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On the computation of the determinant of vector-valued Siegel modular forms

Sho Takemori

Abstract

Let $A^0(\Gamma_2)$ denote the ring of scalar-valued Siegel modular forms of degree two, level 1 and even weights. In this paper, we prove the determinant of a basis of the module of vector-valued Siegel modular forms $\bigoplus_{k\equiv\epsilon \bmod 2} A_{\det^k\otimes \mathrm{Sym}(j)}(\Gamma_2)$ over $A^0(\Gamma_2)$ is equal to a power of the cusp form of degree two and weight 35 up to a constant. Here $j = 4, 6$ and $\epsilon = 0, 1$. The main result in this paper was conjectured by Ibukiyama (*Comment. Math. Univ. St. Pauli* 61 (2012) 51–75).

1. *Introduction*

Let $\mathrm{Sym}(j)$ be the symmetric tensor representation of $\mathrm{GL}_2(\mathbb{C})$ of degree j and Γ_2 the full Siegel modular group of degree two. We denote by $A_{k,j}(\Gamma_2)$ the space of vector-valued Siegel modular forms of degree two, level 1 and weight $\det^k\otimes\mathrm{Sym}(j)$. When $j = 0$, we simply denote $A_k(\Gamma_2) = A_{k,0}(\Gamma_2)$. We put

$$A^0_{\mathrm{Sym}(j)}(\Gamma_2) = \bigoplus_{k\equiv 0 \bmod 2} A_{k,j}(\Gamma_2), \quad A^1_{\mathrm{Sym}(j)}(\Gamma_2) = \bigoplus_{k\equiv 1 \bmod 2} A_{k,j}(\Gamma_2).$$

When $j = 0$, we denote $A^0(\Gamma_2) = A^0_{\mathrm{Sym}(0)}(\Gamma_2)$. Then $A^0_{\mathrm{Sym}(j)}(\Gamma_2)$ and $A^1_{\mathrm{Sym}(j)}(\Gamma_2)$ become $A^0(\Gamma_2)$ modules. As is well known, Igusa [8] gave explicit generators and the structure of the ring of Siegel modular forms of degree two:

$$A^0(\Gamma_2) = \mathbb{C}[\phi_4, \phi_6, \chi_{10}, \chi_{12}];$$

the modular forms $\phi_4, \phi_6, \chi_{10}, \chi_{12}$ are algebraically independent over $\mathbb{C}$;

$$\bigoplus_k A_k(\Gamma_2) = A^0(\Gamma_2) \oplus \chi_{35} A^0(\Gamma_2).$$

Here ϕ_k is the Siegel–Eisenstein series of weight k (for $k = 4, 6$) and χ_k is a cusp form weight k (for $k = 10, 12, 35$). It is known that there exists a unique irreducible polynomial P of four variables which satisfies $\chi_{35}^2 = \chi_{10} P(\phi_4, \phi_6, \chi_{10}, \chi_{12})$. Thus we also have

$$\bigoplus_k A_k(\Gamma_2) \cong \mathbb{C}[t_4, t_6, t_{10}, t_{12}, t_{35}]/(t_{35}^2 - t_{10}P(t_4, t_6, t_{10}, t_{12})),$$

where t_i $(i = 4, 6, 10, 12, 35)$ are variables.

The generators of the modules $A^0_{\mathrm{Sym}(j)}(\Gamma_2)$ and $A^1_{\mathrm{Sym}(j)}(\Gamma_2)$ and the fundamental relations among them have been examined by several authors. Satoh [11] proved the structure theorem for $A^0_{\mathrm{Sym}(2)}(\Gamma_2)$. To prove the linear independence of some functions over $A^0(\Gamma_2)$, he showed

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the nonvanishing of the following generalized Wronskian:

$$\det\begin{pmatrix} 4\phi_4 & 6\phi_6 & 10\chi_{10} & 12\chi_{12} \\ \partial_1\phi_4 & \partial_1\phi_6 & \partial_1\chi_{10} & \partial_1\chi_{12} \\ \partial_2\phi_4 & \partial_2\phi_6 & \partial_2\chi_{10} & \partial_2\chi_{12} \\ \partial_3\phi_4 & \partial_3\phi_6 & \partial_3\chi_{10} & \partial_3\chi_{12} \end{pmatrix}. \tag{1.1}$$

Here $Z = \left(\begin{smallmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{smallmatrix}\right)$ is the parameter of the Siegel upper half space $\mathbb{H}_2$ of degree two and

$$\partial_1 = \frac{\partial}{\partial z_{11}}, \quad \partial_2 = \frac{\partial}{\partial z_{12}}, \quad \partial_3 = \frac{\partial}{\partial z_{22}}.$$

Aoki and Ibukiyama [1] proved the Wronskian (1.1) is a non-zero constant multiple of χ_{35}.

