(ii) $$\psi(t) - \log t + \frac{1}{2t} + \frac{1}{12t^2} = -\frac{1}{6}\sum_{j=0}^{\infty}\int_t^{\infty}\frac{dx}{(x+j)^3(x+j+1)^3}.$$

Proof: Equation (5) implies (i) and the difference, (5) – (8), implies (ii).

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107.23 Location of the inarc circle and its point of contact with the circumcircle

The inarc circle of a triangle

An *inarc circle* of a triangle is a circle tangent to two sides of a triangle and internally to the circumcircle of the triangle, see Figure 1. In this note we consider first the interesting problem of locating the *inarc centre*, the centre of this circle, L_A, and then as a second problem we locate the point of tangency T of the inarc circle and the circumcircle. In [1] the first problem is solved geometrically by beautiful application of inversion. We will use simple algebra, one well-known theorem and one famous formula.

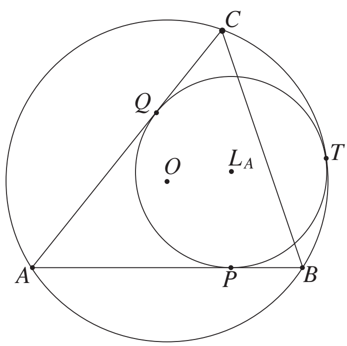

FIGURE 1: An inarc circle

In a triangle ABC with circumcentre O, circumradius R, incentre I and inradius r, we consider the inarc circle opposite vertex A. Let the line through the incentre I perpendicular to AI meet the sides AB and AC at the points P and Q respectively. Let the intersection of the line through P perpendicular to AB and the line AI be the point L_A. We claim that the circle with centre L_A and radius $\rho_A = L_AP$ is the inarc circle, see Figure 2. To prove this claim, it is enough to show that the distance OL_A is equal to $R - \rho_A$.

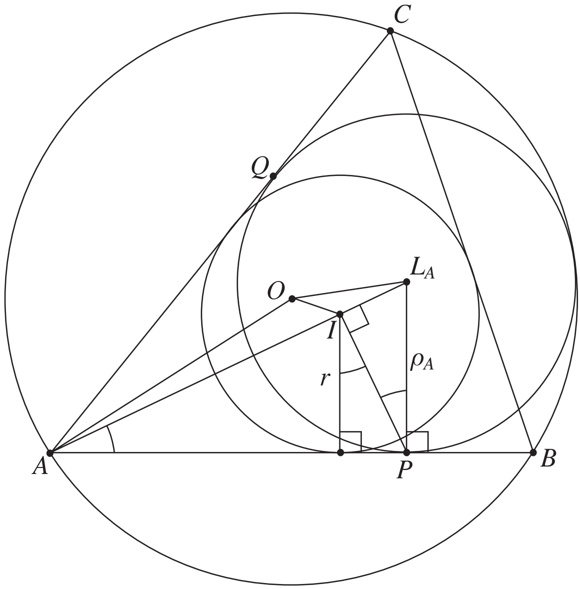

FIGURE 2: Construction of inarc centre L_A

We have

$$AI = \frac{r}{\sin \frac{1}{2}A}.$$

Moreover, in triangle PL_AI from $PI = r / \cos \frac{1}{2}A$, follows

$$\rho_A = \frac{PI}{\cos \frac{1}{2}A} = \frac{r}{\cos^2 \frac{1}{2}A}$$

and

$$L_AI = \rho_A \sin \tfrac{1}{2}A = \frac{r \tan \frac{1}{2}A}{\cos \frac{1}{2}A}.$$

We will make use of Stewart's theorem [2], which says that in triangle ABC with sides $BC = a$, $CA = b$, $AB = c$, cevian $AD = d$ and segments $BD = m$, $DC = n$, the relationship

$$b^2m + c^2n = a\left(d^2 + mn\right)$$

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holds, see Figure 3. Since $a = m + n$, this can be written as

$$\begin{aligned} b^2 &= d^2 + mn + \frac{n}{m}\left(d^2 + mn - c^2\right) \\ &= d^2 + n^2 + \frac{n}{m}\left(m^2 + d^2 - c^2\right). \end{aligned} \tag{1}$$

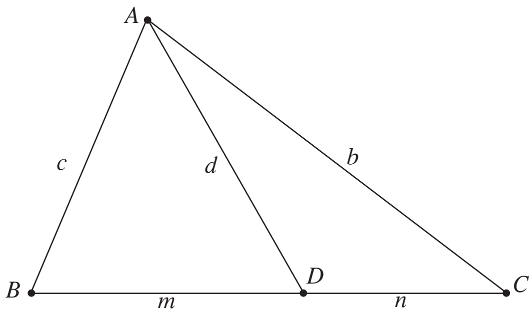

FIGURE 3: Stewart's theorem: $b^2 m + c^2 n = a\left(d^2 + mn\right)$

Stewart's theorem is to be applied to triangle OAL_A with cevian OI. Replacing $b = OL_A$, $d = OI$, $n = L_A I$, $m = AI$, $c = OA$ in (1) and recalling the famous Euler's formula [1] for the distance $OI^2 = R(R - 2r)$, we obtain

