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ASYMPTOTIC FORMULAE FOR THE EIGENVALUES OF A TWO PARAMETER SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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1. **Introduction.** The origin and importance of multiparameter Sturm-Liouville problems in mathematical physics has recently been discussed by Atkinson [1, sects. 3 and 4], [2, introduction]. In spite of the importance of such problems, and in spite of the work done in this field by such early investigators as Klein and Hilbert, Atkinson points out that in recent years this field has been relatively neglected in contrast to the single parameter case. As an example, he states that as opposed to the one parameter case, the detailed behaviour of the eigenvalues and eigenfunctions in the multiparameter Sturm-Liouville case is still far from clear.

In light of the above remarks, we wish to present here some results concerning the asymptotic developments for the eigenvalues of the simultaneous two parameter systems

(1a) $$y_1''+(\lambda a_1(x_1)+\mu b_1(x_1)+q_1(x_1))y_1 = 0, \qquad 0 \le x_1 \le 1, \qquad ' = d/dx_1,$$

(1b) $$y_1(0)\cos \alpha_1 - y_1'(0)\sin a_1 = 0, \qquad 0 \le \alpha_1 < \pi,$$
$$y_1(1)\cos \beta_1 - y_1'(1)\sin \beta_1 = 0, \qquad 0 < \beta_1 \le \pi,$$

and

(2a) $$y_2''+(\lambda a_2(x_2)+\mu b_2(x_2)+q_2(x_2))y_2 = 0, \qquad 0 \le x_2 \le 1, \qquad ' = d/dx_2,$$

(2b) $$y_2(0)\cos \alpha_2 - y_2'(0)\sin \alpha_2 = 0, \qquad 0 \le \alpha_2 < \pi,$$
$$y_2(1)\cos \beta_2 - y_2'(1)\sin \beta_2 = 0, \qquad 0 < \beta_2 \le \pi,$$

where we shall assume, unless otherwise stated, that for $i=1, 2$, a_i, b_i, and q_i are real-valued, continuous functions in $0 \le x_i \le 1$. Furthermore, it is also assumed that

(3) $$[a_1(x_1)b_2(x_2) - a_2(x_2)b_1(x_1)] > 0$$

for $0 \le x_i \le 1$, $i=1, 2$; and since this implies that for at least one i, $b_i \ne 0$ in $0 \le x_i \le 1$, we see that there is no loss of generality in assuming in the sequel that $b_2 > 0$ in $0 \le x_2 \le 1$. For clearly this can always be achieved, if necessary, by interchanging the order of systems (1a, 1b) and (2a, 2b) as well as introducing an obvious orthogonal transformation in the parameters λ and μ. In section 2 we summarize

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known results for systems (1a, 1b), (2a, 2b) which we utilize in section 3 to arrive at our main result, namely theorem 5.

2. **Preliminaries.** We now collect some well-known facts concerning systems (1a, 1b), (2a, 2b) which we require for later use. We first need the following definitions. We shall call the tuple (λ^*, μ^*) an eigenvalue of system (1a, 1b) (resp. (2a, 2b)) if (1a) (resp. (2a)), with $\lambda=\lambda^*$ and $\mu=\mu^*$, has a non-trivial solution satisfying (1b) (resp. (2b)); if both λ^* and μ^* are real, then we shall call (λ^*, μ^*) a real eigenvalue. Furthermore, for $i=1, 2$ and for arbitrary values of the parameters λ and μ, we shall denote by $y_i(x_i, \lambda, \mu)$ the solution of (ia) satisfying $y_i(0, \lambda, \mu)=\sin \alpha_i$, $y_i'(0, \lambda, \mu)=\cos \alpha_i$, where $'=d/dx_i$.

