THE NULLITY OF A WILD KNOT IN A COMPACT 3-MANIFOLD

Dedicated to the memory of Hanna Neumann

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The nullity of the Alexander module of the fundamental group of the complement of a knot in S^3 was one of the invariants of wild knot type defined and investigated by E. J. Brody in [1], in which he developed a generalised elementary divisor theory applicable to infinitely generated modules over a unique factorisation domain. Brody asked whether the nullity of a knot with one wild point was bounded above by its enclosure genus; for knots in S^3, the present author showed in [6] that this was indeed the case. In [7], it was (prematurely) stated by the author that this was also the case for knots k embedded in a 3-manifold M so that $H_1(M - k)$ was torsion-free.

The aim of this paper is to set the record straight. Let k be an oriented knot in the interior of a connected compact triangulated 3-manifold M, and suppose that k has only one wild point. Let $P(k, p)$ be the penetration index of k at this point p. Let $\pi_1(M - k)$ be the fundamental group of $M - k$; the first Betti group $B_1(M - k)$, being the torsion-free part of $H_1(M - k)$, is a free abelian group of finite rank. Let JB be the integer group ring of $B_1(M - k)$; JB is therefore a unique factorisation domain, which means that we can apply the results of [1] and [6] to obtain invariants of $M - k$, and therefore of the embedding type of k in M. The Betti module of k is the unique JB-module determined by $\pi_1(M - k)$; the main theorem of this paper, theorem 1, states that the nullity ν of the Betti module of k is bounded above by the expression

$$\tfrac{1}{2}P(k, p) + n(M) - 1,$$

in which $n(M)$ denotes the maximum possible nullity of the Betti module of a tame knot in M, and is an invariant of M. The paper closes with some questions about $n(M)$, and the conjecture that

$$\nu \leqq e(k) + n(M) - 1$$

for all compact 3-manifolds M, where $e(k)$ denotes the enclosure genus of k (cf. [6], p. 550).

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Proof of theorem 1

It will be assumed throughout that readers are familiar with the terminology, notation, and results of [1] and [6]; in particular, if G is a group and $\tau: G \to G/G'$ the canonical map, $\beta: G/G' \to B$ the canonical map whose kernel is the torsion subgroup of G/G', then the Jacobian JB-module of G evaluated at $\beta\tau$ (cf. [1], p. 146, and [6], p. 546 f.) will be called the *Betti module of G*.

PROPOSITION 1. *There exists a closed 3-cell neighbourhood U of p such that the map $H_1(M-k-U) \to H_1(M-k)$, induced by inclusion, is an epimorphism.*

PROOF. Let N be any tame closed 3-cell neighbourhood of p, whose boundary is in general position with respect to k and meets k in $2n$ ($\neq 0$) points. The reduced Mayer-Vietoris sequence

$$\cdots \to H_1(N-k) \oplus H_1(M-k-N) \to H_1(M-k) \to \tilde{H}_0(\mathrm{Bd}N-k) = 0$$

shows that $H_1(M-k)$ is a quotient of the direct sum of $H_1(N-k)$ (which is a free group on n generators) and $H_1(M-k-N)$. This last group is finitely generated, as it is isomorphic to the first homology group of the compact space obtained by removing an open regular neighbourhood of $k \cup N$ from M; this means that $H_1(M-k)$ is finitely generated.

Let $\alpha_1, \cdots, \alpha_m$ be a set of paths whose homology classes generate $H_1(M-k)$, and let the distance of k from $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_m$, measured with the barycentric metric on M, be ε. Then any 3-cell neighbourhood U of p which lies in the interior of a ball of radius ε and centred at p will have the property we require. Q.E.D.

Since theorem 1 is trivial if $P(k,p)$ is infinite, we may assume that $P = P(k,p)$ is finite. Let U be a 3-cell of the type given by proposition 1. An *admissible sequence of* 3-*cells for the knot* k is a sequence $U_1 \supset U_2 \supset U_3 \supset \cdots$ such that, for each $i = 1, 2, \cdots$,

(i) U_i is a tame closed 3-cell neighbourhood of p whose boundary is in general position with respect to k and meets k in precisely P points,

(ii) $U_{i+1} \subset \mathrm{Int}\, U_i$ and $U_1 \subset \mathrm{Int}\, U$, and

(iii) $\cap\, U_i = \{p\}$.

