ON THE BUSY PERIOD IN THE QUEUEING SYSTEM *GI/G/1*

P. D. FINCH

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1. Introduction

A number of authors have studied busy period problems for particular cases of the general single-server queueing system. For example, using the now standard notation of Kendall [7], $GI/M/1$ was studied by Conolly [3] and Takacs [18]. Earlier work on $M/M/1$ includes that of Ledermann and Reuter [8] and Bailey [1]. Kendall [6], Takacs [17], and Prabhu [11] have considered $M/G/1$.

More recently Conolly [4] has given a detailed study of the busy period in $GI/E_k/1$ and promises a further study of $E_k/G/1$. In [19] Takacs has also considered the system $E_k/G/1$.

We shall shew that results for the busy period of $GI/G/1$ can be obtained by means of a combinatorial lemma due to Spitzer [15]. Combinatorial methods in the study of sums of random variables were introduced by Sparre Andersen [13], [14]. His methods were, however, very difficult and were simplified by Spitzer [15]. In [16] Spitzer applied the method to the Wiener-Hopf integral equation obtained by Lindley [9] for the limiting distributing of waiting time in the queueing system $GI/G/1$.

Recently Feller [5] has shewn how the combinatorial methods of Spitzer and Sparre Andersen can be simplified considerably. In the next section we follow the treatment of Feller and give the fundamental Combinatorial lemma due to Spitzer. This lemma will be used in our study of the busy period of $GI/G/1$. It is possible to extend the methods of this paper to study the transient behaviour of $GI/G/1$ and we shall do so in a later paper in which we use the results obtained below.

2. Combinatorial Considerations

Let $(X_1, X_2, \cdots, X_n)$ be an ordered n-tuple of real numbers and define the partial sums S_k by the equations

$$(2.1) \qquad S_0 = 0, \quad S_k = \sum_{j=1}^{k} X_j \qquad k = 1, 2, \cdots, n.$$

We say that $k \geqq 1$ is a ladder index of the partial sums $(S_0, S_1 \cdots, S_n)$ if

$$S_k \geqq S_j \qquad j = 0, 1, \cdots k-1. \quad k \geqq 1. \tag{2.2}$$

If the inequality in (2.2) is a strict inequality we say that k is a strict ladder index.

A (strict) ladder index is said to be the mth ladder index if it is preceded by $(m-1)$ other (strict) ladder indices.

For any integral a, $0 \leqq a < n$, we define a cyclic rearrangement $(X_1(a), X_2(a), \cdots, X_n(a))$ of the n-tuple $(X_1, X_2, \cdots, X_n)$ by the equations

$$\begin{aligned} X_i(a) &= X_{i+a} \qquad i = 1, 2, \cdots (n-a). \\ X_i(a) &= X_{i-n+a} \quad i = (n-a+1), \cdots, n. \end{aligned} \tag{2.3}$$

Write $S_0(a) = 0$ and $S_k(a) = \sum_{j=1}^{k} X_j(a)$. We shall continue to write X_i, S_i, in the case $a = 0$. From (2.3) we obtain

$$\begin{aligned} S_k(a) &= S_{a+k} - S_a \qquad\qquad k = 1, 2 \cdots, (n-a) \\ S_k(a) &= S_n - S_a + S_{k-n+a} \quad k = (n-a+1), \cdots, n. \end{aligned} \tag{2.4}$$

The following theorem is proved by Feller [5].

Theorem F.

If $S_n \geqq 0$ there is at least one a, $0 \leqq a < n$, that is at least one cyclic rearrangement, such that n is a ladder index of the partial sums $(S_0(a), S_1(a), \cdots, S_n(a))$. If n is the mth ladder index of this sequence of partial sums then there exist m such cyclic rearrangements. If $S > 0$ the same statement is true of strict ladder indices. This is essentially Feller's theorem 5 and we indicate briefly its proof. If $S_n \geqq 0$ there is at least one a such that $S_a \geqq S_j$ for all $j = 0, 1 \cdots n$.

It is easily verified from (2.4) that for such an a the sequence of partial sums $(S_0(a), S_1(a) \cdots, S_n(a))$ has n as a ladder index. It follows from (2.4) that n is a ladder index of $(S_0(a), S_1(a), \cdots, S_n(a))$ if and only if a is a ladder index of $(S_0, S_1, \cdots, S_n)$ and the theorem follows readily from this fact.

