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THE CURVED A_∞-COALGEBRA OF THE KOSZUL CODUAL OF A FILTERED DG ALGEBRA

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Abstract. The goal of this article is to study the coaugmented curved A_∞-coalgebra structure of the Koszul codual of a filtered dg algebra over a field k. More precisely, we first extend one result of B. Keller that allowed to compute the A_∞-coalgebra structure of the Koszul codual of a nonnegatively graded connected algebra to the case of any unitary dg algebra provided with a nonnegative increasing filtration whose zeroth term is k. We then show how to compute the coaugmented curved A_∞-coalgebra structure of the Koszul codual of a Poincaré-Birkhoff-Witt (PBW) deformation of an N-Koszul algebra.

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1. Introduction. The objective of this article is twofold:

(i) extend one result of B. Keller that allowed to compute the A_∞-coalgebra structure of the Koszul codual of a nonnegatively graded connected graded algebra to the case of any unitary dg algebra provided with a nonnegative increasing filtration whose zeroth term is the field k (see Theorem 5.2);
(ii) apply the previous result to compute the coaugmented curved A_∞-coalgebra structure of the Koszul codual of a PBW deformation of an N-Koszul algebra (see Theorems 6.2 and 6.4).

The paper is organised as follows. In Section 2, we first recall some of the basic definitions on curved A_∞-coalgebras, following essentially [**9**]. The only new result is Proposition 2.1, which is only an exercise in specializing the general definition to a particular case. In Sections 3 and 4, we present the rudimentary and well-known facts we will need on generalised Koszul algebras and their PBW deformations, respectively.

In Section 5, we present the first two main results of this article, which generalise the mentioned theorem by Keller (see Proposition 5.1 and Theorem 5.2).

In Section 6, we apply the previous results to recursively compute the coaugmented curved A_∞-coalgebra of the Koszul codual of a PBW deformation U of an N-Koszul algebra A. More precisely, we first show that a PBW deformation U of an N-Koszul algebra A determines a unique coaugmented curved A_∞-coalgebra structure on $\mathrm{Tor}^A_\bullet(k,k)$ satisfying some assumptions (see Theorem 6.2). The results required to prove this theorem are relegated to Sections 6.2–6.5, and they follow but also complete the ideas in [**3**, Section 3]. Indeed, using the notation of that article, the authors never proved that $m_1 \circ d = 0 = d \circ m_1$ (cf. Lemma 6.20). Moreover, they also used the previously mentioned result of Keller, which only applies to nonnegatively graded connected algebras, even though the PBW deformations they consider are not nonnegatively graded (they are just connected). Our Theorem 5.2 fills this gap as well. Furthermore, some of our proofs are shorter and clearer (cf. [**3**, Lemmas 3.6 and 3.7] and our Lemmas 6.19 and 6.21, resp.), and mostly with less signs (cf. [**3**, Lemma 3.5] and our Fact 6.18). Finally, as an application of Theorem 5.2, we prove that the previous coaugmented curved A_∞-coalgebra structure on $\mathrm{Tor}^A_\bullet(k,k)$ is filtered quasi-equivalent to the curved bar construction of U (see Theorem 6.4). In particular, this also gives a description of a 'small' projective resolution of the standard U-bimodule U, and it generalises the Koszul bimodule complex given by R. Berger and V. Ginzburg for a particular case in [**2**, Section 5] (see Remark 6.5).

2. Some definitions and a basic result. In what follows, k will denote a field. For A_∞-(co)algebras, we shall use the conventions and terminology given in [**5**, Section 2.1]. We recall that, if $V = \oplus_{n\in\mathbb{Z}} V^n$ is a (cohomological) graded vector space, $V[m]$ is the graded vector space over k whose n-th homogeneous component $V[m]^n$ is given by V^{m+n}, for all $n, m \in \mathbb{Z}$, and it is called the *shift* of V. We will denote by $s_V : V \to V[1]$ the *suspension morphism*, whose underlying map is the identity of V. We are not going to consider any shift on other gradings, such as the Adams grading. All morphisms between modules will be k-linear (satisfying further requirements if the modules are decorated). All unadorned tensor products $\otimes$ would be over k.