Let $U_1 \supset U_2 \supset U_3 \supset \cdots$ be an admissible sequence for k, and let G_i and G be the fundamental groups of $M-k-U_i$ and $M-k$ respectively. Then the inclusion maps $M-k-U_i \subset M-k-U_{i+1}$ and $M-k-U_i \subset M-k$ induce maps $\phi_i: G_i \to G_{i+1}$ and $\psi_i: G_i \to G$ such that $\psi_{i+1}\phi_i = \psi_i$; note that G is the direct limit of the sequence

$$\cdots \to G_i \xrightarrow{\phi_i} G_{i+1} \to \cdots,$$

by [3].

Let $\mathfrak{a}: \mathscr{G} \to \mathscr{A}$ be the abelianisation functor, given by $\mathfrak{a}(G) = G/G'$ for all $G \in \mathscr{G}$. Note that $\mathfrak{a}(G_i) \cong H_1(M-k-U_i)$ and $\mathfrak{a}(G) \cong H_1(M-k)$, so we have a commuting diagram

$$\begin{tikzcd} G_i \arrow[rr, "\psi_i"] \arrow[d, "\tau_i"'] & & G \arrow[d, "\tau"] & \\ H_1(M-k-U_i) \arrow[rr, "\eta\mathfrak{a}(\psi_i)\eta_i^{-1}"] & & H_1(M-k) \arrow[r, "\beta"] & B_1(M-k) = B \end{tikzcd}$$

in which τ_i is the composition of the Hurewicz isomorphism $\eta_i: G_i/G_i \to H_1(M-k-U_i)$ with the canonical map $G_i \to G_i/G_i'$ (and $\eta: G/G' \to H_1(M-k)$ is another Hurewicz isomorphism).

We shall use ζ_i to denote the map $\eta\mathfrak{a}(\psi_i)\eta_i^{-1}$; note that ζ_i is really just the homology map induced by the inclusion $M-k-U_i \subset M-k$. Since $U_i \subset \operatorname{Int} U$, proposition 1 shows that ζ_i is an epimorphism. We may therefore make G_i into a B-group $[G_i, \alpha_i]$ by setting $\alpha_i = \beta\zeta_i\tau_i$; then ϕ_i and ψ_i become morphisms in the category $\mathscr{B}$ of B-groups, because the diagrams

$$\begin{tikzcd} G_i \arrow[rr, "\phi_i"] \arrow[dr, "\alpha_i"'] & & G_{i+1}, \arrow[dl, "\alpha_{i+1}"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} G_i \arrow[rr, "\psi_i"] \arrow[dr, "\alpha_i"'] & & G \arrow[dl, "\beta\tau"] \\ & B & \end{tikzcd}$$

commute (cf. [6], p. 545).

PROPOSITION 2. *$[G, \beta\tau]$ is the direct limit in $\mathscr{B}$ of the sequence*

$$\cdots \to [G_i, \alpha_i] \xrightarrow{\phi_i} [G_{i+1}, \alpha_{i+1}] \to \cdots$$

PROOF. (Note that this proposition is analogous to (5.1) of [6]; a proof is included here because the proof of (5.1) contains a lacuna.)

Notice that $[G, \beta\tau]$ and the maps $\psi_i: [G_i, \alpha_i] \to [G, \beta\tau]$ satisfy all the commuting requirements for a direct limit. Let $[L, \lambda]$ be a B-group and $\lambda_i: [G_i, \alpha_i] \to [L, \lambda]$ a family of B-homomorphisms such that $\lambda_{i+1}\phi_i = \lambda_i$ for each i; we wish to show that there exists a unique homomorphism γ in $\mathscr{B}$ such that $\lambda_i = \gamma\psi_i$ for each i, where $\gamma: [G, \beta\tau] \to [L, \lambda]$.

Now G is the direct limit of the sequence in $\mathscr{G}$, so there exists a unique homomorphism $\gamma: G \to L$ such that the diagram

$$\begin{tikzcd}
G_i \arrow[dr, "\psi_i"] \arrow[drr, "\lambda_i"] \arrow[dd, bend right, "\phi_i"'] & & \\
& G \arrow[r, "\gamma"] & L \\
G_{i+1} \arrow[ur, "\psi_{i+1}"] \arrow[urr, "\lambda_{i+1}"'] & &
\end{tikzcd}$$

commutes for each i; we shall show that $\lambda\gamma = \beta\tau$.