We give now the Combinatorial lemma we require for our study of the busy period. We formulate it for strict ladder indices, a similar result holds for non-strict ladder indices.

Combinatorial Lemma.

Suppose that $X_1 + X_2 + \cdots + X_n > 0$ and consider the $n!$ permutations of $(X_1, X_2, \cdots, X_n)$. Let $F(X_1, X_2, \cdots, X_n)$ be some functional relation between the X_i which is invariant under the symmetric group of permutations on n numbers, that is $F(X_{1'}, X_{2'}, \cdots, X_{n'}) = F(X_1, X_2, \cdots, X_n)$ for any permutation $(1', 2', \cdots, n')$ of $(1, 2, \cdots, n)$. Suppose that $F(X_1, X_2, \cdots, X_n)$ is the case and let $n!\,\pi_n^{(m)}(F_n)$ be the number of permutations in which n is the mth strict ladder index of the corresponding sequence of partial sums. Then we have

$$\frac{1}{n} = \sum_{m=1}^{n} \frac{1}{m} \pi_n^{(m)}(F_n). \tag{2.5}$$

Before indicating the proof of the lemma we remark that we assume that it is not the case that the relation $F(X_1, X_2, \cdots, X_n)$ implies the inequality $X_1 + X_2 + \cdots + X_n \leqq 0$. Apart from this restriction the functional relation F_n is arbitrary. For example, we could have

$$F_n(X_1, X_2, \cdots, X_n) \equiv \text{“}0 < \sum_{i=1}^{n} X_i \leqq X\text{”}.$$

Later in this paper we shall suppose that $X_i = Y_i - Z_i, i = 1, 2, \cdots, n$ and shall take for F_n relations of the form

$$F_n(X_1, X_2, \cdots, X_n) \equiv \text{“}0 < \sum_{i=1}^{n} Y_i \leqq Y\text{”}$$

$$F_n(X_1, X_2, \cdots, X_n) \equiv \text{“}0 < \sum_{i=1}^{n} Z_i \leqq Z\text{”}$$

The Combinatorial lemma is proved by noting that we can divide the $n!$ permutations into n groups, the mth group, $m = 1, 2, \cdots, n$, consisting of the permutations where n appears as the mth ladder index and the cyclic rearrangements of these permutations. Because of Theorem F. any permutation must appear in one and only one group. Further within the mth group there are $n!\,\pi_n^{(m)}(F_n)/m$ permutations from distinct cyclic rearrangements. Thus $n! \sum_{m=1}^{n} 1/m\pi_n^{(m)}(F_n)$ is the number of classes of cyclic rearrangements among the $n!$ permutations, that is $(n-1)!$ Hence we obtain (2.5).

3. The single-server queue

We consider now the single-server queueing system $GI/G/1$. Customers arrive at the instants $t_1, t_2, \cdots, t_n, \cdots$ where the inter-arrival intervals $\tau_n = t_{n+1} - t_n, n \geqq 1, t_1 = 0$ are independently and identically distributed non-negative random variables with common d.f. (distribution function) $P(\tau_n \leqq x) = A(x)$. Customers are served in the order of their arrival and it is never the case that customers are waiting for service and the server is idle. The service-times of successive customers form a sequence (σ_n) of independently and identically distributed non-negative random variables, independent also of the input process (t_n), with common d.f. $P(\sigma_n \leqq x) = B(x)$.

We shall suppose that the first customer arrives to find the server idle at the instant $t_1 = 0$ and starts service immediately. By the busy period we mean the time interval from t_1 until the server is next idle. By the busy cycle we mean the time interval from t_1 until it next happens that a customer arrives to find the server idle.

Let $\pi_n^{(1)}(x)$, $x > 0$, $n \geqq 1$ be the joint probability that during a busy period exactly n customers are served and the duration of the busy period does not exceed x.