Finally, $\mathbb{N}$ will denote the set of (strictly) positive integers, whereas $\mathbb{N}_0$ will be the set of nonnegative integers. Similarly, for $N \in \mathbb{N}$, we denote by $\mathbb{N}_{\geq N}$ the set of positive integers greater than or equal to N. The analogous conventions hold for other inequality symbols. When working with $\mathbb{N}^p$ or $\mathbb{N}_0^p$, we will denote by $e_j \in \mathbb{N}_0^p$ the p-tuple whose i-th coordinate is $\delta_{i,j}$, the Kronecker's delta. We also recall the convention that a sum over an empty set of indices is zero.

2.1. Basics on curved (strongly homotopic) coalgebras. We will now recall the basic definitions of curved (strongly homotopic) coalgebras, since they are not so widely known. We follow the basic conventions of [**9**] (see also [**5, 10**]), to which we refer.

Let $(C, \Delta_C, \epsilon_C)$ be a *counitary (coassociative) graded coalgebra*, i.e., the homogeneous maps $\Delta_C : C \to C \otimes C$ and $\epsilon_C : C \to k$ of degree zero satisfy that $(\Delta_C \otimes \mathrm{id}_C) \circ \Delta_C = (\mathrm{id}_C \otimes \Delta_C) \circ \Delta_C$, $(\epsilon_C \otimes \mathrm{id}_C) \circ \Delta_C = \mathrm{id}_C = (\mathrm{id}_C \otimes \epsilon_C) \circ \Delta_C$. Let (M, ρ) be a *(counitary) graded bicomodule over* C. We recall that this means that we are given a homogeneous map $\rho : M \to C \otimes M \otimes C$ of degree zero satisfying that $(\Delta_C \otimes \mathrm{id}_M \otimes \Delta_C) \circ \rho = (\mathrm{id}_C \otimes \rho \otimes \mathrm{id}_C) \circ \rho$ and $(\epsilon_C \otimes \mathrm{id}_M \otimes \epsilon_C) \circ \rho = \mathrm{id}_M$. Setting $\rho_\ell : M \to C \otimes M$ and $\rho_r : M \to M \otimes C$ by $\rho_\ell = (\mathrm{id}_C \otimes \mathrm{id}_M \otimes \epsilon_C) \circ \rho$ and $\rho_r = (\epsilon_C \otimes \mathrm{id}_M \otimes \mathrm{id}_C) \circ \rho$, respectively, we see that they satisfy that (M, ρ_ℓ) is a left comodule over C and (M, ρ_r) is a right comodule over C, i.e., $(\Delta_C \otimes \mathrm{id}_M) \circ \rho_\ell = (\mathrm{id}_C \otimes \rho_\ell) \circ \rho_\ell$, $(\mathrm{id}_M \otimes \Delta_C) \circ \rho_r = (\rho_r \otimes \mathrm{id}_C) \circ \rho_r$, $(\epsilon_C \otimes \mathrm{id}_M) \circ \rho_\ell = \mathrm{id}_M = (\mathrm{id}_M \otimes \epsilon_C) \circ \rho_r$. It is clear that a C-bicomodule is a left and right comodule over C satisfying the compatibility $(\mathrm{id}_C \otimes \rho_r) \circ \rho_\ell = (\rho_\ell \otimes \mathrm{id}_C) \circ \rho_r$.

Assume that C is concentrated in even degrees and M is concentrated in odd degrees. Define $D = C[M]$ the counitary graded coalgebra whose underlying graded space is $C \oplus M$, with the induced grading, the comultiplication given by $\Delta_D = \Delta_C \circ \pi_C + \rho_\ell \circ \pi_M + \rho_r \circ \pi_M$ and the counit $\epsilon_D = \epsilon_C \circ \pi_C$, where $\pi_C : D \to C$

and $\pi_M : D \to M$ are the canonical projections. It is easy to see that, if $\eta_C : k \to C$ is a coaugmentation of C, i.e., a homogeneous map such that $\Delta_C \circ \eta_C = (\eta_C \otimes \eta_C) \circ \Delta_k$, where $\Delta_k : k \to k \otimes k$ is the obvious isomorphism, and $\epsilon_C \circ \eta_C = \mathrm{id}_k$, then $\eta_D = i_C \circ \eta_C$ is a coaugmentation of $(D, \Delta_D, \epsilon_D)$, where $i_C : C \to D$ is the canonical inclusion.