Consider the diagram

$$\begin{tikzcd}
G_i \arrow[rr, "\lambda_i"] \arrow[dr, "\psi_i"] \arrow[ddr, "\alpha_i"'] & & L \arrow[ddl, "\lambda"] \\
& G \arrow[ur, "\gamma"] \arrow[d, "\beta\tau"'] & \\
& B &
\end{tikzcd}$$

in which every triangle commutes, with the possible exception of GLB. Some chasing shows that $\lambda\gamma\psi_i = \beta\tau\psi_i$, and this is true for all i. Since the group G is generated by the image groups $\psi_i(G_i)$, it follows that $\lambda\gamma = \beta\tau$; that is, γ is a $\mathscr{B}$-map. This shows that $[G, \beta\tau]$ is the direct limit of the sequence

$$\cdots \to [G_i, \alpha_i] \xrightarrow{\phi_i} [G_{i+1}, \alpha_{i+1}] \to \cdots$$

in $\mathscr{B}$. Q.E.D.

From proposition 2, and theorem 1 of [6], it follows that the Betti module $M[G, \beta\tau]$ of G is the direct limit of the Jacobian modules of the G_i evaluated at α_i; since G is generated by the image groups $\psi_i(G_i)$, $M[G, \beta\tau]$ is the union of its submodules $M\psi_i(M[G_i, \alpha_i])$, and each of these submodules is finitely generated because each G_i is. We therefore have a sequence

$$M\psi_1(M[G_1, \alpha_1]) \subset \cdots \subset M\psi_i(M[G_i, \alpha_i]) \subset M\psi_{i+1}(M[G_{i+1}, \alpha_{i+1}]) \subset \cdots$$

whose union is $M[G, \beta\tau]$. Then the nullity of $M[G, \beta\tau]$ as a JB-module is the supremum of the nullities of the $M\psi_i(M[G_i, \alpha_i])$; by (2.2) of [1], it follows that

$$\nu(M[G, \beta\tau]) \leqq \sup \nu(M[G_i, \alpha_i]).$$

This brings us to the statement of the main theorem.

THEOREM 1. *Let k be an oriented knot with one wild point p, lying in the interior of the compact 3-manifold M: let $P(k, p)$ be the (3-cell) penetration index of k at p (see [7], p. 176). If ν denotes the nullity of the Betti module of k, and if $n(M)$ is the largest integer n such that there exists a tame knot in M whose Betti module has nullity n (or $n(M) = \infty$ if no largest such n exists), then*

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$$\nu \leqq \tfrac{1}{2}P(k, p) + n(M) - 1.$$

Proof. There is nothing to prove if $P(k, p)$ is infinite, so we assume that $P(k, p) = P$ is finite and takes the value $2n$. Then to prove the theorem it is sufficient to prove that

$$\nu(M[G_i, \alpha_i]) \leqq n + n(M) - 1$$

for each i.

We join the P points of $k \cap \mathrm{Bd}U_i$ by n oriented arcs $\beta_1, \cdots, \beta_n$ whose interiors lie in Int U_i, so that the components of k in $M - U_i$ form a consistently oriented knot k^* when joined together by the β_j's. We choose the β_j's so that they are unknotted and do not link each other; that is, so that $\pi_1(U_i - \cup \beta_j)$ is a free group on n generators. The fact that we may choose the β_j's in this way can be proven by induction on n.

Let N be an open regular neighbourhood of $\mathrm{Bd}U_i - k^* = \mathrm{Bd}U_i - k$ in $M - k^*$; the open sets $N \cup (M - U_i - k^*)$ and $N \cup (U_i - \cup\beta_j)$ have the same homotopy types as $M - U_i - k^*$ and $U_i - \cup\beta_j$ respectively, so we can apply the van Kampen theorem to show that the diagram

$$\begin{tikzcd}
& \pi_1(U_i - \cup \beta_j) \arrow[dr, "\omega_1^*"] & \\
\pi_1(N) \arrow[ur, "\xi_1"] \arrow[dr, "\xi_2"'] & & \pi_1(M - k^*) \\
& \pi_1(M-k^*-U_i) = G_i \arrow[ur, "\omega_2^*"'] &
\end{tikzcd}$$

is a pushout in $\mathscr{G}$, where all the maps are induced by inclusion maps. Since the functor $\mathfrak{a}$ preserves pushouts, the diagram

$$\begin{tikzcd}
& H_1(U_i - \cup \beta_j) \arrow[dr, "\mathfrak{a}(\omega_1^*)"] & \\
H_1(N) \arrow[ur, "\mathfrak{a}(\xi_1)"] \arrow[dr, "\mathfrak{a}(\xi_2)"'] & & H_1(M-k^*) \\
& H_1(M-k^*-U_i) \arrow[ur, "\mathfrak{a}(\omega_2^*)"'] \arrow[d, equal] & \\
& H_1(M-k-U_i) &
\end{tikzcd}$$