The Wronskian (1.1) can be interpreted as follows. Let $\bigwedge^{j+1}\mathrm{Sym}(j)$ be the $j+1$th exterior product of $\mathrm{Sym}(j)$. Then as representations of $\mathrm{GL}_2(\mathbb{C})$, we have $\bigwedge^{j+1}\mathrm{Sym}(j) \cong \det^{1/2j(j+1)}$. Take $j+1$ non-negative integers $k_1, \ldots, k_{j+1}$ and $j+1$ vector-valued Siegel modular forms $f_i \in A_{k_i,j}(\Gamma_2)$ for $1 \leqslant i \leqslant j+1$. We define $\det(f_1, \ldots, f_{j+1})$ by $f_1 \wedge \ldots \wedge f_{j+1}$. Then $\det(f_1, \ldots, f_{j+1})$ is a vector-valued Siegel modular of weight $\det^{k_1+\ldots+k_{j+1}} \otimes \bigwedge^{j+1}\mathrm{Sym}(j)$. We fix a basis of the one-dimensional space $\bigwedge^{j+1}\mathrm{Sym}(j)$ and consider $\det(f_1, \ldots, f_{j+1})$ as an element of $A_k(\Gamma_2)$, where $k = \sum_{i=1}^{j+1} k_i + 1/2j(j+1)$. Let $f_1 = [\phi_4, \phi_6]$, $f_2 = [\phi_4, \chi_{10}]$ and $f_3 = [\phi_4, \chi_{12}]$ be the first three generators of $A^0_{\mathrm{Sym}(2)}(\Gamma_2)$ given by Satoh [11]. Then, it is easy to see that (1.1) is a non-zero constant multiple of $\phi_4^{-2}\det(f_1, f_2, f_3)$. Thus we see that $\det(f_1, f_2, f_3)$ is a non-zero constant multiple of $\phi_4^2\chi_{35}$.

For $V = A^0_{\mathrm{Sym}(j)}(\Gamma_2)$ (respectively $A^1_{\mathrm{Sym}(j)}(\Gamma_2)$), we put $j(V) = j$ and $\epsilon(V) = 0$ (respectively 1). Define sets $\mathfrak{S}$ and $\mathfrak{T}$ by

$$\mathfrak{S} = \{A^0_{\mathrm{Sym}(4)}(\Gamma_2), A^1_{\mathrm{Sym}(4)}(\Gamma_2), A^0_{\mathrm{Sym}(6)}(\Gamma_2)\},$$
$$\mathfrak{T} = \{A^1_{\mathrm{Sym}(6)}(\Gamma_2), A^0_{\mathrm{Sym}(8)}(\Gamma_2), A^1_{\mathrm{Sym}(8)}(\Gamma_2)\}.$$

In [7], Ibukiyama gave generators and fundamental relations for $V = A^1_{\mathrm{Sym}(2)}(\Gamma_2)$ and $V \in \mathfrak{S}$. He proved V is a free $A^0(\Gamma_2)$ module of rank $j(V)+1$ for $V \in \mathfrak{S}$ (in § 3, we shall prove that if $A^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma_2)$ is a free $A^0(\Gamma_2)$ module then its rank is equal to $j+1$). It is proved that V is also free of rank $j(V)+1$ for $V \in \mathfrak{T}$. This work was done by van Dorp [14] for $V = A^1_{\mathrm{Sym}(6)}(\Gamma_2)$ and by Kiyuna [9] for $V = A^0_{\mathrm{Sym}(8)}(\Gamma_2)$ and $A^1_{\mathrm{Sym}(8)}(\Gamma_2)$. To prove the linear independence of generators, Ibukiyama proved the nonvanishing of the determinant of generators of $V \in \mathfrak{S}$. By the calculation of weights and Fourier–Jacobi expansion, he conjectured the following statement.

CONJECTURE 1.1 (Ibukiyama). For $V \in \mathfrak{S} \cup \mathfrak{T}$, let $f_1, \ldots, f_{j+1}$ be the basis of V over $A^0(\Gamma_2)$. Then the determinant $\det(f_1, \ldots, f_{j+1})$ is a non-zero constant multiple of $\chi_{35}^{j(V)/2+\epsilon(V)}$.

In this paper, we prove the following statement.

THEOREM 1.2. *With the notation above, the determinant* $\det(f_1, \ldots, f_{j+1})$ *is a non-zero constant multiple of* $\chi_{35}^{j(V)/2+\epsilon(V)}$ *if* $(j(V), \epsilon(V)) = (4,0), (4,1), (6,0)$ *or* $(6,1)$.

To prove the main result, we use Sage [12] and a Sage package for Siegel modular forms of degree two written by the author. The package can be found at https://github.com/stakemori/degree2.

C. Citro, A. Ghitza, N.-P. Skoruppa, M. Raum, N. Ryan and G. Tornarìa also wrote a Sage package for Siegel modular forms of degree two (see http://trac.sagemath.org/ticket/8701).

They implemented a function that computes the multiplication of two Fourier expansions of Siegel modular forms. But it seems that they did not implement a function that computes the multiplication of two Fourier expansions that are not necessarily Siegel modular forms. In this paper, we have to compute many Rankin–Cohen type differential operators. And this computation needs a function that computes the multiplication of derivatives of Siegel modular forms. Since it is not safe to modify such a low-level function as computes the multiplication of Fourier expansions, the author wrote his own package.

