leftrightarrow f$ in G, that is to say, $T_G(b) = T_G(f) = \{x \in G \mid x^{-1}fx = f\}$. If $H \lhd G$ and if B and b are block ideals of FG and FH, respectively, then we say that B covers b in the sense of **[8,** §6] (cf. **[2,** p. 196]). When M_R is an R-module, we write $\mathrm{End}(M_R)$ for the ring of all R-module-endomorphisms of M_R. We write $Z(G)$ for the centre of G. We use the notation $O_{p'}(G)$, $O_p(G)$ and $O_{p',p}(G)$ as in **[1,** p. 397]. Further notation and terminology follow the books of Dornhoff **[1]** and Gorenstein **[5]**.

First of all, we state Fong's results ([3], [4]) which are useful in the proof of our main result.

Lemma 1 (Fong). *Assume that F is an algebraically closed field of characteristic $p>0$. Let $H \lhd G$, let $b \leftrightarrow f$ be a block of FH, and let $T=T_G(f)$. Let $G=\bigcup_{i=1}^t Tg_i$ be a coset decomposition of T in G, let $f_i=g_i^{-1}fg_i$ for each i, and let $e=\sum_{i=1}^t f_i$. Then we have the following:*

(1) *f is a central idempotent of FT.*

(2) *$f_1,\dots,f_t$ are pairwise orthogonal centrally primitive idempotents of FH.*

(3) *e is a central idempotent of FG and $ef=fe=f$.*

(4) *FGf is a free right FTf-module of rank t.*

(5) $\mathrm{End}(FGf_{FTf})\cong \mathrm{Mat}(t, FTf)$ *as F-algebras.*

(6) *For each $x\in FGe$ and $y\in FGf$, define $\varphi(x)\in \mathrm{End}(FGf_{FTf})$ by $[\varphi(x)](y)=xy$. Then $\varphi: FGe\to \mathrm{End}(FGf_{FTf})$ is an F-algebra-isomorphism.*

(7) *Let $\tilde{B}_1\leftrightarrow\tilde{e}_1,\dots,\tilde{B}_m\leftrightarrow\tilde{e}_m$ be blocks of FT such that $f=\sum_{j=1}^m \tilde{e}_j$, and let $B_1\leftrightarrow e_1,\dots,B_n\leftrightarrow e_n$ be blocks of FG such that $e=\sum_{k=1}^n e_k$. Then*

(i) *$m=n$, $\tilde{B}_1,\dots,\tilde{B}_m$ are all block ideals of FT which cover b, and $B_1,\dots,B_m$ are all block ideals of FG which cover b.*

For suitable indexing of $\tilde{B}_j$ and B_j, we get for each $j=1,\dots,m$ that

(ii) *$B_j\cong \mathrm{Mat}(t,\tilde{B}_j)$ as F-algebras.*

(iii) *$\tilde{e}_je_j=e_j\tilde{e}_j=\tilde{e}_j$.*

(iv) *$\tilde{B}_j^G=B_j$.*

(v) *$\delta(B_j)\cong\delta(\tilde{B}_j)$.*

(vi) *If $\tilde{S}$ is a simple FT-module in $\tilde{B}_j$, then $\tilde{S}^G$ is a simple FG-module in B_j where $\tilde{S}^G=\tilde{S}\otimes_{FT}FG$.*

Proof. (1) Obvious.

(2) Clearly, $f_1,\dots,f_t$ are distinct centrally primitive idempotents of FH. Hence $f_if_j=0$ if $i\neq j$.

(3) By (2), $e^2=e$ and $ef=fe=f$. Hence $e\neq 0$. Take any $g\in G$. Since $G=\bigcup_{i=1}^t Tg_ig$ is also a coset decomposition of T in G, we get $g^{-1}eg=e$, so that $e\in Z(FG)$.

(4) Since $FGf=\bigoplus_{i=1}^t g_i^{-1}FTf$ and $g_i^{-1}FTf\cong FTf$ as right FTf-modules for all i, we get (4).

(5) Trivial from (4).