Write $\eta_j = \tau_j - \sigma_j$ then we have

$$\begin{aligned}\pi_n^{(1)}(x) = P(\eta_1 \leqq 0, \eta_1 + \eta_2 \leqq 0, \cdots, \eta_1 + \eta_2 + \cdots \\ + \eta_{n-1} \leqq 0, \eta_1 + \eta_2 \cdots + \eta_n > 0, 0 < \sigma_1 + \sigma_2 + \cdots + \sigma_n \leqq x).\end{aligned} \tag{3.1}$$

Let $\pi^{(1)}(x)$ be the d.f. of a busy period then

$$\pi^{(1)}(x) = \sum_{n=1}^{\infty} \pi_n^{(1)}(x). \tag{3.2}$$

Let $\pi_n = \pi_n^{(1)}(\infty)$ be the probability that n customers are served in a busy period, then

$$\begin{aligned}\pi_n = P(\eta_1 \leqq 0, \eta_1 + \eta_2 \leqq 0, \cdots, \eta_1 + \eta_2 + \cdots + \eta_{n-1} \leqq 0, \\ \eta_1 + \eta_2 + \cdots + \eta_n > 0).\end{aligned} \tag{3.3}$$

Similarly let $\gamma_n^{(1)}(x)$, $x > 0$, $n \geqq 1$ be the joint probability that during a busy cycle exactly n customers are served and the duration of the busy cycle does not exceed x. We have

$$\begin{aligned}\gamma_n^{(1)}(x) = P(\eta_1 \leqq 0, \eta_1 + \eta_2 \leqq 0, \cdots, \eta_1 + \eta_2 + \cdots + \eta_{n-1} \leqq 0, \\ \eta_1 + \eta_2 + \cdots + \eta_n > 0, 0 < \tau_1 + \tau_2 + \cdots + \tau_n \leqq x)\end{aligned} \tag{3.4}$$

If $\gamma^{(1)}(x)$ is the d.f. of the busy cycle then

$$\gamma^{(1)}(x) = \sum_{n=1}^{\infty} \gamma_n^{(1)}(x). \tag{3.5}$$

Evidently $\gamma_n^{(1)}(\infty) = \pi_n^{(1)}(\infty) = \pi_n$.

In deriving the above equations we have used the convention that if a customer arrives at the instant the previous customer departs then we regard the busy period (cycle) as still in progress. The opposite convention admits a parallel treatment and is obtained, for example, by replacing the inequalities by strict inequalities and conversely in (3.3). If $P(\eta_1 + \eta_2 + \cdots + \eta_n = 0)\ \gamma = 0$ for all n there is no distinction between the conventions.

Write $S_0 = 0$ and $S_k = \sum_{j=1}^{k} \eta_j$ then in the terminology of section 2 $\pi_n^{(1)}(x)$ is the joint probability that n is the first strict ladder index of the partial sums $(S_0, S_1, \cdots, S_n)$ and that $0 < \sigma_1 + \sigma_2 + \cdots + \sigma_n \leqq x$. Let $\pi_n^{(m)}(x)$ be the joint probability that n is the mth strict ladder index of the partial sums $(S_0, S_1, \cdots, S_n)$ and that $0 < \sigma_1 + \sigma_2 + \cdots + \sigma_n \leqq x$. Because of the independence of the random variables τ_j, σ_j a short calculation shews that

(3.6) $$\pi_n^{(m)}(x) = \sum_{r=1}^{n-m+1} \int_0^x \pi_{n-r}^{(m-1)}(x-y)d\pi_r^{(1)}(y)$$

Introduce the Laplace-Stieltjes transform

$$*\pi_n^{(m)}(s) = \int_0^\infty e^{-sx} d\pi_n^{(m)}(x)$$

and the generating function

$$*\pi^{(m)}(s, z) = \sum_{n=1}^{\infty} *\pi_n^{(m)}(s) z^n \qquad |z| \leqq 1.$$

From (3.6) we obtain

(3.7) $$*\pi^{(m)}(s, z) = \{*\pi^{(1)}(s, z)\}^m.$$

Defining $*\gamma^{(m)}(s, z)$ similarly we have

(3.8) $$*\gamma^{(m)}(s, z) = \{*\gamma^{(1)}(s, z)\}^m.$$

4. The busy period and the busy cycle

Write $a_n(x) = P(\eta_1 + \eta_2 + \cdots + \eta_n > 0, \sigma_1 + \sigma_2 + \cdots + \sigma_n \leqq x)$ and $b_n(x) = P(\eta_1 + \eta_2 + \cdots + \eta_n > 0, \tau_1 + \tau_2 + \cdots + \tau_n \leqq x)$ so that

(4.1) $$a_n(x) = \int_0^x \{1 - A^{(n)}(y)\} dB^{(n)}(y)$$

(4.2) $$b_n(x) = \int_0^x B^{(n)}(y) dA^{(n)}(y)$$

where $A^{(n)}(x)$, $B^{(n)}(x)$ are the n-fold iterated convolutions of $A(x)$, $B(x)$ with themselves. Let $*a_n(s)$, $*b_n(s)$ be the Laplace-Stieltjes transforms of $a_n(x)$, $b_n(x)$ respectively. We prove

THEOREM 1.