We recall that a *noncounitary curved A_∞-coalgebra* is a derivation D_C of cohomological degree 1 on the unitary graded tensor algebra $T(C[-1])$ provided with the concatenation product, such that $D_C \circ D_C = 0$. The previous unitary dg algebra is called the *(noncounitary curved) cobar construction* of C and is typically denoted by $\Omega_{nc}(C)$. If $n \in \mathbb{N}$, we will typically denote an element $s^{-1}(c_1) \otimes \cdots \otimes s^{-1}(c_n) \in C[-1]^{\otimes n}$ by $\langle c_1 | \dots | c_n \rangle$, where $c_1, \dots, c_n \in C$, and $s = s_{C[-1]} : C[-1] \to C$ is the suspension on $C[-1]$.

Since $\Omega_{nc}(C)$ is a free graded algebra, D_C is uniquely determined by its restriction to $C[-1]$, which we denote by $d = \sum_{i \in \mathbb{N}_0} d_i$ for $d_i : C[-1] \to C[-1]^{\otimes i}$. Set $\Delta_i : C \to C^{\otimes i}$ by means of $d_i = (-1)^i (s_{C[-1]}^{\otimes i})^{-1} \circ \Delta_i \circ s_{C[-1]}$. Then, the collection of maps $\Delta_i : C \to C^{\otimes i}$ for $i \in \mathbb{N}_0$ is locally finite, each of homological degree $i - 2$, and satisfy the following identities:

$$\sum_{(r,s,t) \in \mathcal{I}_n} (-1)^{rs+t} (\mathrm{id}_C^{\otimes r} \otimes \Delta_s \otimes \mathrm{id}_C^{\otimes t}) \circ \Delta_{r+1+t} = 0, \tag{SI(n)}$$

for $n \in \mathbb{N}_0$, where $\mathcal{I}_n = \{(r, s, t) \in \mathbb{N}_0^3 : r + s + t = n\}$. Reciprocally, starting from a locally finite collection of maps $\Delta_i : C \to C^{\otimes i}$ fulfilling the previous properties we obtain a noncounitary curved A_∞-coalgebra structure. A *noncounitary curved dg coalgebra* is a noncounitary curved A_∞-coalgebra such that $\Delta_i = 0$ for all $i \geq 3$. We recall that a noncounitary curved A_∞-coalgebra $(C, \Delta_\bullet)$ is called *cocomplete* (or *conilpotent*) if the cobar construction $\Omega_{nc}(C)$ is a cofibrant dg algebra with respect to the model structure constructed by V. Hinich in [7].

Given two noncounitary curved A_∞-coalgebras C and C', a *morphism* $f_\bullet : C \to C'$ is a morphism of unitary dg algebras $\Omega_{nc}(f_\bullet) : \Omega_{nc}(C) \to \Omega_{nc}(C')$ of the cobar constructions. Since $\Omega_{nc}(C)$ is a free graded algebra, such a morphism is completely determined by its restriction to $C[-1]$, which we denote by $F = \sum_{i \in \mathbb{N}_0} F_i$, where $F_i : C[-1] \to C'[-1]^{\otimes i}$. Define $f_i : C \to (C')^{\otimes i}$ by $F_i = (-1)^{i+1} (s_{C'[-1]}^{\otimes i})^{-1} \circ f_i \circ s_{C[-1]}$, for $i \in \mathbb{N}_0$. Then, $f_i : C \to (C')^{\otimes i}$ is a locally finite collection of maps, each of homological degree $i - 1$ for $i \in \mathbb{N}_0$, satisfying

$$\sum_{(r,s,t) \in \mathcal{I}_n} (-1)^{rs+t} (\mathrm{id}_D^{\otimes r} \otimes \Delta_s^D \otimes \mathrm{id}_D^{\otimes t}) \circ f_{r+1+t} = \sum_{q \in \mathbb{N}_0} \sum_{\bar{\imath} \in \mathbb{N}_0^{q,n}} (-1)^{w'} (f_{i_1} \otimes \cdots \otimes f_{i_q}) \circ \Delta_q^C, \tag{MI(n)}$$

for $n \in \mathbb{N}_0$, where $w' = \sum_{j=1}^{q} (j - 1)(i_j + 1)$ and $\mathbb{N}_0^{q,n}$ is the subset of elements $\bar{\imath}$ of $\mathbb{N}_0^q$ satisfying that $|\bar{\imath}| = i_1 + \cdots + i_q = n$, and the term of the right member of (MI(n)) for $q = 0$ is $\delta_{n,0} \Delta_0^C$. Reciprocally, starting from a locally finite collection of maps $f_i : C \to (C')^{\otimes i}$ fulfilling the previous properties we obtain a morphism of coaugmented A_∞-coalgebras. If $f_\bullet : C \to C'$ and $g_\bullet : C' \to D$ are morphisms of noncounitary curved A_∞-coalgebras, we can consider their composition $\Omega_{nc}(g_\bullet) \circ \Omega_{nc}(f_\bullet)$. Using the previous comments we see that $\Omega_{nc}(g_\bullet) \circ \Omega_{nc}(f_\bullet) = \Omega_{nc}(h_\bullet)$ where $\{h_n : C \to D^{\otimes n}\}_{n \in \mathbb{N}_0}$ is of the