(6) Obviously, φ is well-defined. Let $E=\mathrm{End}\,(FG\,f_{FTf})$. By (3), $\varphi(e)$ is the identity map of FGf, so that φ is an F-algebra-homomorphism.

Assume $\varphi(x)=0$ for some $x\in FGe$. Then $xy=0$ for all $y\in FGf$. Hence $0=\sum_{i=1}^t xg_i^{-1}fg_i=xe=x$. Thus φ is monomorphic.

Take any $\sigma\in E$. Let $x=[\sum_{i=1}^t \sigma(g_i^{-1}f)g_if_i]e\in FGe$. Then by (2), $x=\sum_{i=1}^t \sigma(g_i^{-1}f)g_if_i$.

Let $y \in FGf$. Then we can write $y = \sum_{j=1}^{t} g_j^{-1} s_j$ where $s_j \in FTf$. By (1), $fs_j = s_j f = s_j$. Thus

$$\sigma(y) = \sum_j \sigma(g_j^{-1} s_j) = \sum_j \sigma(g_j^{-1} f s_j) = \sum_j \sigma(g_j^{-1} f) s_j$$

since $\sigma \in E$. On the other hand, since $fs_j = s_j$, we get by (2)

$$\begin{aligned}[\varphi(x)](y) = xy &= \sum_i \sum_j \sigma(g_i^{-1} f) g_i f_i g_j^{-1} s_j \\ &= \sum_i \sum_j \sigma(g_i^{-1} f) g_i f_i (g_j^{-1} f g_j) g_j^{-1} s_j \\ &= \sum_i \sigma(g_i^{-1} f) g_i f_i g_i^{-1} s_i = \sum_i \sigma(g_i^{-1} f) f s_i \\ &= \sum_i \sigma(g_i^{-1} f) s_i.\end{aligned}$$

Hence $\sigma(y) = [\varphi(x)](y)$, so that $\sigma = \varphi(x)$. Hence φ is epimorphic.

(7) By [**2**, V Lemma 3.3] (cf. [**8**, §6]), $B_1, \dots, B_n$ are all block ideals of FG which cover b. Similarly, $\tilde{B}_1, \dots, \tilde{B}_m$ are all block ideals of FT which cover b. Then $m = n$ by [**2**, V Theorem 2.5]. Since $FGe = \bigoplus_{j=1}^{m} FGe_j$ and $FTf = \bigoplus_{j=1}^{m} FT\tilde{e}_j$, by (5) and (6) for suitable indexing of e_j and $\tilde{e}_j$ we have the F-algebra-isomorphisms

$$\begin{array}{ccccc} FGe_j & \xrightarrow{\approx} & \mathrm{End}\,[(FG\tilde{e}_j)_{FT\tilde{e}_j}] & \xrightarrow{\approx} & \mathrm{Mat}\,(t, FT\tilde{e}_j) \\ \cup\!\!\!\!\! \in & & \cup\!\!\!\!\! \in & & \\ x & \longmapsto & [\varphi(x): y \mapsto xy] & & \end{array}$$

for $j = 1, \dots, m$. Let us fix any j. Since e_j is the unit element of the ring FGe_j, $\varphi(e_j)$ is the identity map of $FG\tilde{e}_j$. Hence $\tilde{e}_j e_j = e_j \tilde{e}_j = \tilde{e}_j$. Let $\tilde{S}$ be a minimal right ideal of $\tilde{B}_j = FT\tilde{e}_j$. Then

$$\tilde{S}^G e_j = \tilde{S} FG e_j = \tilde{S}\tilde{e}_j FG e_j = \tilde{S}\tilde{e}_j e_j FG = \tilde{S}\tilde{e}_j FG = \tilde{S} FG = \tilde{S}^G.$$

Hence $\tilde{S}^G \subseteq FGe_j = B_j$. Thus the correspondence $\tilde{B}_j \leftrightarrow B_j$ is the same as that of [**2**, V Theorem 2.5]. Therefore (7) is proved by [**2**, V Theorem 2.5].