$$\begin{aligned} OL_A^2 &= OI^2 + L_A I^2 + \frac{L_A I}{AI}\left(AI^2 + OI^2 - OA^2\right) \\ &= R^2 - 2Rr + \frac{r^2 \tan^2 \frac{1}{2}A}{\cos^2 \frac{1}{2}A} + \tan^2 \tfrac{1}{2}A\left(\frac{r^2}{\sin^2 \frac{1}{2}A} - 2Rr\right) \\ &= R^2 - \frac{2Rr}{\cos^2 \frac{1}{2}A} + \frac{r^2}{\cos^4 \frac{1}{2}A} \\ &= \left(R - \frac{r}{\cos^2 \frac{1}{2}A}\right)^2 = (R - \rho_A)^2. \end{aligned}$$

It follows that $OL_A = R - \rho_A$.

Since $R = OL_A + \rho_A$, the circle $\left(L_A, \rho_A\right)$ is internally tangent to the circle (O, R), i.e. to the circumcircle, and this is sufficient for the claim that the point L_A is the inarc centre.

The point of tangency of the inarc circle and the circumcircle

Next we locate the point of tangency of the inarc circle and the circumcircle, which we call T. The theorem which follows uses the following result from triangle geometry:

Lemma

In the triangle DEF, let DD_1 be the median at D and let X be the intersection of the tangents to the circumcircle at E and F. Then $\angle FDD_1 = \angle EDX$. DX is known as the symmedian at D, see Figure 4.

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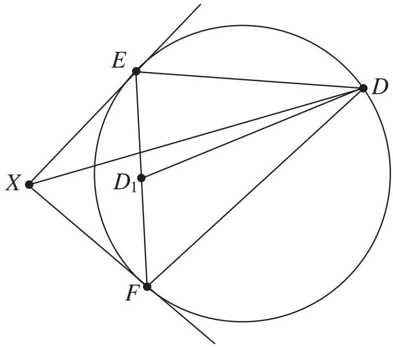

FIGURE 4: Characterisation of symmedian at D in terms of tangents

Proof: A proof can be found in [1, p.101].

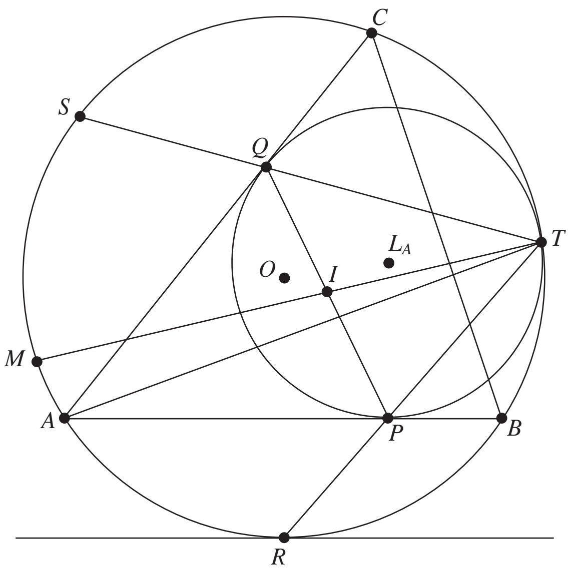

FIGURE 5: The inarc circle and its point of tangency T with the circumcircle

Theorem

Let M, R and S be the other intersections of TI, TP and TQ with the circumcircle. Then these three points are the midpoints of the arcs $\overset{\frown}{BC}$, $\overset{\frown}{AB}$ and $\overset{\frown}{CA}$.

Proof: Consider the homothecy with centre T which sends the inarc circle to the circumcircle, see Figure 5. It maps P to R and the image of AB, which is tangent to the inarc circle, is tangent to the circumcircle and parallel to AB.

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Hence R is the midpoint of the arc $\widehat{AB}$. The same argument works for the point S.

Now we prove that M is the midpoint of arc $\widehat{BC}$ by showing that $\angle BTI = \angle CTI$.

First we see that

$$\angle BTP = \angle BTR = \angle BCR = \tfrac{1}{2}\angle C.$$

Now we note that TI is a median of triangle TQP and, by the Lemma above, AT is a symmedian. It follows that

$$\angle PTI = \angle QTA = \angle STA = \angle SBA = \tfrac{1}{2}\angle B.$$

Therefore $\angle BTI = \frac{1}{2}\left(\angle B + \angle C\right)$ and as exactly the same argument works for $\angle CTI$, we are finished.

There are interesting inequalities for the distances of the inarc centres L_A, L_B, L_C to the vertices, to the incentre, and also inequalities for the inarc radii of the three inarc circles, ρ_A, ρ_B and ρ_C. Examples of such inequalities involving the circumradius and the inradius of the triangle are

$$8r \leqslant AL_A + BL_B + CL_C \leqslant 3R + 2r,$$

$$2r \leqslant IL_A + IL_B + IL_C \leqslant R,$$

$$4r \leqslant \rho_A + \rho_B + \rho_C \leqslant R + 2r,$$

see [3], [4] and [5].

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