2. *Definition*

We review the definition of vector-valued Siegel modular forms of degree two. Define the symplectic group of degree two by

$$\mathrm{Sp}_2(\mathbb{R}) = \{g \in \mathrm{GL}_4(\mathbb{R}) \mid {}^t g w_2 g = w_2\},$$

where $w_2 = \left(\begin{smallmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{smallmatrix}\right)$. Put $\Gamma_2 = \mathrm{Sp}_2(\mathbb{R}) \cap \mathrm{GL}_4(\mathbb{Z})$. We denote by $(\mathrm{Sym}(j), V_j)$ the symmetric tensor representation of $\mathrm{GL}_2(\mathbb{C})$. We identify V_j with the space of homogeneous polynomials $P(u_1, u_2)$ in $u_1,\ u_2$ of degree j. The action of $g \in \mathrm{GL}_2(\mathbb{C})$ is given by $(\mathrm{Sym}(j)(g)P)(u) = P(ug)$ where $u = (u_1, u_2)$. For a V_j-valued function F on the Siegel upper half space $\mathbb{H}_2$ of degree two, non-negative integer k and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$, we put

$$(F|_{k,j}[g])(Z) = \det(cZ+d)^{-k}\mathrm{Sym}(j)(cZ+d)^{-1}F(gZ) \qquad \text{for } Z \in \mathbb{H}_2.$$

A V_j-valued holomorphic function F on $\mathbb{H}_2$ is said to be a vector-valued Siegel modular form of weight $\det^k \otimes \mathrm{Sym}(j)$ if and only if $F|_{k,j}[\gamma] = F$ for all $\gamma \in \Gamma_2$. We call k the determinant weight of F. It is easy to see that $A_{k,j}(\Gamma_2) = 0$ if j is odd.

3. *Hilbert–Poincaré series*

In this section, we prove that the rank of $A^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma_2)$ is equal to $j+1$ if $A^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma_2)$ is a free $A^0(\Gamma_2)$ module for $\epsilon = 0, 1$. And we compute the weight of the determinant of the basis.

For $\epsilon = 0, 1$, let $h_{j,\epsilon}(t)$ be the Hilbert–Poincaré series of $A^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma_2)$

$$h_{j,\epsilon}(t) = \sum_{k \equiv \epsilon \bmod 2} \dim_{\mathbb{C}} A_{k,j}(\Gamma_2) t^k.$$

We put

$$f_{j,\epsilon}(t) = (1-t^4)(1-t^6)(1-t^{10})(1-t^{12})h_{j,\epsilon}(t).$$

Then by the Hilbert–Serre theorem, we have $f_{j,\epsilon}(t) \in \mathbb{Z}[t]$.

We can compute the values of $f_{j,\epsilon}$ and its differential $f'_{j,\epsilon}$ at $t = 1$.

PROPOSITION 3.1. *For $j \in 2\mathbb{Z}_{\geqslant 0}$, we have*

$$f_{j,\epsilon}(1) = j+1, \quad f'_{j,\epsilon}(1) + 1/2j(j+1) = 35 \times \begin{cases} j/2 & \text{if } \epsilon = 0, \\ j/2+1 & \text{if } \epsilon = 1. \end{cases}$$

Proof. For $k, j \in \mathbb{Z}_{\geqslant 0}$, we put $a_{k,j} = 0$ if j is odd and let $a_{k,2j}$ be the value given by Tsushima [**13**, Theorem 4]. Then we have $a_{k,j} = \dim_{\mathbb{C}} S_{k,j}(\Gamma_2)$ if $k \geqslant 5$. Here $S_{k,j}(\Gamma_2)$ is the cuspidal

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subspace of $A_{k,j}(\Gamma_2)$. By [13, Theorem 4], there exists a polynomial $g(t,s) \in \mathbb{C}[t,s]$ such that

$$\sum_{k,j=0}^{\infty} a_{k,j} t^k s^j = \frac{g(t,s)}{q(t)r(s)}.$$

Here $q(t)$ and $r(s)$ are defined by

$$q(t) = (1-t^4)(1-t^6)(1-t^{10})(1-t^{12}),$$
$$r(s) = (1-s^6)(1-s^8)(1-s^{10})(1-s^{12}).$$

The polynomial $g(t,s)$ is of degree 31 with respect to t, of degree 34 with respect to s and has 354 terms. Let $g(t,s) = \sum_{i,j} c_{ij} t^i s^j$ with $c_{ij} \in \mathbb{Z}$. For $\epsilon = 0, 1$, we define a polynomial $g_\epsilon(t,s)$ by $g_\epsilon(t,s) = \sum_{i \equiv \epsilon \bmod 2} c_{ij} t^i s^j$. Then $g_\epsilon(t,s)$ satisfies the equations

$$\frac{g_\epsilon(1,s)}{r(s)} = \frac{1+s^2}{(1-s^2)^2} = \sum_{j \in 2\mathbb{Z}_{\geqslant 0}} (j+1)s^j, \quad \text{for } \epsilon = 0, 1, \tag{3.1}$$

$$\begin{aligned} \frac{\partial g_\epsilon}{\partial t}(1,s)/r(s) + \sum_{j \in 2\mathbb{Z}_{\geqslant 0}} 1/2j(j+1)s^j &= \frac{35 s^{2(1-\epsilon)}}{(1-s^2)^2} \\ &= 35 \times \begin{cases} \sum_{j \in 2\mathbb{Z}_{\geqslant 0}} j/2 s^j & \text{if } \epsilon = 0, \\ \sum_{j \in 2\mathbb{Z}_{\geqslant 0}} (j/2+1)s^j & \text{if } \epsilon = 1. \end{cases} \end{aligned} \tag{3.2}$$