Lemma 2 (Fong). *Assume that F is an algebraically closed field of characteristic $p > 0$. Let $H \lhd G$ such that $p \nmid |H|$, and let b be a block ideal of FH covered by a block ideal B of FG. If $T_G(b) = G$, then there are a finite group $\tilde{G}$ and an exact sequence*

$$1 \longrightarrow Z \longrightarrow \tilde{G} \stackrel{f}{\longrightarrow} G \longrightarrow 1 \qquad (*)$$

which satisfy the following:

(1) *Z is cyclic, $Z \subseteq Z(\tilde{G})$ and $|Z| \,\big|\, |H|^2$.*

(2) *$\tilde{G}$ has a normal subgroup $\tilde{H}$ such that $\tilde{H} \cong H$ and $Z\tilde{H} = Z \times \tilde{H} = f^{-1}(H)$.*

(3) *$F(\tilde{G}/\tilde{H})$ has a block ideal B^* such that $B \cong \mathrm{Mat}(n, B^*)$ as F-algebras for an integer $n \geqq 1$ and that $\delta(B^*) \cong \delta(B)$.*

(4) *Let $X = \tilde{G}/\tilde{H}$. Especially, if G is p-solvable, $p \mid |G|$ and $H = O_{p'}(G)$, then we get the following:*

(i) *X is also p-solvable.*

(ii) *$O_{p'}(X) \subseteqq Z(X)$.*

(iii) *X has a normal p-subgroup Q such that $O_{p',p}(X) = O_{p'}(X) \times Q$.*

(iv) *$O_p(X) \neq 1$.*

(v) *Every block ideal of FX has full defect.*

Proof. By [**2,** X Lemma 1.1 and Theorem 1.2], [**12,** §1] and [**10,** Theorem 2], we have an exact sequence (*) which satisfies (1), (2) and (3).

(4) (i) is clear. Since $p \nmid |\tilde{H}|$, $O_{p'}(X) = O_{p'}(\tilde{G})/\tilde{H}$. By (1) and (2), $O_{p'}(\tilde{G}) = Z \times \tilde{H}$. Hence $O_{p'}(X) \subseteqq Z(X)$ by (1). Since $O_{p',p}(X)$ is p-nilpotent, we get (iii) from (ii). Since $p \mid |X|$, by (i) and (iii) we have $1 \neq Q \subseteqq O_p(X)$ (cf. [**2,** p. 416]). (v) is obtained from [**2,** X Lemma 1.4].

The next lemma has been essentially proved by Wallace [**11,** Theorem 2.4].

Lemma 3 (Wallace). *Let F be any field of characteristic $p > 0$ and P a normal p-subgroup of G, and let $\bar{G} = G/P$. Let $FG \xrightarrow{f} F\bar{G}$ be the canonical ring-epimorphism such that $f(g) = gP$ for each $g \in G$, and let $B \leftrightarrow e$ be a block of FG. Then we can write $f(B) = \bigoplus_{i=1}^n \bar{B}_i$ for an integer $n \geqq 1$ where each $\bar{B}_i$ is a block ideal of $F\bar{G}$. Moreover, we have the following:*

(1) *$t(B) \leqq t(P) \cdot m$ where $m = \max\{t(\bar{B}_i) \mid i = 1, \ldots, n\}$.*

(2) *$t(B) \geqq t(P) + t(\bar{B}_i) - 1$ for all $i = 1, \ldots, n$.*

Proof. The proof is similar to that of Wallace [**11,** Theorem 2.4]. Let $G = \bigcup_{j=1}^q g_j P$ be a coset decomposition of P in G. Then $FG = \bigoplus_{j=1}^q g_j FP$, so that $FG \cdot J(FP) = \bigoplus_{j=1}^q g_j J(FP)$. By [**8,** Lemma 4.5] and [**6,** Theorem 1.2], $\mathrm{Ker}\, f = J(FP)FG = FG \cdot J(FP)$, so that $\mathrm{Ker}\, f$ is a nilpotent ideal of FG. Hence $\mathrm{Ker}\, f \subseteqq J(FG)$. Then $f(e) \neq 0$ since $\mathrm{Ker}\, f$ is nilpotent. Thus we can write $f(e) = \sum_{i=1}^n \bar{e}_i$ for an integer $n \geqq 1$ where each $\bar{e}_i$ is a centrally primitive idempotent of $F\bar{G}$. Let $\bar{B}_i = F\bar{G}\bar{e}_i$ for each i, then $f(B) = \bigoplus_{i=1}^n \bar{B}_i$.