For $GI/G/1$ the generating function $*\pi^{(1)}(s, z)$ is given by

(4.3) $$*\pi^{(1)}(s, z) = 1 - \exp\left\{-\sum_{n=1}^{\infty} \frac{*a_n(s)}{n} z^n\right\}$$

The Laplace-Stieltjes transform $*\pi_n^{(1)}(s)$ is given explicitly by

(4.4) $$*\pi_n^{(1)}(s) = \sum \frac{(-)^{k_1+k_2+\cdots+k_n+1}}{k_1!k_2!\cdots k_n!} \left(\frac{*a_1(s)}{1}\right)^{k_1} \left(\frac{*a_2(s)}{2}\right)^{k_2} \cdots \left(\frac{*a_n(s)}{n}\right)^{k_n}$$

where the summation is over all non-negative integers k_i such that $k_1 + 2k_2 + \cdots + nk_n = n$, that is over all the partitions of n, and

(4.5) $$*a_n(s) = \int_0^\infty e^{-sx} \{1 - A^{(n)}(x)\} dB^{(n)}(x).$$

A similar result holds for ${}^*\gamma^{(1)}(s, z)$, ${}^*\gamma_n^{(1)}(s)$ with ${}^*a_n(s)$ replaced by ${}^*b_n(s)$ where

$$ {}^*b_n(s) = \int_0^\infty e^{-sx} B^{(n)}(x) dA^{(n)}(x). \tag{4.6} $$

In order to prove the theorem we apply the Combinatorial lemma of section 2 with $X_j = \eta_j$ and $F_n(\eta_1, \eta_2, \cdots, \eta_n) \equiv$ "$0 < \sigma_1 + \sigma_2 + \cdots + + \sigma_n \leqq x$" to obtain

$$ \frac{a_n(x)}{n} = \sum_{m=1}^{n} \frac{1}{m} \pi_n^{(m)}(x) \tag{4.7} $$

Equation (4.3) follows at once from (3.7) and (4.7). Equation (4.4) follows from (4.3) by a standard result in Combinatorial Analysis (e.g. Riordan [12] Chap. 4. eq. (3a).) Equations (4.5), (4.6) follow at once from (4.1), (4.2).

We mention the following important corollary.

Corollary.

The probabilities π_n for the number of customers served in a busy period are given by

$$ \pi_n = \sum \frac{(-)^{k_1+k_2+\cdots+k_n+1}}{k_1!k_2!\cdots k_n!} \left(\frac{a_1}{1}\right)^{k_1} \left(\frac{a_2}{2}\right)^{k_2} \cdots \left(\frac{a_n}{n}\right)^{k_n} \qquad n \geqq 1 \tag{4.8} $$

where the summation is over all the partitions of n and

$$ a_n = \int_0^\infty \{1 - A^{(n)}(x)\} dB^{(n)}(x). $$

(4.8) is proved by noting that $\pi_n = \pi_n^{(1)}(\infty) = {}^*\pi_n^{(1)}(0)$.

For example we have

$$ \begin{aligned} 1!\pi_1 &= a_1 \\ 2!\pi_2 &= a_2 - a_1^2 \\ 3!\pi_3 &= 2a_3 - 3a_1a_2 + a_1^3 \\ 4!\pi_4 &= 6a_4 - 8a_1a_3 - 3a_2^2 + 6a_1^2a_2 - a_1^4 \\ 5!\pi_5 &= 24a_5 - 30a_1a_4 - 20a_2a_3 + 15a_1^2a_3 - 10a_1^3a_2 + a_1^5 \\ 6!\pi_6 &= 120a_6 - 144a_1a_5 - 90a_2a_4 + 40a_3^2 + 90a_1^2a_4 + 20a_1a_2a_3 + 15a_2^3 \\ &\qquad - 15a_3^2 - 40a_1^3a_3 - 45a_1^2a_2^2 + 15a_1^4a_2 - a_1^6. \end{aligned} $$

It can be verified that these expressions agree with those given by Conolly [3] who obtained $\pi_n, n \leqq 5$ for $D/M/1$.