Suppose $k > 4$ is even. Then by Klingen [10] and Arakawa [2], we have

$$A_{k,j}(\Gamma_2) \cong \begin{cases} \mathbb{C}\phi_k \oplus S_{k,0}(\Gamma_2) \oplus S_k(\Gamma_1) & \text{if } j = 0, \\ S_{k,j}(\Gamma_2) \oplus S_{k+j}(\Gamma_1) & \text{if } j > 0. \end{cases}$$

Here $S_{k+j}(\Gamma_1)$ is the space of elliptic cusp forms of level 1 and we put $\phi_k = 0$ if k is odd. Since the denominator of the Hilbert–Poincaré series of $\{S_{k+j}(\Gamma_1)\}_k$ is equal to $(1-t^4)(1-t^6)$, we have

$$\frac{g_\epsilon(t,s)}{q(t)r(s)} = \sum_{j=0}^{\infty} \left(\frac{f_{j,\epsilon}(t)}{q(t)} + \frac{P_j(t)}{(1-t^4)(1-t^6)} + \frac{Q_j(t)}{(1-t^2)} \right) s^j,$$

where $P_j(t)$ and $Q_j(t)$ are polynomials. Therefore by equations (3.1) and (3.2), we obtain the assertion of the proposition. □

By Hilbert's syzygy theorem, we have the following corollary (see [5, Exercise 19.14]).

COROLLARY 3.2. *Assume $A^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma_2)$ is a finitely generated $A^0(\Gamma_2)$-module. Put $K = \mathrm{Frac}\, A^0(\Gamma_2)$. Then the following statements hold:*

(i) $\dim_K A^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma_2) \otimes K = j+1$ *for* $\epsilon = 0, 1$;

(ii) *if $A^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma_2)$ is free and $\{f_1, \dots, f_{j+1}\}$ is its basis then $\det(f_1, \dots, f_{j+1}) \in A_{35n}(\Gamma_2)$, where*

$$n = \begin{cases} j/2 & \text{if } \epsilon = 0, \\ j/2+1 & \text{if } \epsilon = 1. \end{cases}$$

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4. *The structure of the modules in the case of* Sym(4) *and* Sym(6)

We recall the results of Ibukiyama [7] and van Dorp [14] on the structure of the modules of $A^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma_2)$ $(j=4,6,\ \epsilon=0,1)$.

4.1. *General theory of Rankin–Cohen differential operators*

For the construction of vector-valued Siegel modular forms, Ibukiyama used Rankin–Cohen type differential operators. In this subsection, we recall the theory of Rankin–Cohen type differential operators in the degree two case.

Let t be a positive integer and $r_{11}^{(i)}, r_{12}^{(i)}, r_{22}^{(i)}$ variables for $1 \leqslant i \leqslant t$. We put $R_i = \left(\begin{smallmatrix} r_{11}^{(i)} & r_{12}^{(i)} \\ r_{12}^{(i)} & r_{22}^{(i)} \end{smallmatrix}\right)$. Let $\mathbb{H}_2^{(1)}, \ldots, \mathbb{H}_2^{(t)}$ be t copies of $\mathbb{H}_2$ and put $Z_i = \left(\begin{smallmatrix} z_{11}^{(i)} & z_{12}^{(i)} \\ z_{12}^{(i)} & z_{22}^{(i)} \end{smallmatrix}\right) \in \mathbb{H}_2^{(i)}$. Let $Q(R_1, \ldots, R_t, u)$ be a $\mathbb{C}$-coefficient polynomial of the components of R_i and u such that Q is homogeneous of degree j in u. For such Q and holomorphic functions $f_i(Z_i)$ on $\mathbb{H}_2^{(i)}$ $(1 \leqslant i \leqslant t)$, we define a function on $\mathbb{H}_2$ by

$$\{f_1, \ldots, f_t\}_Q(Z) = \left(Q(\partial_{Z_1}, \ldots, \partial_{Z_t}, u)\left(\prod_{i=1}^{t} F_i(Z_i)\right)\right)\bigg|_{Z_i=Z}.$$

Here

$$\partial_Z = \left(\frac{1+\delta_{ij}}{2(2\pi i)}\frac{\partial}{\partial z_{ij}}\right)$$

and $|_{Z_i=Z}$ means substituting Z_i by Z for all $1 \leqslant i \leqslant t$.

THEOREM 4.1 (Ibukiyama [6]). *Fix $k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant 1}$ and $a \in \mathbb{Z}_{\geqslant 0}$. Then Q satisfies*

$$\{f_1|_{k_1}[g], \ldots, f_t|_{k_t}[g]\}_Q = \{f_1, \ldots, f_t\}_Q|_{k_1+\ldots+k_t+a,j}[g]$$

for any holomorphic functions f_i $(1 \leqslant i \leqslant t)$ and for any $g \in \mathrm{Sp}_2(\mathbb{R})$ if and only if Q satisfies the following two conditions.