(1) Let $\tilde{f} = f|_B : B \to f(B)$. Then $\mathrm{Ker}\, \tilde{f} = \mathrm{Ker}\, f \cap B = (\mathrm{Ker}\, f)e = J(FP)B$, so that $\mathrm{Ker}\, \tilde{f} = J(FP)B = B \cdot J(FP) \subseteqq J(B)$. Thus $\tilde{f}$ induces a ring-isomorphism

$$\bigoplus_{i=1}^n \bar{B}_i = f(B) \cong B/\mathrm{Ker}\, \tilde{f} = B/J(FP)B.$$

Since $J[B/J(FP)B] = [J(B) + J(FP)B]/J(FP)B = J(B)/J(FP)B$, we have

$$\bigoplus_{i=1}^n J(\bar{B}_i) = J\left(\bigoplus_{i=1}^n \bar{B}_i\right) \cong J(B)/J(FP)B.$$

Then since $[\bigoplus_{i=1}^n J(\bar{B}_i)]^m = \bigoplus_i J(\bar{B}_i)^m = 0$, we get $J(B)^m \subseteqq J(FP)B = B \cdot J(FP)$. Thus we have $J(B)^{m \cdot t(P)} = 0$, so that $t(B) \leqq m \cdot t(P)$.

(2) Fix any i $(1 \leqq i \leqq n)$, and let $\bar{B} = \bar{B}_i$ and $t = t(\bar{B})$. Since $J(\bar{B})^{t-1} \neq 0$, we get

$$\tilde{f}[J(B)^{t-1}] = [\tilde{f}(J(B))]^{t-1} = [J(\tilde{f}(B))]^{t-1} = \bigoplus_{k=1}^{n} J(\bar{B}_k)^{t-1} \neq 0.$$

Then $J(B)^{t-1} \nsubseteqq \operatorname{Ker} \tilde{f} = J(FP)B$, so that there is some $w \in J(B)^{t-1} - J(FP)B$. We can write $w = \sum_{j=1}^q g_j s_j$ where $s_j \in FP$. Clearly, $w \notin J(FP)FG = FG \cdot J(FP)$. Thus we may assume $s_1 \notin J(FP)$. We can write $s_1 = \sum_{x \in P} c_x x$ where $c_x \in F$. Without the assumption that F is algebraically closed, the result of Wallace [**11**, Lemma 2.3] holds (cf. [**6**]). Hence by [**11**, Lemma 2.3(3)], $\sum_{x \in P} c_x \neq 0$. Let $\hat{P} = \sum_{x \in P} x$ in FG.

Next, we want to claim that $w\hat{P} \neq 0$. Suppose $w\hat{P} = 0$. Since $w\hat{P} = (\sum_j g_j s_j)\hat{P} = \sum_j g_j(s_j\hat{P})$ and since $s_j\hat{P} \in FP$ for all j, we have $s_j\hat{P} = 0$ for all j. Thus $0 = s_1\hat{P} = (\sum_{x \in P} c_x x)\hat{P} = (\sum_{x \in P} c_x)\hat{P}$, so that $\sum_{x \in P} c_x = 0$, a contradiction.

Hence $w\hat{P} \neq 0$. Since $J(FP)^{t(P)-1} = F\hat{P}$ by [**11**, Lemma 2.3(2)] and since $e \cdot J(FP)^h = J(FP)^h e \subseteqq J(B)^h$ for any integer $h \geqq 0$, we have $w\hat{P} \in J(B)^{t+t(P)-2}$. Thus $t(B) \geqq t + t(P) - 1$.

Now, we are ready to prove the following main result of this paper.