2.1. System (2a, 2b). We know from the Sturm theory that for each λ, $-\infty<\lambda<\infty$, the totality of values of μ which renders (2a, 2b) soluble, and with a corresponding non-trivial solution, form a countably infinite set of real numbers which we shall denote by $\{\mu_n(\lambda)\}_{n=0}^{\infty}$, where $\mu_0(\lambda)<\mu_1(\lambda)<\dots$, and $\mu_n(\lambda)\to\infty$ as $n\to\infty$. From Richardson [8, sects. 2 and 3] and [9, sect. 2] we then have the following information (we remark that Richardson considers the case where a_2, b_2, and q_2 are analytic in [0, 1] and $\alpha_2=0$, $\beta_2=\pi$; however for the results stated below, his arguments are in no way vitiated for the system under consideration here). For $n\geq 0$, $\mu_n(\lambda)$ is analytic in $-\infty<\lambda<\infty$; and if $c_n=\{(\lambda, \mu_n(\lambda)) \mid -\infty<\lambda<\infty\}$, then the totality of the real eigenvalues of system (2a, 2b) is precisely the union of the disjoint subsets c_n, $n=0, 1, \dots$, of E_2 (real Euclidean 2-space). Moreover, if $(\lambda^*, \mu^*)\in c_n$, then $y_2(x_2, \lambda^*, \mu^*)$ is a solution of (2a, 2b) having exactly n zeros in (0, 1).

We may now look upon the sets c_n as curves in the (λ, μ)-plane. In this plane we introduce angle in the usual way, and denote by ϕ the angle which a ray emanating from the origin makes with the positive λ-axis. Put

$$g(x_2) = (a_2(x_2)/b_2(x_2)), \qquad 0 \leq x_2 \leq 1,$$

$$G = \sup_{0\leq x_2\leq 1} g(x_2), \qquad G^* = \inf_{0\leq x_2\leq 1} g(x_2),$$

$$\phi_1 = \tan^{-1}\{-G\}, \quad \text{and} \quad \phi_2 = \tan^{-1}\{-G^*\},$$

where the principal branch of the inverse tangent is taken. Denoting by $c_n(\lambda)$ the point of c_n having λ as abscissa, we have

THEOREM 1. *Let ε be any number satisfying* $0<\varepsilon<\min\{(\phi_1+\pi/2), (-\phi_1+\pi/2)\}$. *Let n be any nonnegative integer. Then there exists the positive number $\lambda_n^\dagger(\varepsilon)$ such that $c_n(\lambda)$ lies in the sector $(\phi_1-\varepsilon)<\phi<(\phi_1+\varepsilon)$ for $\lambda\geq\lambda_n^\dagger(\varepsilon)$.*

Proof. Consider the solution $y(x_2, \lambda)$ of the equation $y''+[\lambda(a_2(x_2)+\Delta(x_2))+q_2(x_2)]y=0$, $0\leq x_2\leq 1$, $'=d/dx_2$, which satisfies the initial conditions $y(0, \lambda)=\sin \alpha_2$, $y'(0, \lambda)=\cos \alpha_2$, and where $\Delta(x_2)=b_2(x_2)\tan(\phi_1+\varepsilon)$. Observing from above that $(a_2+\Delta)>0$ in at least some proper subinterval of [0, 1], we conclude from

[7, p. 227] that the number of zeros of y in (0, 1) exceeds n for all λ sufficiently large. Hence if $\{\lambda_j\}_{j=1}^{\infty}$ is an increasing sequence of positive numbers tending to infinity with j such that $\mu_n(\lambda_j)\geq\lambda_j\tan(\phi_1+\varepsilon)$ for each j, then we know from the Sturm comparison theorem that the number of zeros of $y_2(x_2, \lambda_j, \mu_n(\lambda_j))$ in (0, 1) exceeds n for all large j. Since this is impossible, we conclude that $c_n(\lambda)$ lies in the sector $-\pi/2<\phi<(\phi_1+\varepsilon)$ for all λ sufficiently large. In a similar manner me can also show that $c_n(\lambda)$ lies in the sector $(\phi_1-\varepsilon)<\phi<\pi/2$ for all λ sufficiently large, and the theorem follows.