(i) *For all $A \in \mathrm{GL}_2$, Q satisfies*

$$Q(AR_1{}^tA, \ldots, AR_t{}^tA, u) = (\det A)^a Q(R_1, \ldots, R_t, uA).$$

(ii) *Let X_i (for $1 \leqslant i \leqslant t$) be $2 \times 2k_i$ matrix variables. Then $Q(X_1{}^tX_1, \ldots, X_t{}^tX_t)$ is pluriharmonic with respect to $X = (x_{ij})$, that is*

$$\sum_{\nu=1}^{2(k_1+\ldots+k_r)} \frac{\partial^2 Q}{\partial x_{i\nu}\partial x_{j\nu}} = 0, \quad \text{for } 1 \leqslant i,j \leqslant 2.$$

We shall give examples for Q satisfying the conditions (i) and (ii) in Theorem 4.1.

4.2. *Examples in the case of $t=2$*

Let $t=2$. We write $R = R_1 = (r_{ij})$, $S = R_2 = (s_{ij})$, $k_1 = k$ and $k_2 = l$. Eholzer and Ibukiyama [4] gave polynomials satisfying the conditions (i) and (ii) in Theorem 4.1 for $a=0$ and $a=2$. We introduce these polynomials.

For $k, l, m \in \mathbb{Z}_{\geqslant 0}$, we put

$$G_{k,l,m}(x,y) = \sum_{i=0}^{m} (-1)^i \binom{m+l-1}{i}\binom{m+k-1}{m-i} x^i y^{m-i}.$$

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For an even j, we put $Q_{k,l,j}(R,S) = G_{k,l,j/2}(r,s)$, where $r = r_{11}u_1^2 + 2r_{12}u_1u_2 + r_{22}u_2^2$ and $s = s_{11}u_1^2 + 2s_{12}u_1u_2 + s_{22}u_2^2$. Then the polynomial $Q_{k,l,j}$ satisfies the conditions (i) and (ii) in Theorem 4.1 for $a = 0$. For $F \in A_k(\Gamma_2)$ and $G \in A_l(\Gamma_2)$, we put

$$\{F,G\}_{\mathrm{Sym}(j)} = \{F,G\}_{Q_{k,l,j}}.$$

Then we have $\{F,G\}_{\mathrm{Sym}(j)} \in A_{k+l,j}(\Gamma_2)$. If $j = 2$, this differential operator was defined by Satoh [11].

We define a polynomial $Q_{k,l,(2,j)}(R,S,u)$ by

$$4^{-1}G_2(R,S)G_{k+1,l+1,j/2}(r,s) + 2^{-1}((2l-1)\det(R)s - (2k-1)\det(S)r) \\ \times \left(\frac{\partial G_{k+1,l+1,j/2}}{\partial x}(r,s) - \frac{\partial G_{k+1,l+1,j/2}}{\partial y}(r,s)\right).$$

Here $G_2(R,S)$ is defined by

$$G_2(R,S) = (2k-1)(2l-1)\det(R+S) - (2k-1)(2k+2l-1)\det(S) \\ - (2l-1)(2k+2l-1)\det(R).$$

Then the polynomial $Q_{k,l,(2,j)}$ satisfies the conditions (i) and (ii) in Theorem 4.1 for $a = 2$. For $F \in A_k(\Gamma_2)$ and $G \in A_l(\Gamma_2)$, we put

$$\{F,G\}_{\det^2 \mathrm{Sym}(j)} = \{F,G\}_{Q_{k,l,(2,j)}}.$$

Then we have $\{F,G\}_{\det^2 \mathrm{Sym}(j)} \in A_{k+l+2,j}(\Gamma_2)$.

4.3. *Example in the case of $t = 3$ and $j = 4$*

Let $t = 3$ and $j = 4$. We put $R = (r_{ij})_{1\leqslant i,j\leqslant 2}$, $S = (s_{ij})_{1\leqslant i,j\leqslant 2}$ and $T = (t_{ij})_{1\leqslant i,j\leqslant 2}$.

We define $Q_{\det \mathrm{Sym}(4)}(R,S,T,u) = \sum_{i=0}^{4} Q_i(R,S,T)u_1^{4-i}u_2^i$. Here $Q_0(R,S,T)$ is defined by

$$Q_0(R,S,T) = (k_2+1)\begin{vmatrix} (k_1+1)r_{11} & k_2 & k_3 \\ r_{11}^2 & s_{11} & t_{11} \\ r_{11}r_{12} & s_{12} & t_{12} \end{vmatrix} - (k_1+1)\begin{vmatrix} k_1 & (k_2+1)s_{11} & k_3 \\ r_{11} & s_{11}^2 & t_{11} \\ r_{12} & s_{11}s_{12} & t_{12} \end{vmatrix}.$$

We omit the definition of $Q_i(R,S,T)$ if $i > 0$. The polynomial $Q_{\det \mathrm{Sym}(4)}$ was given by Ibukiyama [7] and satisfies conditions (i) and (ii) in Theorem 4.1 for $a = 1$ and $j = 4$. For $f_i \in A_{k_i}(\Gamma_2)$ $(i = 1,2,3)$, we put

$$\{f_1,f_2,f_3\}_{\det \mathrm{Sym}(4)} = \{f_1,f_2,f_3\}_{Q_{\det \mathrm{Sym}(4)}}.$$

Then we have $\{f_1,f_2,f_3\}_{Q_{\det \mathrm{Sym}(4)}} \in A_{k_1+k_2+k_3+1,4}(\Gamma_2)$.