Theorem. *Let F be any field of characteristic $p > 0$, G a finite p-solvable group and B a block ideal of FG with defect d. Then we have*

$$t(B) \geqq d(p-1) + 1.$$

Proof. Let E be the algebraic closure of F. By [**8**, Lemma 12.9], we can write $E \otimes_F B = \bigoplus_{i=1}^n B_i^*$ for an integer $n \geqq 1$ where each B_i^* is a block ideal of EG with the same defect d. By [**8**, Corollary 12.12], for any integer $m \geqq 1$ $E \otimes_F J(B)^m = J(E \otimes_F B)^m = \bigoplus_i J(B_i^*)^m$. So $t(B) \geqq t(B_i^*)$ for all i. Thus we may assume that F is algebraically closed.

We prove the theorem by double induction on d and $|G|$.

If $d = 0$, then $J(B) = 0$ (cf. [**1**, Theorem 62.5]), so that it is easy. Thus we may assume $d \geqq 1$, so that $p \mid |G|$.

If $G = \delta(B)$, then $B = FG$, so that it is proved by [**11**, Lemma 2.3(1)].

Let $H = O_{p'}(G)$. Then there is a block ideal b of FH covered by B. Let $T = T_G(b)$. By Lemma 1(7), FT has a block ideal $\tilde{B}$ with the same defect d and $t(\tilde{B}) = t(B)$.

If $G \neq T$, then since $|T| < |G|$ we get the result by induction. Hence we may assume $G = T$.

Then by Lemma 2, there is a finite p-solvable group X such that $O_p(X) \neq 1$ and FX has a block ideal B^* with the same defect d and $t(B^*) = t(B)$. Let $P = O_p(X)$, $|P| = p^r$ and $\bar{X} = X/P$. By [**2**, V Lemma 4.4] and Lemma 3(2), $F\bar{X}$ has a block ideal $\bar{B}$ with defect $d - r$ and $t(B^*) \geqq t(P) + t(\bar{B}) - 1$. By [**11**, Lemma 2.3(1)], $t(P) \geqq r(p-1) + 1$. Since $d - r < d$, we get by induction that $t(\bar{B}) \geqq (d-r)(p-1) + 1$. Therefore

$$t(B) = t(B^*) \geqq t(P) + t(\bar{B}) - 1 \geqq d(p-1) + 1.$$

This completes the proof of the theorem.

Corollary (Wallace [**11,** Theorem 3.3]). *Let F be any field of characteristic $p>0$, G a finite p-solvable group, p^a the order of Sylow p-subgroups of G, and $B_0(G)$ the principal block ideal of FG. Then*

$$t(G) \geqq t(B_0(G)) \geqq a(p-1)+1.$$

Proof. Since $d(B_0(G))=a$, it is clear from Theorem.

Remark. W. Willems [**13**] has also improved the result of Wallace [**11,** Theorem 3.3] (cf. [**13,** 3.5 Theorem and 3.6 Corollary]). But our theorem is not contained in that of Willems.

Let G be a finite p-solvable group such that G has no proper normal subgroups of index prime to p and that FG has a non-principal block ideal B with full defect d, so that $v_p(|G|)=d$ where we use the notation $v_p(n)$ for an integer $n \geqq 1$ as in [**1,** p. 376]. Let S be a simple FG-module in B, and let $K=\operatorname{Ker} S$ where $\operatorname{Ker} S$ is the kernel of S in G.

Assume $v_p(|K|)=d$. Then there is a Sylow p-subgroup D of G such that $D \subseteqq K$. Let $M=\langle g^{-1}Dg \mid g \in G\rangle$. Since $K \lhd G$, $M \subseteqq K$. Since $M \lhd G$ and $p \nmid |G:M|$, $G=M$. Thus $K=G$, so that S is the trivial FG-module. Hence B is the principal block ideal of FG, a contradiction.

Thus for any simple FG-module S in B we get $v_p(|\operatorname{Ker} S|)<d$, so that $v_p(|\operatorname{Ker} S|)\cdot(p-1)+1<d(p-1)+1$. Thus our theorem is not contained in [**13,** 3.5 Theorem (b)].

In fact, there is a finite p-solvable group G which satisfies the above conditions. See our previous example [**7,** Example 3 (pp. 229–230)].

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