From theorem 1 we conclude that for $n\geq 0$, $\{\mu_n(\lambda)/\lambda\}\to -G$ as $\lambda\to\infty$. In [4] we have sharpened this result by imposing certain restrictions on the coefficients of (2a); and we have

THEOREM 2. *Let both $a_2(x_2)$ and $b_2(x_2)$ belong to C_4 in* [0, 1]. *Let $g(x_2)=G$ at precisely the finite set of points $\{h_j\}_{j=1}^{p}$, where $p\geq 1$, $0\leq h_1<h_2<\cdots<h_p\leq 1$, and $g'(h_j)=0$, $g''(h_j)<0$, $j=1,\ldots,p$. Then for each $n\geq 0$ there exists the numbers $G_{i,n}$, $i=1, 2, 3$, such that*

$$\mu_n(\lambda) = -\lambda G+\lambda^{1/2}G_{1,n}+\lambda^{1/4}G_{2,n}+G_{3,n}+o(1)$$

as $\lambda\to\infty$, (and not uniformly with respect to n).

2.2. System (1a, 1b). Noting from (3) that $\psi(x_1, \phi)=(a_1(x_1)+b_1(x_1)\tan\phi)>0$ in $0\leq x_1\leq 1$ and $\phi_1\leq\phi\leq\phi_2$, it then follows from the works of Richardson cited above that the totality of the real eigenvalues of system (1a, 1b) is the union of a countably infinite number of disjoint analytic curves in E_2 which we shall denote by S_n, $n=0, 1, \ldots$. Moreover, if $(\lambda^*, \mu^*)\in S_n$, then $y_1(x_1, \lambda^*, \mu^*)$ is a solution of (1a, 1b) having exactly n zeros in (0, 1); and if d_n denotes the minimum distance from the zero element of E_2 to S_n, then $d_n\to\infty$ as $n\to\infty$.

We now choose the numbers ϕ_1^* and ϕ_2^*, where $-\pi/2<\phi_1^*<\phi_1\leq\phi_2<\phi_2^*<\pi/2$, so that $\psi>0$ in $0\leq x_1\leq 1$ and $\phi_1^*\leq\phi\leq\phi_2^*$. Hence, from Richardson we know that if $\phi_1^*\leq\phi\leq\phi_2^*$, then in the (λ, μ)-plane a straight line through the origin with slope $\tan\phi$ intersects each S_n in exactly one point, which we denote by $(\lambda_n(\phi), \mu_n(\phi))$, $n=0, 1, \ldots$, where $\lambda_0(\phi)<\lambda_1(\phi)<\ldots$, and $\lambda_n(\phi)\to\infty$ as $n\to\infty$. Moreover, for $n\geq 0$, $\lambda_n(\phi)$ and $\mu_n(\phi)$ are analytic in $[\phi_1^*, \phi_2^*]$; and if N_1 denotes the smallest integer greater than the number of zeros of $y_1(x_1, 0, 0)$ in $0<x_1<1$, then $\lambda_n(\phi)>0$ for $n\geq N_1$. Finally, we observe that if for $n\geq N_1$ we denote by S_n^+ the subset of S_n lying in the sector $\phi_1^*\leq\phi\leq\phi_2^*$, then S_n^+ is a Jordan arc and can be represented parametrically by putting $\lambda=\lambda_n(\phi)$, $\mu=\mu_n(\phi)$, $\phi_1^*\leq\phi\leq\phi_2^*$.

2.3. Systems (1a, 1b), (2a, 2b). Referring to the simultaneous systems (1a, 1b), (2a, 2b) as system (1-2), we shall call the tuple (λ^*, μ^*) an eigenvalue of this system if, with $\lambda=\lambda^*$ and $\mu=\mu^*$, (*ia*) has a non-trivial solution satisfying (*ib*) for $i=1, 2$. From the results of Atkinson [3, p. 551, problem 16 and pp. 160–168] (see also [7, pp. 248–251]) it readily follows that the eigenvalues of system (1-2) form an