4.4. *The structure of $A^0_{\mathrm{Sym}(4)}(\Gamma_2)$, $A^1_{\mathrm{Sym}(4)}(\Gamma_2)$ and $A^0_{\mathrm{Sym}(6)}(\Gamma_2)$*

In this subsection, we recall the result of Ibukiyama [7].

THEOREM 4.2 (Ibukiyama).

(1) *The module $A^0_{\mathrm{Sym}(4)}(\Gamma_2)$ is free over $A^0(\Gamma_2)$ of rank 5 and is generated by the elements*

$$\{\phi_4,\phi_4\}_{\mathrm{Sym}(4)},\ \{\phi_4,\phi_6\}_{\mathrm{Sym}(4)},\ \{\phi_4,\phi_6\}_{\det^2 \mathrm{Sym}(4)}, \\ \{\phi_4,\chi_{10}\}_{\mathrm{Sym}(4)},\ \{\phi_6,\chi_{10}\}_{\mathrm{Sym}(4)}.$$

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(2) *The module* $A^1_{\mathrm{Sym}(4)}(\Gamma_2)$ *is free over* $A^0(\Gamma_2)$ *of rank* 5 *and is generated by the elements*

$$\{\phi_4,\phi_4,\phi_6\}_{\det \mathrm{Sym}(4)},\ \{\phi_4,\phi_6,\phi_6\}_{\det \mathrm{Sym}(4)},\ \{\phi_4,\phi_4,\chi_{10}\}_{\det \mathrm{Sym}(4)},$$
$$\{\phi_4,\phi_4,\chi_{12}\}_{\det \mathrm{Sym}(4)},\ \{\phi_4,\phi_6,\phi_{12}\}_{\det \mathrm{Sym}(4)}.$$

(3) *The module* $A^0_{\mathrm{Sym}(6)}(\Gamma_2)$ *is free over* $A^0(\Gamma_2)$ *of rank* 7 *and is generated by the elements*

$$E_{6,6},\ X_{8,6},\ X_{10,6},\ \{\phi_4,\phi_6\}_{\det^2 \mathrm{Sym}(6)},\ \{\phi_4,\chi_{10}\}_{\mathrm{Sym}(6)},$$
$$\{\phi_4,\chi_{12}\}_{\mathrm{Sym}(6)},\ \{\phi_6,\chi_{12}\}_{\mathrm{Sym}(6)}.$$

Here $E_{6,6}\in A_{6,6}(\Gamma_2)$ *is the Klingen–Eisenstein series associated with the Ramanujan Delta function* Δ, *which Arakawa* [**2**] *defined in the general case. The modular forms* $X_{8,6}\in A_{8,6}(\Gamma_2)$ *and* $X_{10,6}\in A_{10,6}(\Gamma_2)$ *are theta series defined in* [**7**, § 6].

Remark 4.3. Generators with small determinant weights often cannot be constructed by differential operators.

4.5. *The structure of* $A^1_{\mathrm{Sym}(6)}(\Gamma_2)$

We briefly recall the result of van Dorp [**14**].

As mentioned in the introduction, van Dorp [**14**] proved the module $A^0_{\mathrm{Sym}(6)}(\Gamma_2)$ is free of rank 7 and gave generators explicitly. He made a recipe for constructing a polynomial satisfying the conditions (i) and (ii) in Theorem 4.1 for $t=3, a=1$ and for all even $j\geqslant 2$. And he constructed polynomials satisfying the conditions (i) and (ii) in Theorem 4.1 for $t=3,\ j=6$ and $a=1$.

He also constructed a Rankin–Cohen differential operator on vector-valued Siegel modular forms, that is, for $f\in A_{k,j}(\Gamma_2)$ and $g\in A_k(\Gamma_2)$, he defined a Rankin–Cohen differential operator $\{f,g\}\in A_{k+l+1,j}(\Gamma_2)$.

He constructed five of the seven generators by Rankin–Cohen type differential operators (on scalar-valued Siegel modular forms) and the remaining two generators by $\{E_{6,6},\phi_4\}$ and $\{X_{8,6},\phi_4\}$. Here modular forms $E_{6,6}$ and $X_{8,6}$ are given in Theorem 4.2.

5. *Proof of the main result*

We prove our main result by numerical computation. We use Sage [**12**] and a Sage package for Siegel modular forms of degree two written by the author. With this package, we can compute generators $\phi_4,\ \phi_6,\ \chi_{10},\ \chi_{12}$ and χ_{35} of the ring of scalar-valued Siegel modular forms and a basis of the space of scalar-valued Siegel modular forms $A_k(\Gamma_2)$. We can also compute the action of Hecke operators on vector valued Siegel modular forms and the differential operators introduced in § 4 from given Siegel modular forms.