The probabilities can be computed also from the following recurrence relation which follows easily from (4.3).

$$ n\pi_n = a_n - \sum_{m=1}^{n-1} a_{n-m}\pi_m \qquad n \geqq 1 $$

We prove now

THEOREM 2.

Suppose that $E(|\eta|) < \infty$. Write $\pi = \sum_{j<\infty} \pi_j$ and $N = \sum_{j<\infty} j\pi_j$ then

$$\pi = \begin{cases} 1 & \text{if} \quad E(\eta) \geqq 0 \\ 1 - \exp\left\{-\sum_{n=1}^{\infty} \frac{a_n}{n}\right\} & \text{if} \quad E(\eta) < 0 \end{cases} \tag{4.11}$$

$$N = \begin{cases} \exp\left\{\sum_{n=1}^{\infty} \frac{1-a_n}{n}\right\} & \text{if} \quad E(\eta) > 0 \\ \infty & \text{if} \quad E(\eta) = 0 \\ \left(\sum_{k=1}^{\infty} a_k\right) \exp\left\{-\sum_{n=1}^{\infty} \frac{a_n}{n}\right\} & \text{if} \quad E(\eta) < 0. \end{cases} \tag{4.12}$$

PROOF. Write $S_n = \eta_1 + \eta_2 + \cdots + \eta_n$ and $P(S_n > 0 \text{ i.o.})$ for the probability that the event $(S_n > 0)$ occurs infinitely often, that is, $P(S_n > 0 \text{ i.o.}) = P(\limsup (S_n > 0))$. By the Borel zero-one criterion (e.g. Loeve [10]) $P(S_n > 0 \text{ i.o.}) = 0$ or 1 according as $\sum a_n < \infty$ or $= \infty$.

Suppose that $E(\eta) < 0$ then $P(S_n > 0 \text{ i.o.}) = 0$ by the strong law of large numbers, hence $\sum a_n < \infty$ and so $\sum a_n/n < \infty$.

If $E(\eta) > 0$ then $\lim a_n = 1$ by the strong law of large numbers, thus $\sum a_n/n = \infty$ and by the preceding argument applied to the sums $S'_n = -S_n$ we have $\sum 1 - a_n/n < \infty$.

If $E(\eta) = 0$ the sequence of partial sums S_n is recurrent in the sense of Chung and Fuchs [2], so that if $P(\eta_n = 0) < 1$ then $P(S_n > 0 \text{ i.o.}) = 1$ and hence $\sum a_n = \infty$ and since $\limsup a_n = 1$, $\sum a_n/n = \infty$. A similar argument shews that $P(S_n \leqq 0 \text{ i.o.}) = 1$ and that $\sum (1 - a_n)/n = \infty$.

Write $\pi(z) = \sum_{j=1}^{\infty} \pi_j z^j = \pi^{(1)}(z)$ then from (4.3) we have

$$\pi(z) = 1 - \exp\left\{-\sum_{n=1}^{\infty} \frac{a_n}{n} z^n\right\} \tag{4.13}$$

$$\pi'(z) = \left(\sum_{k=1}^{\infty} a_k z^{k-1}\right) \exp\left\{-\sum_{n=1}^{\infty} \frac{a_n}{n} z^n\right\}. \tag{4.14}$$

Since the coefficients of powers of z in the expansion of $\pi(z)$ and $\pi'(z)$ are non-negative we can apply the converse of Abel's theorem on power series to obtain

$$\pi = \pi(1) = \lim_{z \to 1} \pi(z)$$

$$N = \pi'(1) = \lim_{z \to 1} \pi'(z)$$

(See, for example, Widder [21] Ch. 5. Cor. 4.56.).

When $E(\eta) \geqq 0$ we have $\sum a_n/n = \infty$ and when $E(\eta) < 0$ we have $\sum a_n/n < \infty$. Thus we obtain (4.11) from (4.13) by letting $z \to 1$.