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infinite subset of E_2, and moreover, if p_1 and p_2 are any pair of nonnegative integers, then there is exactly one eigenvalue of system (1-2), say (λ^*, μ^*), such that $y_i(x_i, \lambda^*, \mu^*)$ has precisely p_i zeros in $0<x_i<1$, $i=1, 2$. We see from this that the set of eigenvalues of system (1-2) may be denoted by $\{(\lambda_{i,j}, \mu_{i,j})\}_{i,j=0}^{\infty}$, where $y_1(x_1, \lambda_{i,j}, \mu_{i,j})$ has precisely i zeros in $0<x_1<1$, and $y_2(x_2, \lambda_{i,j}, \mu_{i,j})$ has precisely j zeros in $0<x_2<1$. Furthermore, it now follows (see Richardson [8, sect. 4]) that the eigenvalues of system (1-2) and the points of intersection of the sets S_n with the sets c_n are identical. Indeed, if i and j are any pair of non-negative integers, then S_i intersects c_j in precisely one point, namely at the eigenvalue of system (1-2), $(\lambda_{i,j}, \mu_{i,j})$.

Finally, for a further discussion of the results of section 2 we refer to [5, chapts. 2 and 3].

3. **Main results.** In this section we shall fix the non-negative integer m and use the results of section 2 to obtain asymptotic developments for $\lambda_{n,m}$ and $\mu_{n,m}$ as $n\to\infty$.

From theorem 1 we see that there is a positive number $\lambda^\dagger$ such that $c_m(\lambda)$ lies in the sector $\phi_1^*<\phi<\phi_2^*$ for $\lambda\geq\lambda^\dagger$. Also, from subsection 2.2 we note that if d_n is the minimum distance from the zero element of E_2 to S_n and $d(\lambda)$ the distance from the zero element to $c_m(\lambda)$, then we can choose the integer $N>N_1$ large enough so that $d_n>2d(\lambda^\dagger)$ for $n\geq N$.

THEOREM 3. *For $n\geq N$, $(\lambda_{n,m}, \mu_{n,m})$, the unique point of intersection of S_n with c_m, lies in the sector $\phi_1^*<\phi<\phi_2^*$. Moreover, $0<\lambda^\dagger<\lambda_{N,m}<\lambda_{N+1,m}<\lambda_{N+2,m}<\dots$, and $\lambda_{n,m}\to\infty$ as $n\to\infty$.*

Proof. Referring to subsection 2.2, we see that for $n\geq N$ we may introduce into E_2 the Jordan curve γ_n given by:

$$\lambda = 3t\lambda_n(\phi_1^*), \qquad \mu = 3t\mu_n(\phi_1^*), \qquad 0\leq t\leq \tfrac{1}{3},$$

$$\lambda = \lambda_n(\phi(t)), \qquad \mu = \mu_n(\phi(t)), \qquad \phi(t) = (2-3t)\phi_1^*+(3t-1)\phi_2^*, \quad \tfrac{1}{3}<t\leq\tfrac{2}{3},$$

$$\lambda = 3(1-t)\lambda_n(\phi_2^*), \qquad \mu = 3(1-t)\mu_n(\phi_2^*), \qquad \tfrac{2}{3}<t\leq 1.$$

We observe that the arc obtained by restricting the parameter t to the interval $\frac{1}{3}\leq t\leq\frac{2}{3}$ is precisely S_n^+. From the Jordan theorem we know that γ_n separates E_2 into two disjoint regions, say Ω_n and Ω_n^*, having γ_n as common boundary; moreover, if Ω_n is the bounded region, then it is clear that

$$\Omega_n = \{(\lambda,\mu) \mid (\lambda,\mu)\in E_2,\ \lambda = \tau\lambda_n(\phi(t)),\ \mu = \tau\mu_n(\phi(t)),\ 0<\tau<1,\ \tfrac{1}{3}<t<\tfrac{2}{3}\}.$$

We note that $c_m(\lambda^\dagger)\in\Omega_n$; and if we denote by c_m^+ the curve $\{c_m(\lambda) \mid \lambda^\dagger\leq\lambda<\infty\}$, then c_m^+ lies in the sector $\phi_1^*<\phi<\phi_2^*$. Since Ω_n is bounded, we conclude that c_m^+ intersects S_n^+. From subsection 2.3 it then follows that c_m^+ intersects S_n^+ in precisely one point, namely $(\lambda_{n,m}, \mu_{n,m})$. Thus $(\lambda_{n,m}, \mu_{n,m})$ lies in the sector $\phi_1^*<\phi<\phi_2^*$ and $c_m(\lambda)\in\Omega_n^*$ for $\lambda>\lambda_{n,m}$.