Put $V=A^{\epsilon}_{\mathrm{Sym}(j)}(\Gamma_2)$. By Ibukiyama [**7**], van Dorp [**14**] and Kiyuna [**9**], if $j(V)=4,6$ or 8, then V is free of rank $j(V)+1$ over $A^0(\Gamma_2)$. Therefore by Corollary 3.2 if $f_1,\dots,f_{j(V)+1}$ is its basis, then $\det(f_1,\dots,f_{j(V)+1})\in A_{k(V)}(\Gamma_2)$, where $k(V)=35(j(V)/2+\epsilon(V))$. We define an element $\det(V)$ of $A_{k(V)}(\Gamma_2)/\mathbb{C}^{\times}$ by

$$\det(V)=[\det(f_1,\dots,f_{j(V)+1})].$$

Here $\mathbb{C}^{\times}$ acts on $A_{k(V)}(\Gamma_2)$ by ordinary multiplication and $[\det(f_1,\dots,f_{j(V)+1})]$ is the class represented by $\det(f_1,\dots,f_{j(V)+1})$.

In order to prove $\det(V)=[\chi_{35}^{j(V)/2+\epsilon(V)}]$, it is enough to calculate a finite number of Fourier coefficients.

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Lemma 5.1. *Let k be a non-negative integer and $f \in A_k(\Gamma_2)$ a scalar-valued Siegel modular form. For $(n, r, m) \in \mathbb{Z}^3$ with $n,\ m,\ 4nm - r^2 \geqslant 0$, we denote by $a((n, r, m), f)$ the (n, r, m)th Fourier coefficient*

$$f(Z) = \sum_{(n,r,m)} a((n, r, m), f)\mathbf{e}(nz_{11} + rz_{12} + mz_{22}).$$

Here $Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \in \mathbb{H}_2$ and $\mathbf{e}(z) = \exp(2\pi i z)$. Assume that $a((n, r, m), f) = 0$ if $n \leqslant [k/10]$ and $m \leqslant [k/10]$, where $[x]$ is the Gauss symbol. Then we have $f = 0$.

Proof. Igusa [8] proves that $A_k(\Gamma_2)$ has a basis of forms with integral coefficients. Therefore we may assume $a((n, r, m), f) \in \mathbb{Z}$ for all (n, r, m). Applying the Sturm type theorem [3] for all primes $p \geqslant 5$, we obtain the assertion of the lemma. □

5.1. *Proof in the case of $A^0_{\mathrm{Sym}(4)}(\Gamma_2)$ and $A^1_{\mathrm{Sym}(4)}(\Gamma_2)$*

Since the proof is the same, we consider only the case of $A^0_{\mathrm{Sym}(4)}(\Gamma_2)$. By Lemma 5.1, we need to compute the Fourier coefficients of a representative of $\det(A_{\mathrm{Sym}(4)}(\Gamma_2))$ and χ_{35}^2 at (n, r, m) for $0 \leqslant n, m \leqslant 7$ and $r^2 - 4nm \leqslant 0$. For the computation of χ_{35}, we use the result of Aoki and Ibukiyama [1], which we mentioned in § 1. We can easily compute the other generators $\phi_4,\ \phi_6,\ \chi_{10}$ and χ_{12} because they are Siegel–Eisenstein series or written as polynomials of Siegel–Eisenstein series. The Fourier coefficients of $\{F_1, \ldots, F_r\}_Q$ can be written by a polynomial of the Fourier coefficients of $F_1, \ldots, F_r$ and the polynomial is determined by Q. Here we use the notation in § 4. Thus we can confirm our statement in this case if we use the package. The following code checks Theorem 1.2 in the case when $V = A^0_{\mathrm{Sym}(4)}(\Gamma_2)$.

```
from degree2.utils import naive_det as det
from degree2.all import (
    rankin_cohen_pair_sym, eisenstein_series_degree2,
    x10_with_prec, x12_with_prec, x35_with_prec,
    rankin_cohen_pair_det2_sym)

prec = 7

phi4 = eisenstein_series_degree2(4, prec)
phi6 = eisenstein_series_degree2(6, prec)
x10 = x10_with_prec(prec)
x12 = x12_with_prec(prec)
x35 = x35_with_prec(prec)

f1 = rankin_cohen_pair_sym(4, phi4, phi4)
f2 = rankin_cohen_pair_sym(4, phi4, phi6)
f3 = rankin_cohen_pair_det2_sym(4, phi4, phi6)
f4 = rankin_cohen_pair_sym(4, phi4, x10)
f5 = rankin_cohen_pair_sym(4, phi6, x10)

x70 = det([f.forms for f in [f1, f2, f3, f4, f5]])
y70 = -19945421021123916595200000^(-1) * x70
assert y70 == x35^2
```

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We explain the code above. In the first five lines, we load the functions used in this computation. In the seventh line, we define the variable prec, whose value is 7. This means we compute the Fourier coefficients of Siegel modular forms for (n, r, m) with $n, m \leqslant 7$. From the ninth line to the thirteenth line, we calculate ϕ_4, ϕ_6, χ_{10}, χ_{12} and χ_{35}. From the fifteenth line to the nineteenth line, we calculate the bases given in Theorem 4.2 and denote them by $f_1, \ldots, f_5$ respectively. In the 21st line, we compute $\det(f_1, \ldots, f_5)$ and denote by x_{70} the determinant. In the 22nd line, we calculate $-19945421021123916595200000^{-1}x_{70}$ and denote it by y_{70}. In the last line, we check whether y_{70} is equal to χ_{35}^2.