In order to establish (4.12) we consider the three cases for $E(\eta)$ separately.

(i) $E(\eta) > 0$.

Write (4.14) in the form

$$\pi'(z) = \left\{(1-z)\sum_{k=1}^{\infty} a_k z^{k-1}\right\} \exp\left\{\sum_{n=1}^{\infty} \frac{1-a_n}{n} z^n\right\}. \tag{4.15}$$

Since $E(\eta) > 0$, $\sum (1-a_n)/n < \infty$. Also $\lim a_n = 1$ so that

$$\lim_{z\to 1} (1-z) \sum a_k z^{k-1} = \lim_{k\to\infty} a_k = 1$$

and we obtain the first equation of (4.12) by letting $z \to 1$ in (4.15).

(ii) $E(\eta) = 0$.

We have $\sum(1-a_n)/n = \infty$, also $\limsup a_n = 1$. Thus from (4.15) $\lim_{z\to 1} \sup \pi'(z) = \infty$. It follows that $N = \pi'(1) = \infty$.

(iii) $E(\eta) < 0$.

We have $\sum a_n < \infty$, $\sum a_n/n < \infty$. By letting $z \to 1$ in (4.14) we obtain the third equation of (4.12).

We remark that although we have formulated theorem 2 under the assumption that $E(|\eta|) < \infty$ it is clear that we could obtain (4.11) and (4.12) when $E(|\eta|) = \infty$ if we formulate the various cases of the theorem in terms of the behaviour of the series $\sum a_n$, $\sum a_n/n$, $\sum(1-a_n)/n$.

We establish now the following

COROLLARY.

Suppose that $E(\sigma) < \infty$, $E(\tau) < \infty$ and denote by Π, Γ respectively the expectations of the length of a busy period and a busy cycle, then

$$\begin{aligned} &\Pi = \Gamma = \infty \quad \text{when} \quad E(\sigma) \geqq E(\tau) \\ &\left.\begin{aligned} \Pi &= NE(\sigma) \\ \Gamma &= NE(\tau) \end{aligned}\right\} \quad \text{when} \quad E(\sigma) < E(\tau) \end{aligned} \tag{4.16}$$

where N is given by the first equation of (4.12).

PROOF. When $E(\eta) \leqq 0$ the expectation of the number of customers in a busy period is infinite, thus $\Pi = \Gamma = \infty$.

When $E(\eta) > 0$ the expectation of the number of customers served in a busy period (cycle) is given by the first equation of (4.12). Thus, for example, $\Pi = E(\sigma_1 + \sigma_2 + \cdots + \sigma_r)$ where r is a random variable taking the values $1, 2, \cdots$ and with expectation $E(r) = N < \infty$. Also $P(r = n) = P(\eta_1 \leqq 0, \eta_1 + \eta_2 \leqq 0, \cdots, \eta_1 + \eta_2 + \cdots + \eta_{n-1} \leqq 0, \eta_1 + \eta_2 + \cdots + \eta_n > 0)$ and this probability does not depend on $\sigma_{n+1}, \sigma_{n+2}, \cdots$. Applying a theorem due to Wald [20] we obtain (4.16). A similar argument applies to Γ.

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5. The busy period in $M/G/1$

When the input process is Poisson we can obtain the probabilities $\pi_n(x)(=\pi_n^{(1)}(x))$ explicitly. We prove

THEOREM 3.

For $M/G/1$ we have

$$\pi_n(x) = \frac{1}{n!}\int_0^x e^{-\lambda y}(\lambda y)^{n-1}dB^{(n)}(y) \tag{5.1}$$

where λ is the parameter of the input process. Recently this result has been obtained also by Prabhu [11] by quite different methods.

In order to prove (5.1) we shall shew that

$$*\pi_n^{(1)}(s) = \frac{1}{n!}\int_0^\infty e^{-(\lambda+s)x}(\lambda x)^{n-1}dB^{(n)}(x). \tag{5.2}$$

When the input process is Poisson, so that $A(x) = 1 - e^{-\lambda x}$, $x \geqq 0$, we have from (4.1)

$$a_n(x) = \sum_{m=0}^{n-1} \frac{1}{m!}\int_0^x e^{-\lambda y}(\lambda y)^m dB^{(n)}(y)$$

thus

$$*a_n(s) = \sum_{m=0}^{n-1} \frac{1}{m!}\int_0^\infty e^{-(\lambda+s)x}(\lambda x)^m dB^{(n)}(x). \tag{5.3}$$

If (5.3) is substituted into (4.4) it is easy to verify that (5.2) is true for $n = 1, 2$. We shall prove (5.2) by induction, to do so we require the following lemma which seems to be new and is not without an independent interest.