Assume now that for some $n \geq N$, $\lambda_{n+1,m} < \lambda_{n,m}$; and for $0 \leq t \leq 1$, denote by $\gamma_n(t)$ the point of γ_n corresponding to t. Then we know from above that there is a t^*, $\frac{1}{3} < t^* < \frac{2}{3}$, such that $\gamma_n(t^*) = (\lambda_{n,m}, \mu_{n,m})$ and $\gamma_n(t^*) \in \Omega^*_{n+1}$. On the other hand, from subsection 2.2 and the definition of Ω_{n+1}, we see that $\gamma_n(t) \in \Omega_{n+1}$ for $\frac{1}{3} < t < \frac{2}{3}$. Thus we must have $\lambda_{n,m} < \lambda_{n+1,m}$. This and the results of subsection 2.2 complete the proof of our theorem.

We now choose the number h_1, where $0 \leq h_1 \leq 1$, so that $g(h_1) = G$ (see subsection 2.1). Put

$$\Delta(x_1) = (b_2(h_1))^{-1}[a_1(x_1)b_2(h_1) - b_1(x_1)a_2(h_1)], \qquad 0 \leq x_1 \leq 1,$$

$$D = \int_0^1 (\Delta(x_1))^{1/2}\, dx_1,$$

where here and in the sequel the positive root of a positive quantity is always taken.

THEOREM 4. *As* $n \to \infty$,

$$\lambda_{n,m} = [n\pi/D]^2[1+o(1)], \qquad \mu_{n,m} = [n\pi/D]^2[-G+o(1)].$$

Proof. From theorem 3 and subsection 2.1 we see that for $n \geq N$, $\mu_{n,m} = -\lambda_{n,m}G + r_n$, where $r_n = o(\lambda_{n,m})$ as $n \to \infty$. For $n \geq N$, put $Q_n = (1 + |r_n|\, B + Q)$, where B (resp. Q) is the supremum of $|b_1|$ (resp. $|q_1|$) in $0 \leq x_1 \leq 1$, and choose the integer $N^* > N$ large enough so that for $n \geq N^*$, $\{Q_n/\lambda_{n,m}\delta^2\} < \frac{1}{2}$, where δ is the infimum of $\Delta^{1/2}$ in $0 \leq x_1 \leq 1$. Now for any $\varepsilon > 0$ there exists the partition of [0, 1], $0 = t_0 < t_1 < \cdots < t_p = 1$, $p > 2$, such that $\sum_{i=1}^p M_i(t_i - t_{i-1}) < (D+\varepsilon)$, and

$$\sum_{i=1}^p m_i(t_i - t_{i-1}) > (D-\varepsilon),$$

where M_i (resp. m_i) is the supremum (resp. infimum) of $\Delta^{1/2}$ in $[t_{i-1}, t_i]$, $i = 1, \ldots, p$. Observing that for $n \geq N^*$, $y_1(x_1, \lambda_{n,m}, \mu_{n,m})$ satisfies the differential equation

$$y'' + (\lambda_{n,m}\Delta(x_1) + r_n b_1(x_1) + q_1(x_1))y = 0, \qquad 0 \leq x_1 \leq 1, \quad ' = d/dx_1,$$

it follows from a simple application of the Sturm fundamental theorem to each of the intervals $[t_{i-1}, t_i]$ that

$$n < \left\{(l_n/\pi) \sum_{i=1}^p M_i(t_i - t_{i-1}) + Q_n/\pi\, \delta l_n + 2p\right\},$$

$$n > \left\{(l_n/\pi) \sum_{i=1}^p m_i(t_i - t_{i-1}) - Q_n/\pi\, \delta l_n - 2p\right\},$$

where $l_n^2 = \lambda_{n,m}$. Hence the limit superior of the sequence $\{n\pi/\lambda_{n,m}^{1/2}D\}_{n=N}^\infty$ does not exceed $(1+\varepsilon/D)$ and the limit inferior is not less than $(1-\varepsilon/D)$. Since ε is arbitrary, our theorem follows.

