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DEPENDENCE OF BEST RATIONAL CHEBYSHEV APPROXIMATIONS ON THE DOMAIN

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Sufficient conditions are given for the error norm and coefficients of best rational Chebyshev approximation on a domain to depend continuously on the domain. Examples of discontinuity are given.

Let W be a space with metric ρ. For X, Y non-empty subsets of W define

$$\text{dist}(X, Y) = \sup\{\inf\{\rho(x, y) : x \in X\} : y \in Y\},$$

and the Hausdorff metric

$$d\ (X, Y) = \max\{\text{dist}(Y, Y), \text{dist}(Y, X)\}.$$

Let $X, X_1, \ldots, X_n, \ldots$ be compact subsets of W. We say $\{X_k\} \to X$ if $d(X, X_k) \to 0$. Let f be a fixed element of $C(W)$. Let $NG = \{\phi_1, \ldots, \phi_n\}$, $DG = \{\psi_1, \ldots \psi_m\}$ be linearly independent subsets of $C(W)$. Define

$$R(A, x) = P(A, x)/Q(A, x) = \sum_{k=1}^{n} a_k\phi_k(x) \Big/ \sum_{k=1}^{m} a_{n+k}\psi_k(x).$$

For a subscript s, define $\| \ \|_s$ to be the Chebyshev norm on X_s and define

$$\sigma(X_s) = \inf\{\|f - R(A, .)\|_s : Q(A, x) \geq 0 \quad \text{for} \quad x \in X_s, Q(A, .) \not\equiv 0\}.$$

A parameter A^* for which the infimum is attained is called best (on X_s). If we use the convention of Goldstein a best approximation always exists, providing there is A with $Q(A, .) > 0$, which we hence forth assume.

As in [1, 484], we normalize rational functions such that

$$\sum_{k=1}^{m} |a_{n+k}| = 1, \tag{1}$$

THEOREM 1. *Let the generators NG and DG be independent on X. Let* $\{X_k\} \to X$ *and* $R(A^k, .)$ *be best to f on* X_k. *Let f have a best approximation* r^* *on X and a closed neighbourhood N of X exist such that* (i) *the denominator of* r^* *is non-negative on N, and* (ii) r^* *is continuous on N. Then* $\sigma(X_k) \to \sigma(X)$, $\{A^k\}$ *has an accumulation point, and any accumulation point is best to f on X.*

Proof. The proof of the corresponding result of [1] can be used except for one point. By continuity of r^* on N, there is a neighbourhood L of X, $L \subset N$,

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such that

$$(2) \qquad \|f-r^*\|_L < \|f-r^*\|_X + \varepsilon$$

and for all k sufficiently large, $X_k \subset L$. We apply this to the bottom inequality of [1, 485] to get a contradiction, proving optimality of accumulation points.

We now show that $\sigma(X_k) \to \sigma(X)$, which was claimed but not explicitly shown in [1]. Let $\varepsilon > 0$ be given. Assume without loss of generality that $\{A^k\} \to A$. By arguments of [1], there is $x \in X$ such that $Q(A, x) > 0$ and

$$|f(x) - R(A, x)| > \|f - R(A, .)\|_X - \varepsilon. \quad \text{Let} \quad \{x_k\} \to x, \, x_k \in X_k,$$

then

$$|f(x_k) - R(A^k, x_k)| \to |f(x) - R(A, x)|$$

hence

$$\liminf_{k\to\infty} \sigma(X_k) \geq \sigma(X).$$

(This fills the gap in the proof of Theorem 1 of [1]). Hence if $\sigma(X_k) \not\to \sigma(X)$, we can assume that

$$\sigma(X_k) > \sigma(X) + \varepsilon$$

and that $X_k \subset L$. As $\sigma(X_k) \leq \|f - r^*\|_k \leq \|f - r^*\|_L$, we have (2) and a contradiction. Hence $\sigma(X_k) \to \sigma(X)$ and the theorem is proven.

REMARK. The closed neighbourhood N of the theorem is easily seen to exist if the denominator of r^* is positive on X.

The independence condition of the theorem cannot be deleted.