LEMMA.

If $f(x)$ is a function of a real variable x which has a derivative of the m th order then

$$D^m f^n = \frac{1}{m+1}\sum_{u=1}^{m+1}\binom{m+1}{u}(D^{u-1}f^u)(D^{m+1-u}f^{n-u}) \quad 0 \leqq m < n \tag{5.4}$$

where $D \equiv d/dx$.

It is easily verified that (5.4) is true for $m = 1$, $n > 1$ and we shall prove it generally by induction. We remark that (5.4) is true also for $m \geqq n$, as follows from (5.5) below, but we shall not use this fact.

Suppose that (5.4) is true for $m = 1, 2, \cdots, r$. (it is trivially true for $m = 0$) and $n > m$. We shall shew that it is true also for $m = r + 1$ and $n > r + 1$. In the course of the proof of the lemma we suppose that any functions considered are differentiable as often as required.

We shew first that the induction hypothesis implies that

$$D^r(gf^n) = \frac{1}{r+1} \sum_{u=1}^{r+1} \binom{r+1}{u} (D^{u-1}f^u)(D^{r+1-u}gf^{n-u}) \tag{5.5}$$

for $n > r$ and an arbitrary function $g(x)$.

We prove (5.5) by noting that

$$\begin{aligned} D^r(gf^n) &= \sum_{t=0}^{r} \binom{r}{t} (D^{r-t}g)(D^t f^n) \\ &= \sum_{t=0}^{r} \binom{r}{t} (D^{r-t}g) \left[\frac{1}{t+1} \sum_{u=1}^{t+1} \binom{t+1}{u} (D^{u-1}f^u)(D^{t+1-u}f^{n-u})\right] \end{aligned}$$

by the induction hypothesis applied to $(D^t f^n)t \leqq r, n > r$.

Interchanging the order of summation we obtain

$$D^r(gf^n) = \frac{1}{r+1} \sum_{u=1}^{r+1} \binom{r+1}{u} (D^{u-1}f^u) \left[\sum_{t=0}^{r+1-u} \binom{r+1-u}{t} (D^{r+1-u-t}g)(D^t f^{n-u})\right]$$

The second sum in this equation is just $D^{r+1-u}(gf^{n-u})$ and so we obtain (5.5).

In (5.5) write $n = r + 1$ and put $g = (r + 2)Df$ then since we now have

$$D^r(gf^{r+1}) = D^{r+1}(f^{r+2})$$

$$D^{r+1-u}(gf^{r+1-u}) = \frac{r+2}{r+2-u} D^{r+2-u}(f^{r+2-u})$$

we obtain

$$(r+1)D^{r+1}f^{r+2} = \sum_{u=1}^{r+1} \binom{r+2}{u} (D^{u-1}f^u)(D^{r+2-u}f^{r+2-u})$$

adding $D^{r+1}f^{r+2}$ to each side of this equation gives

$$D^{r+1}f^{r+2} = \frac{1}{r+2} \sum_{u=1}^{r+2} \binom{r+2}{u} (D^{u-1}f^u)(D^{r+2-u}f^{r+2-u})$$

By the induction hypothesis

$$D^m f^{r+2} = \frac{1}{m+1} \sum_{u=1}^{m+1} \binom{m+1}{u} (D^{u-1}f^u)(D^{m+1-u}f^{r+2-u})$$

for $m = 0, 1, \cdots, r$. We deduce that

$$D^{r+1}(gf^{r+2}) = \frac{1}{r+2} \sum_{u=1}^{r+2} \binom{r+2}{u} (D^{u-1}f^u)(D^{r+2-u}gf^{r+2-u}) \tag{5.6}$$

for an arbitrary function $g(x)$. Equation (5.6) is established by an argument similar to that used to derive (5.5). Writing $g = f^k$, $k \geqq 0$ in (5.6) we obtain

(5.4) for $m = r + 1$ and $n > r + 1$. This completes the proof of the lemma.