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We note that the proof of theorem 4 depends upon the fact that $\mu_m(\lambda)=\lambda(-G+o(1))$ as $\lambda\to\infty$. Since in [4] we have been able to sharpen this result, it is natural to ask whether the results of [4] can be used to improve the results of theorem 4. In the sequel we show that this can be done provided that certain conditions are fulfilled. Hence in the following theorem it will be assumed that (i) the hypotheses of theorem 2 are satisfied, and (ii) both $a_1(x_1)$ and $b_1(x_1)$ belong to C_2 in [0, 1]. Let $\Delta(x_1)$ and D be defined as in theorem 4 (where now h_1 is given by theorem 2) and put

$$D^* = \int_0^1 b_1(x_1)(\Delta(x_1))^{-1/2}\, dx_1,$$

$$F = D\left[\int_0^1 (\Delta(x_1))^{-1/2}[q_1(x_1)+(\Delta(x_1))^{1/4}d^2[(\Delta(x_1))^{-1/4}]/dx_1^2]\, dx_1\right],$$

$$F^\dagger = 2D[(\Delta(1))^{-1/2}\cot\beta_1+(\tfrac{1}{4})\Delta'(1)(\Delta(1))^{-3/2}] \quad \text{if} \quad \beta_1\neq\pi,$$

$$F^* = -2D[(\Delta(0))^{-1/2}\cot\alpha_1+(\tfrac{1}{4})\Delta'(0)(\Delta(0))^{-3/2}] \quad \text{if} \quad \alpha_1\neq 0,$$

where $'=d/dx_1$. We shall also put (see theorem 2),

$$D_{1,m} = D^*G_{1,m}/2, \qquad D_{2,m} = D^*G_{2,m}/2,$$

$$D_{3,m} = D^*G_{3,m}/2-(G_{1,m}^2/8)\int_0^1 (b_1(x_1))^2(\Delta(x_1))^{-3/2}\, dx_1,$$

$$A_{1,m} = -2D_{1,m}, \qquad A_{2,m} = -2D^{1/2}D_{2,m}, \qquad A_{3,m} = (D_{1,m}^2-2DD_{3,m}-F),$$

$$B_{1,m} = (DG_{1,m}-GA_{1,m}), \qquad B_{2,m} = (D^{3/2}G_{2,m}-GA_{2,m}),$$

$$B_{3,m} = (D^2G_{3,m}-DG_{1,m}D_{1,m}-GA_{3,m}).$$

THEOREM 5. *As* $n\to\infty$,

(i) $$\lambda_{n,m} = [(n+1)\pi/D]^2[1+A_{1,m}/(n+1)\pi+A_{2,m}/((n+1)\pi)^{3/2} + A_{3,m}/((n+1)\pi)^2+o(1/n^2)],$$

$$\mu_{n,m} = [(n+1)\pi/D]^2[-G+B_{1,m}/(n+1)\pi+B_{2,m}/((n+1)\pi)^{3/2} + B_{3,m}/((n+1)\pi)^2+o(1/n^2)],$$

if $\alpha_1=0$ *and* $\beta_1=\pi$;

(ii) $$\lambda_{n,m} = [(n+\tfrac{1}{2})\pi/D]^2[1+A_{1,m}/(n+\tfrac{1}{2})\pi+A_{2,m}/((n+\tfrac{1}{2})\pi)^{3/2} +(A_{3,m}-F^\dagger)/((n+\tfrac{1}{2})\pi)^2+o(1/n^2)],$$

$$\mu_{n,m} = [(n+\tfrac{1}{2})\pi/D]^2[-G+B_{1,m}/(n+\tfrac{1}{2})\pi+B_{2,m}/((n+\tfrac{1}{2})\pi)^{3/2} +(B_{3,m}+GF^\dagger)/((n+\tfrac{1}{2})\pi)^2+o(1/n^2)],$$