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is a pushout in $\mathscr{A}$ (where we have translated every group L/L' by the Hurewicz isomorphism to the first homology group of the appropriate space; this means that in the diagram above, the maps between homology groups have all been induced by inclusion maps). Similarly, the diagram

$$\begin{tikzcd} & H_1(U_i-k) \arrow[dr, "\mathfrak{a}(\omega_1)"] & \\ H_1(N) \arrow[ur, "\mathfrak{a}(\chi)"] \arrow[dr, "\mathfrak{a}(\xi_2)"'] & & H_1(M-k) \\ & H_1(M-k-U_i) \arrow[ur, "\mathfrak{a}(\omega_2)"'] & \end{tikzcd}$$

in which the maps are induced by inclusion maps, is a pushout in $\mathscr{A}$.

Now we choose a starting point on k lying in $M - U_i$, and order the points of $k \cap \mathrm{Bd}U_i$ with the order in which we meet them as we travel along k in its positive direction. If the points of $k \cap \mathrm{Bd}U_i = k^* \cap \mathrm{Bd}U_i$ are $x_1, x_2, \cdots, x_{2n}$, then x_{2m-1} and x_{2m} are joined by an arc of k; we may also assume that they are joined by the subarc β_m of k^*. Let $t_1, t_2, \cdots, t_{2n-1}$ be $2n-1$ disjoint simple loops on $\mathrm{Bd}U_i - k$, chosen so that each t_j bounds a disc D_j on $\mathrm{Bd}U_i$ which contains no other t_m in its interior, and whose interior contains x_j but no other points of $k \cap \mathrm{Bd}U_i$.

Let the n components of $k \cap U_i$ be ordered so that k_j starts at x_{2j-1} and finishes at x_{2j}. Then $H_1(U_i - k)$ is generated by the homology classes of the loops t_{2j-1} in $U_i - k$; in fact, $H_1(U_i - k)$ is the free abelian group generated by the elements $\mathfrak{a}(\chi)(t_1)$, $\mathfrak{a}(\chi)(t_3)$, $\mathfrak{a}(\chi)(t_5), \cdots, \mathfrak{a}(\chi)(t_{2n-1})$. Similarly, $H_1(U_i - \cup\beta_j$ is the free abelian group on the n generators $\mathfrak{a}(\xi_1)(t_1)$, $\mathfrak{a}(\xi_1)(t_3), \cdots, \mathfrak{a}(\xi_1)(t_{2n-1})$. Notice also that $\mathfrak{a}(\chi)(t_{2m}) = \mathfrak{a}(\chi)(t_{2m-1})$ and $\mathfrak{a}(\xi_1)(t_{2m}) = \mathfrak{a}(\xi_1)(t_{2m-1})$ for each m.

There is therefore an isomorphism $\omega\colon H_1(U_i - \cup\beta_j) \to H_1(U_i - k)$ which makes the diagram

$$\begin{tikzcd} & H_1(U_i - \cup\beta_j) \arrow[dd, "\omega"] \\ H_1(N) \arrow[ur, dash, "\mathfrak{a}(\xi_1)"] \arrow[dr, dash, "\mathfrak{a}(\chi)"'] & \\ & H_1(U_i-k) \end{tikzcd}$$

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commute. Some chasing around the diagram

$$\begin{tikzcd}
H_1(U_i - \cup \beta_j) \arrow[r, "\mathfrak{a}(\omega_1^*)"] & H_1(M-k^*) \\
H_1(N) \arrow[u, "\mathfrak{a}(\xi_1)"] \arrow[r, "\mathfrak{a}(\xi_2)"] \arrow[d, "\mathfrak{a}(\chi)"'] & H_1(M-k-U_i) \arrow[u, "\mathfrak{a}(\omega_2^*)"'] \arrow[d, "\mathfrak{a}(\omega_2)"] \\
H_1(U_i-k) \arrow[r, "\mathfrak{a}(\omega_1)"] & H_1(M-k)
\end{tikzcd}$$