EXAMPLE 1. Let $X = \{0\}$, $X_k = \{1/k\}$, $f = 1$, and $R(a, x) = ax$. The unique coefficient of best approximation on X_k is $a^k = k$, $\sigma(X_k) = 0$ and since $R(a, 0) = 0$, $\sigma(X) = 1$.

The hypothesis of a non-negative denominator in the theorem cannot be weakened.

EXAMPLE 2. Let us approximate $f = 1$ by $R(A, x) = a_1 x/(a_2 + a_3 x)$. f is approximated with zero error on $[0, 1]$ by x/x. In approximation on $X_k = [-1/k, 1]$, the denominator must be positive at 0, hence all permitted approximants vanish at zero, and 0 is a best approximation with error norm of 1.

COROLLARY. *Suppose in addition f has a unique best approximation $R(A, .)$ on X which has a unique representation on X under the normalization* (1) *and $Q(A, x) > 0$ for $x \in X$. Then $\{A^k\} \to A$, $Q(A^k, x) > 0$ for $x \in X_k \cup X$ and all k sufficiently large, and $\{R(A^k, .)\}$ converges uniformly to $R(A, .)$ on X.*

If we merely have $R(A,.)$ a unique best approximation and $Q(A, x)>0$ for $x \in X$, uniform convergence may not occur (see the example at the end of [1]).

Examples 1 and 2 show that σ need be neither lower semi-continuous, nor upper semi-continuous.

Let us next consider approximation by admissible rational functions (denominators are greater than zero). Define

$$\sigma_+(X_s) = \inf\{\|f - R(A,.)\|_s : Q(A, x) > 0 \quad \text{for} \quad x \in X_s\}.$$

A result comparable to Theorem 1 does not hold even when $\{X_k\} \subset X$.

EXAMPLE 3. Let $X = [0, 1]$, $X_k = [1/k, 1]$, $f = 1$, $R(A, x) = a_1 x/(a_2 + a_3 x)$. $x/x = 1 = f$ is best to f on X_k and $\sigma_+(X_k) = 0$. x/x is not admissible on X and since $R(A, 0) = 0$ for all admissible A, 0 is best to f on X and $\sigma_+(X) = 1$. We, however, have

THEOREM 2. *Let the generators NG and DG be independent on X. Let f have a unique best admissible approximation $R(A,.)$ on X which has a unique representation on X under normalization* (1). *Let $\{X_k\} \to X$. For all k sufficiently large there is a best admissible approximation $R(A^k,.)$ to f on X_k (it is also admissible on X), $\{A^k\} \to A$, and $\{R(A^k,.)\}$ converges uniformly to $R(A,.)$ on X.*

Proof. It is shown in [1, middle 486] that $R(A,.)$ is best in rationals with non-negative denominators. We then apply the earlier results of this paper.

It is seen from earlier results that $\sigma_+(X_k) \to \sigma_+(X)$ under the hypotheses of the above theorem.

Without the unique representation hypothesis of Theorem 2, we may not have existence (see the example at the end of [2]) or uniform convergence (see the example at the end of [1]), even when $X_k \subset X$.

Sentence two of Theorem 2 can be replaced by "Let $R(A,.)$ be a best admissible approximation to f on X and $S(A)$ be a Haar subspace of dimension $n+m-1$ on X", where

$$S(A) = \{R(A,.)Q(B,.) + P(B,.)\}.$$

That $R(A,.)$ is uniquely best on X follows from classical uniqueness results. If $R(A,.)$ had another representation, $S(A)$ would be of dimension less than $n+m+1$.

A case of special interest is where $W = [\alpha, \beta]$, a closed finite interval, and R is the rational approximating function for ordinary rational approximation. The examples at the end of papers [1; 2] show respectively that uniform convergence need not occur nor best admissible approximations exist on subsets. Whether σ and σ_+ are continuous in this case is an open question. It is open even for the case of approximation by constants divided by first-degree polynomials.

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The possibility of discontinuity of σ was first shown by the author in [3]. Dependence of best linear approximations on the domain is treated implicitly by Kripke in [5] and explicitly by the author in [4]. Riha [6] considers the case of linear approximation on an interval.

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