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if $\alpha_1=0$ *and* $\beta_1\neq\pi$;

$$\text{(iii)}\quad \lambda_{n,m} = [n\pi/D]^2[1+A_{1,m}/n\pi+A_{2,m}/(n\pi)^{3/2} +(A_{3,m}-F^\dagger-F^*)/(n\pi)^2+o(1/n^2)],$$

$$\mu_{n,m} = [n\pi/D]^2[-G+B_{1,m}/n\pi+B_{2,m}/(n\pi)^{3/2} +(B_{3,m}+GF^\dagger+GF^*)/(n\pi)^2+o(1/n^2)],$$

if $\alpha_1\neq 0$ *and* $\beta_1\neq\pi$;

$$\text{(iv)}\quad \lambda_{n,m} = [(n+\tfrac{1}{2})\pi/D]^2[1+A_{1,m}/(n+\tfrac{1}{2})\pi+A_{2,m}/((n+\tfrac{1}{2})\pi)^{3/2} +(A_{3,m}-F^*)/((n+\tfrac{1}{2})\pi)^2+o(1/n^2)],$$

$$\mu_{n,m} = [(n+\tfrac{1}{2})\pi/D]^2[-G+B_{1,m}/(n+\tfrac{1}{2})\pi+B_{2,m}/((n+\tfrac{1}{2})\pi)^{3/2} +(B_{3,m}+GF^*)/((n+\tfrac{1}{2})\pi)^2+o(1/n^2)],$$

if $\alpha_1\neq 0$ *and* $\beta_1=\pi$.

Proof. We shall only consider the case $\alpha_1=0$, $\beta_1=\pi$; the other cases may be similarly treated Now we have seen that $c_m(\lambda)$ lies in the sector $\phi_1^*<\phi<\phi_2^*$ for $\lambda\geq\lambda^\dagger$; hence from subsections 2.1 and 2.2 it follows that

$$L(\lambda) = \int_0^1 (\lambda a_1(x_1)+\mu_m(\lambda)b_1(x_1))^{1/2}\,dx_1$$

is analytic in $\lambda^\dagger\leq\lambda<\infty$. If for $\lambda\geq\lambda^\dagger$ we put

$$L(\lambda) = \lambda^{1/2}D+D_1+\lambda^{-1/4}D_2+\lambda^{-1/2}D_3+h(\lambda),$$

where $D_i=D_{i,m}$, $i=1, 2, 3$, then $h(\lambda)$ is analytic in $\lambda^\dagger\leq\lambda<\infty$; also, a simple computation involving the use of theorem 2 shows that $h(\lambda)=o(\lambda^{-1/2})$ as $\lambda\to\infty$. From theorem 3 it then follows that for $n\geq N$,

$$(4)\qquad \begin{aligned} L_n &= \int_0^1 (\lambda_{n,m}a_1(x_1)+\mu_{n,m}b_1(x_1))^{1/2}\,dx_1 \\ &= \lambda_{n,m}^{1/2}D+D_1+\lambda_{n,m}^{-1/4}D_2+\lambda_{n,m}^{-1/2}D_3+h(\lambda_{n,m}), \end{aligned}$$

where $h(\lambda_{n,m})=o(\lambda_{n,m}^{-1/2})$ as $n\to\infty$.

Now from subsection 2.2 we know that $\psi(x_1, \phi)>0$ in $0\leq x_1\leq 1$ and $\phi_1^*\leq\phi\leq\phi_2^*$. Hence there are positive numbers δ_1, δ_2 such that $\delta_1\leq\psi\leq\delta_2$ in this rectangle. For $n\geq N$ put $\psi_n(x_1)=\psi(x_1, \phi_n)$, $0\leq x_1\leq 1$, where $\phi_n=\tan^{-1}\{\mu_{n,m}/\lambda_{n,m}\}$ and the principal branch of the inverse tangent is taken. Then $\delta_1\leq\psi_n\leq\delta_2$ in $0\leq x_1\leq 1$. Moreover, both $|\psi_n'|$ and $|\psi_n''|$ remain less than some bound independent of n and x_1, where $'=d/dx_1$.