We return to the proof of theorem 3. Equations (5.2), (5.3) can be written in the form

$$(5.7) \qquad {}^*\pi_n^{(1)}(s) = \frac{1}{n!}(-\lambda D)^{n-1}\{{}^*B(s+\lambda)\}^n$$

$$(5.8) \qquad {}^*a_n(s) = \sum_{m=0}^{n-1} \frac{1}{m!}(-\lambda D)^m\{{}^*B(s+\lambda)\}^n$$

where $D \equiv d/ds$ and ${}^*B(s) = \int_0^\infty e^{-sx} dB^{(1)}(x)$.

Suppose that (5.1) is true for $n = 1, 2, \cdots, k$ and substitute from (5.7) and (5.8) into the equation

$$(k+1){}^*\pi_{k+1}^{(1)}(s) = {}^*a_{k+1}(s) - \sum_{m=1}^{k} {}^*a_{k+1-m}(s){}^*\pi_m^{(1)}(s)$$

which follows from (4.3) (compare equation (4.10)). Because of (5.4) it will be found that the resulting expression for ${}^*\pi_{k+1}^{(1)}(s)$ becomes (5.2). This proves the theorem.

References

[1] Bailey, N. T. J., A continuous time treatment of a simple queue using generating functions. J. R. Statist. Soc. B. 16. (1954) 288—291.

[2] Chung, K. L. and Fuchs, W. H. J., On the distribution of values of sums of random variables. Memoirs of the Amer. Math. Soc. No. 6 (1951).

[3] Conolly, B. W., The busy period in relation to the queueing system GI/M/1. Biometrika, 46. (1959) 246—251.

[4] Conolly, B. W., The busy period in relation to the single-server queueing system with general independent arrivals and Erlangian service-time. J. R. Statist. Soc. B. 22, (1960) 89—96.

[5] Feller, W., On combinatorial methods in fluctuation theory. Probability and Statistics. Ed. U. Grenander. J. Wiley and Sons, Inc. (1959) 75—91.

[6] Kendall, D. G., Some problems in the theory of queues. J. R. Statist. Soc. B. (1951) 13, 151—173.

[7] Kendall, D. G., Stochastic processes occurring in the theory of queue and their analysis by the method of the imbedded Markov Chain. Ann. Math. Statist., (1953) 24, 338—354.

[8] Lederman, W., and Reuter, G. E. H., Spectral theory for the differential equations of simple birth and death processes. Phil. Trans. A. 246, (1954) 321—369.

[9] Lindley, D. V., The theory of queues with a single server. Proc. Camb. Soc., 48, (1952) 277—289.

[10] Loeve, M., Probability Theory. D. Van Nostrand Co., Inc. (1955).

[11] Prabhu, N. U., Some results for the queue with Poisson arrivals. J. R. Statist. Soc. B. 22, (1960) 104—107.

[12] Riordan, J., An Introduction to Combinatory analysis. J. Wiley & Sons, Inc. (1958).

[13] Sparre Andersen, E., On the fluctation of sums of random variables. I. Math. Scand. 1, (1953) 263—285.

[14] Sparre Andersen, E., On the fluctuations of sums of random variables, II. Math. Scand. 2, (1954) 195—223.

[15] Spitzer, F., A Combinatorial lemma and its applications to Probability theory. Trans. Amer. Math. Soc. 82, (1956) 323—339.

https://doi.org/10.1017/S1446788700026690 Published online by Cambridge University Press

[16] Spitzer, F., The Wiener-Hopf equation whose kernel is a probability density.Duke. Math. Journ. 24, (1957) 327—343.
[17] Takacs, L., Investigation of waiting time problems by reduction to Markov Processes. Acta. Math. Acad. Sci. Hung. 6, (1955) 101—128.
[18] Takacs, L., Transient behaviour of single-server queueing processes with recurrent input and exponentially distributed service times. Opns. Res. 8, (1960) 231—245.
[19] Takacs, L., Transient behaviour of single queueing processes with Erlang input. To be published.
[20] Wald., A., Sequential tests of statistical hypothesis. Ann. Math. Statist. 16, (1945) 117—186.
[21] Widder, D. V., The Laplace Transform. Princeton Univ. Press. (1946).

University of Melbourne.