(in which the small rectangles are both pushouts in $\mathscr{A}$) shows that this isomorphism ω induces an isomorphism $\Omega: H_1(M-k^*) \to H_1(M-k)$ such that $\Omega\mathfrak{a}(\omega_1^*) = \mathfrak{a}(\omega_1)\omega$ and $\Omega\mathfrak{a}(\omega_2^*) = \mathfrak{a}(\omega_2)$. If $\beta^*: H_1(M-k^*) \to B_1(M-k^*)$ is the canonical map to the Betti group of k^*, then Ω induces an isomorphism $\Omega^*: B_1(M-k^*) \to B_1(M-k)$ such that $\Omega^*\beta^* = \beta\Omega$. We therefore have a commutative diagram

$$\begin{tikzcd}
G_i \arrow[r, "\tau_i"] \arrow[dr, "\alpha_i"'] & H_1(M-k-U_i) \arrow[r, "\beta^*\mathfrak{a}(\omega_2^*)"] \arrow[d, "\beta\zeta_i"] & B_1(M-k^*) \arrow[dl, "\Omega^*"] \\
& B_1(M-k) = B &
\end{tikzcd}$$

(because $\zeta_i = \mathfrak{a}(\omega_2)$ is the map induced by the inclusion of $M-k-U_i$ in $M-k$).

This means that any Jacobian matrix of G_i, evaluated at α_i, is the image of a Jacobian matrix of G_i evaluated at $\beta^*\mathfrak{a}(\omega^*)\tau_i$ under the ring isomorphism $J\Omega^*: J(B_1(M-k^*)) \to JB$; this is the result we will use to find an upper bound for $\nu(M[G_i, \alpha_i])$.

Now we direct our attention to finding a Betti matrix for $\pi_1(M-k^*)$. Since $\pi_1(M-k^*)$ is the quotient of the free product $G_i^*\pi_1(U_i - \cup\beta_j)$ by the normal subgroup generated by the $2n-1$ elements of the form $\omega_1^*\xi_1(t_m)\omega_2^*\xi_2(t_m^{-1})$, it follows that $\pi_1(M-k^*)$ has a Betti matrix of the form

$$\left(\frac{A\;0}{C}\right),$$

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where A is a Jacobian matrix of G_i evaluated at $\beta^*\mathfrak{a}(\omega_2)\tau_i$, the matrix 0 has n columns of zeros, and the matrix C has $2n-1$ rows, one for each of the relators $\omega_1^*\xi_1(t_m)\omega_2^*\xi_2(t_m^{-1})$, $m = 1, 2, \cdots, 2n-1$. Then by (4.1) of [6], the nullity ν^* of the Betti module of $\pi_1(M-k^*)$ is at least as great as the nullity of the matrix $(A\ 0)$ reduced by the rank $r(C)$ of C. That is,

$$\nu^* \geqq \nu(A) + n - r(C) \geqq \nu(A) + n - (2n-1);$$

hence

$$\nu(A) \leqq \nu^* + n - 1.$$

Since $J\Omega^*$ is an isomorphism and $\Omega^*\beta^*\mathfrak{a}(\omega_2^*)\tau_i = \alpha_i$, the nullity of a Jacobian matrix of G_i evaluated at $\beta^*\mathfrak{a}(\omega^*)\tau_i$ is the nullity of any Betti matrix of G_i. Hence

$$\begin{aligned} \nu(M[G_i, \alpha_i]) &= \nu(A) \leqq \nu^* + n - 1 \\ &\leqq n(M) + n - 1. \end{aligned}$$

As mentioned earlier, this upper bound on $\nu(M[G_i, \alpha_i])$ yields an upper bound for $\nu[M[G, \beta\tau])$, and the theorem is proved. Q.E.D.

Some open questions

Question 1. *How can one compute $n(M)$ for a given compact 3-manifold M? Is there an algorithm for computing $n(M)$ from a triangulation of M? Does there exist an M such that $n(M) = \infty$?*

We can at least give a lower bound for $n(M)$, as follows.

Theorem 2: *For every compact 3-manifold M,*

$$n(M) \geqq 1 + \nu(M),$$

where $\nu(M)$ is the nullity of the Betti module of M.