We can run this code if we install Sage [12] and the package 'degree2'. Running this code returns nothing. It means the main result for $V = A^0_{\mathrm{Sym}(4)}(\Gamma_2)$ is true. If the main result for this case were false, then the last line of the code would cause an error.

5.2. *Proof in the case of* $A^0_{\mathrm{Sym}(6)}(\Gamma_2)$ *and* $A^1_{\mathrm{Sym}(6)}(\Gamma_2)$

Since the proof is the same, we consider only the case of $A^0_{\mathrm{Sym}(6)}(\Gamma_2)$. Let $E_{6,6}$ and $X_{8,6}$ be the vector-valued Klingen–Eisenstein series and the theta series that appeared in Theorem 4.2. Since the computation of theta series is slow and we do not know the Fourier coefficients of vector-valued Klingen–Eisenstein series explicitly, we compute $\phi_6 E_{6,6}$ and $\phi_4 X_{8,6}$ instead. By Theorem 4.2, $\{\phi_6 E_{6,6}, \phi_4 X_{8,6}, F_{12}\}$ is a basis of $A_{12,6}(\Gamma_2)$ over $\mathbb{C}$. Here we put $F_{12} = \{\phi_4, \phi_6\}_{\det^2 \mathrm{Sym}(6)}$. We give another computable basis of $A_{12,6}(\Gamma_2)$ by F_{12}, $F_{12}|T(2)$ and a vector modular form constructed by a differential operator, where $T(2)$ is the Hecke operator. This method was already used by van Dorp [14].

Define $F_k \in A_{k,6}(\Gamma_2)$ for $k = 10, 14, 16, 18$ as follows:

$$F_{10} = \{\phi_4, \phi_6\}_{\mathrm{Sym}(6)}, \quad F_{14} = \{\phi_4, \chi_{10}\}_{\mathrm{Sym}(6)},$$
$$F_{16} = \{\phi_4, \chi_{12}\}_{\mathrm{Sym}(6)}, \quad F_{18} = \{\phi_6, \chi_{12}\}_{\mathrm{Sym}(6)}.$$

Then by Theorem 4.2, $\{E_{6,6}, X_{8,6}, X_{10,6}, F_{12}, F_{14}, F_{16}, F_{18}\}$ is a basis of $A^0_{\mathrm{Sym}(6)}(\Gamma_2)$ over $A^0(\Gamma_2)$. We define $G_{12}, H_{12} \in A_{12,6}(\Gamma_2)$ by

$$G_{12} = F_{12}|T(2), \quad H_{12} = \{\phi_4, \phi_4^2\}_{\mathrm{Sym}(8)}.$$

Here $T(2)$ is the Hecke operator. As mentioned above, $\{\phi_6 E_{6,6}, \phi_4 X_{8,6}, F_{12}\}$ is a basis of $A_{12,6}(\Gamma_2)$ over $\mathbb{C}$. It can be checked that $\{F_{12}, G_{12}, H_{12}\}$ is also a basis. By Theorem 4.2, $\dim_{\mathbb{C}} A_{10,6}(\Gamma_2) = 2$ and $\{\phi_4 E_{6,6}, X_{10,6}\}$ is a basis. It can be checked that $\{\phi_4 E_{6,6}, F_{10}\}$ is also a basis. Therefore we have

$$\phi_4\phi_6 \det(A^0_{\mathrm{Sym}(6)}(\Gamma_2)) = [\det(F_{10}, F_{12}, G_{12}, H_{12}, F_{14}, F_{16}, F_{18})],$$

in $A_{115}(\Gamma_2)/\mathbb{C}^\times$. Thus in order to prove the main result for this case, it is enough to prove that

$$[\det(F_{10}, F_{12}, G_{12}, H_{12}, F_{14}, F_{16}, F_{18})] = [\phi_4\phi_6\chi_{35}^3]. \tag{5.1}$$

Since $F_{10}, F_{12}, H_{12}, F_{14}, F_{16}$ and F_{18} are constructed by differential operators, we can compute these vector-valued Siegel modular forms. We can compute G_{12} by [2, (2.5)] if we compute Fourier coefficients of F_{12}. Since the computation of the Fourier coefficient of G_{12} at (n, r, m) requires the computation of the Fourier coefficient of F_{12} at $(2n, 2r, 2m)$, we need to compute more Fourier coefficients of F_{12} than the other vector-valued Siegel modular forms. We can prove (5.1) in the same way as the case where $V = A^0_{\mathrm{Sym}(4)}(\Gamma_2)$. The source code to check (5.1) is slightly too long to show here. The source code can be found at https://github.com/stakemori/det_vec_vald_SMFs.

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Sho Takemori
Department of Mathematics
Kyoto University
Kitashirakawa-Oiwake-Cho
Sakyo-Ku, Kyoto, 606-8502
Japan

takemori@math.kyoto-u.ac.jp