We now show that as $n\to\infty$,

$$(5)\qquad L_n = (n+1)\pi-\gamma_n/(n+1)\pi+o(1/n),$$

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where, with

$$H_n(x_1)=\int_0^{x_1}\psi_n^{1/2}\,d\tau,\qquad H_n^*(x_1)=[q_1+\psi_n^{1/4}(\psi_n^{-1/4})''],\qquad 0\le x_1\le 1,\qquad '=d/dx_1,$$

$$\gamma_n=(H_n(1)/2)\int_0^1\psi_n^{-1/2}H_n^*\,dx_1.$$

We argue as in [7, pp. 271–273]. For $n\ge N$ put

$$t_n=H_n^{-1}(1)H_n(x_1),\qquad 0\le x_1\le 1,\qquad Y_n(t_n)=(\psi_n(x_1))^{1/4}y_1(x_1,\lambda_{n,m},\mu_{n,m}),$$

$$\rho_n^2=H_n^2(1)\lambda_{n,m},\qquad Q_n(t_n)=(\psi_n(x_1))^{-1}H_n^2(1)H_n^*(x_1),\qquad \xi_n=\rho_nt_n,$$

and

$$\delta_n^*=(\psi_n(0))^{-1/4}H_n(1).$$

Then for $0\le t_n\le 1$,

$$Y_n(t_n)=\rho_n^{-1}\left[\delta_n^*\sin\xi_n-\int_0^{t_n}\sin(\xi_n-\rho_n\tau)Q_n(\tau)Y_n(\tau)\,d\tau\right]. \tag{6}$$

In light of the above remarks concerning bounds for ψ_n and its first two derivatives, it now follows from the Gronwall lemma that the supremum in [0, 1] of $|Y_n|$, of $|Y_n'|$ $('=d/dt_n)$, and of $\rho_n^2\,|Y_n-\delta_n^*\rho_n^{-1}\sin\xi_n|$ all remain less than some bound independent of n. Hence, using an argument similar to that used in the proof of the Riemann-Lebesgue lemma, we deduce from (6) that

$$Y_n(t_n)=\rho_n^{-1}\delta_n^*\left[\sin\xi_n+(2\rho_n)^{-1}\left(\int_0^{t_n}Q_n(\tau)\,d\tau\right)\cos\xi_n+\rho_n^{-1}z_{1,n}(t_n)\right],$$

$$Y_n'(t_n)=\delta_n^*[\cos\xi_n+z_{2,n}(t_n)],$$

where, for $i=1, 2$, the supremum in [0, 1] of $|z_{i,n}|$ tends to zero as $n\to\infty$. Thus, since Y_n has n zeros in (0, 1) and $Y_n(1)=0$, a standard argument now shows that $\rho_n=\pi(n+1+\varepsilon_n)$, where

$$\tan\pi\varepsilon_n=\left[-(2\rho_n)^{-1}\int_0^1Q_n(t_n)\,dt_n+\rho_n^{-1}\varepsilon_n^*\right]$$

and $\varepsilon_n^*=o(1)$ as $n\to\infty$. From this (5) follows.

From (4), (5), and theorems 2 and 3 we conclude that as $n\to\infty$, $(\lambda_{n,m}^{1/2}D/(n+1)\pi)\to 1$ and $(\mu_{n,m}/\lambda_{n,m})=(-G+O(1/n))$. A simple calculation now shows that $\gamma_n=((F/2)+O(1/n))$ as $n\to\infty$. Hence, from (4) and (5), we see that if for $n\ge N$ we put

$$\lambda_{n,m}^{1/2}=[(n+1)\pi/D][1-D_1/(n+1)\pi-D^{1/2}D_2/((n+1)\pi)^{3/2}$$
$$-(DD_3+F/2)/((n+1)\pi)^2+\varepsilon_n^\dagger],$$

then $\varepsilon_n^\dagger=o(1/n^2)$ as $n\to\infty$. Our results follow from this and theorems 2 and 3.

To conclude, we remark that the formulae given in theorem 5 have been obtained to an accuracy determined by the conditions which we have imposed upon the coefficients of the differential equations (1a) and (2a) and under suitable conditions, these formulae may be further developed. This follows from the fact that our results depend upon the formulae of theorem 2 and upon formulae of the type given in (5) (see [7, pp. 272–273]), and in the manner described in [4, sect. 1] and [6], these formulae may be further developed for suitable coefficients in our differential equations.

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