Proof. Let C be a tame closed 3-cell in the interior of M, and k a tame knot in the interior of C. Then

$$\begin{aligned} \pi_1(M-k) &\cong \pi_1(M)^*\pi_1(C-k), \text{ and} \\ B_1(M-k) &\cong B_1(M) \oplus Z. \end{aligned}$$

Let β be the canonical map from $H_1(M-k)$ to $B_1(M-k)$; the symbol β will also be used to denote the induced map of integer group rings. If A is a matrix $\| a_{ij} \|$, $\beta(A)$ is the matrix $\| \beta(a_{ij}) \|$. Now by [4], p. 206, $\pi_1(M-k)$ has an Alexander matrix of the form

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 is an Alexander matrix for $\pi_1(M)$, and A_2 an Alexander matrix for $\pi_1(C-k)$; since $B_1(C-k) = H_1(C-k)$, it follows that $\beta(A_2) = A_2$ and therefore that $\pi_1(M-k)$ has a Betti matrix of the form

$$\begin{pmatrix} \beta(A_1) & 0 \\ 0 & A_2 \end{pmatrix}.$$

Then

$$\nu(k) = \nu(\beta(A_1)) + \nu(A_2) = \nu(\beta(A_1)) + 1 = 1 + \nu(M),$$

for every tame knot in a 3-cell has (Alexander) nullity equal to 1. Hence

$$n(M) \geqq \nu(k) = 1 + \nu(M). \qquad \text{Q.E.D.}$$

Let $\chi(\mathrm{Bd}M)$ denote the Euler characteristic of the boundary Bd M of the 3-manifold M. Theorem 2 above then yields the corollary:

COROLLARY: *If M is compact, then*

(i) *if* Bd M *is empty*, or if Bd M is not empty *but* $\chi(\mathrm{Bd}M) \geqq 3$, *then* $n(M) \geqq 1$;

(ii) *if* Bd M *is not empty but* $\chi(\mathrm{Bd}M) \leqq 2$, *then*

$$n(M) \geqq 2 - \tfrac{1}{2}\chi(\mathrm{Bd}M).$$

PROOF. The nullity of any Jacobian module of a group G of deficiency d is bounded below by 0 if d is negative, or by d if $d \geqq 0$. A lower bound for the deficiency of $\pi_1(M)$ will therefore yield a lower bound for $n(M)$, by theorem 2; we have used the lower bounds given in (6.2) of [5] (cf. theorem 2.2 of [2]) to obtain the lower bounds for $n(M)$. Q.E.D.

Note that if M is orientable and $\chi(\mathrm{Bd}M) \geqq 3$, the lower bound for $n(M)$ given above can be improved by using theorem 2.2 of [2] rather then (6.2) of [5].

QUESTION 2. *If M is a closed oriented 3-manifold different from S^3 then, by theorem 1 of [8], M can be written as the connected sum of prime 3-manifolds $M_1, M_2, \cdots, M_m$. If M is not closed, the pair $(M, \mathrm{Bd}M)$ can, by theorem 4.1 of [9], be written as the "connected sum along the boundary" of boundary-prime 3-manifold pairs $(M_1, \mathrm{Bd}M_1)$, $(M_2, \mathrm{Bd}M_2), \cdots, (M_m, \mathrm{Bd}M_m)$. How is the number $n(M)$ related to the numbers $n(M_1,), n(M_2), \cdots, n(M_m)$?*

QUESTION 3. *Can the upper bound of theorem 1 be replaced by $e(k) + n(M) - 1$, where $e(k)$ is the enclosure genus of k?*

In S^3, or in a 3-cell, the answer is yes, by theorem 2 of [6]; I conjecture that the answer is "yes" for all compact 3-manifolds.

If k is a knot with a finite or empty set of wild points, a *meridian* of k is a simple loop in $M - k$ which bounds a disc in M which meets k in precisely one

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point. Then the proof of theorem 2 of [6] can be modified to yield the following partial answer to question 3.

THEOREM 3. *Let k be an oriented knot with one wild point, lying in the interior of the compact 3-manifold M, and let $e(k)$ be the enclosure genus of k. Let ν be the nullity of the Betti module of k. Then if $B_1(M-k)$ is the infinite cyclic group generated by a meridian of k,*

$$\nu \leqq e(k) + n(M) - 1.$$

For we can still use the results of p. 549 of [6], since in the diagram

$$\begin{array}{ccc} & & \pi_1(S^3 - k^* - U) \\ & \overset{\zeta}{\nearrow} & \\ \pi_1((S^3 - U - k^*) \cap (U - \cup \beta_j)) & & \\ & \underset{\eta}{\searrow} & \\ & & \pi_1(U - \cup \beta_j) \end{array}$$

which occurs on page 551, all the groups are $\boldsymbol{Z}$-groups and all the maps are $\boldsymbol{Z}$-homomorphisms. Q.E.D.

We conclude with another partial answer to question 3, which may be obtained by modifying the proof of theorem 2 above, and using theorem 2 of [6]: *If k lies in the interior of a 3-cell in M, then*

$$\nu \leqq e(k) + \nu(M).$$

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