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form

$$h_n = \sum_{q\in\mathbb{N}_0} \sum_{\bar{i}\in\mathbb{N}_0^{q,n}} (-1)^{w'} (g_{i_1} \otimes \cdots \otimes g_{i_q}) \circ f_q, \tag{1}$$

where $w' = \sum_{j=1}^{q} (j-1)(i_j+1)$ and the term with $q = 0$ is $\delta_{n,0} f_0$. A morphism $\{f_\bullet\}_{\bullet\in\mathbb{N}_0}$ of noncounitary A_∞-coalgebras is called *strict* if $f_n = 0$ for all $n \in \mathbb{N}_0 \setminus \{1\}$, and it is said to be a *quasi-equivalence* if the map $\Omega_{nc}(f_\bullet)$ is a quasi-isomorphism. The identity morphism of a noncounitary A_∞-coalgebra C is the strict morphism satisfying that $f_1 = \mathrm{id}_C$. Moreover, if C and C' are noncounitary curved dg coalgebras, a *morphism* from C to C' is a morphism of noncounitary A_∞-coalgebras $f_\bullet$ such that $f_n = 0$ for all $n \in \mathbb{N} \setminus \{1\}$. It is easy to check that the previous composition rule preserves morphisms of curved dg coalgebras.

A noncounitary curved A_∞-coalgebra $(C, \Delta_\bullet)$ is called *(strictly) counitary* if there exists a homogeneous linear map $\epsilon_C : C \to k$ of degree zero such that $(\mathrm{id}_C^{\otimes r} \otimes \epsilon_C \otimes \mathrm{id}_C^{\otimes t}) \circ \Delta_i$ vanishes for all $i \in \mathbb{N} \setminus \{2\}$ and all $r, t \in \mathbb{N}_0$ such that $r + 1 + t = i$, and $(\mathrm{id}_C \otimes \epsilon_C) \circ \Delta_2 = \mathrm{id}_C = (\epsilon_C \otimes \mathrm{id}_C) \circ \Delta_2$. A *morphism of (strictly) counitary curved A_∞-coalgebras* $f_\bullet : C \to C'$ is a morphism of noncounitary curved A_∞-coalgebras such that $(\mathrm{id}_{C'}^{\otimes(j-1)} \otimes \epsilon_{C'} \otimes \mathrm{id}_{C'}^{\otimes(i-j)}) \circ f_i$ vanishes for all $i \geq 2$ and $j \in \{1, \dots, i\}$, and $\epsilon_{C'} \circ f_1 = \epsilon_C$. A counitary curved A_∞-coalgebra $(C, \Delta_\bullet, \epsilon_C)$ is said to be *(strictly) coaugmented* if there is homogeneous linear map $\eta_C : k \to C$ satisfying that $\epsilon_C \circ \eta_C = \mathrm{id}_k$, $\Delta_2 \circ \eta_C(1_k) = \eta_C(1_k)^{\otimes 2}$ and $\Delta_i \circ \eta_C(1_k) = 0$ for all $i \in \mathbb{N}_0 \setminus \{2\}$. This is tantamount to say that η_C is a strict morphism of counitary curved A_∞-coalgebras, where k has the trivial structure given by $\Delta_i = 0$ if $i \in \mathbb{N}_0 \setminus \{2\}$, Δ_2 is the obvious map $k \to k \otimes k$ and ϵ_k is the identity. A *morphism of (strictly) coaugmented curved A_∞-coalgebras* $f_\bullet : C \to C'$ is a morphism of counitary curved A_∞-coalgebras such that its composition with $\epsilon_{C'}$ gives ϵ_C. Given a coaugmented curved A_∞-coalgebra, we will denote the cokernel of η_C by J_C. Let us consider the functor from the category of coaugmented curved A_∞-coalgebras to the category of noncounitary curved A_∞-coalgebras given by sending $(C, \Delta_\bullet, \epsilon_C, \eta_C)$ to J_C provided with the comultiplications $\bar{\Delta}_\bullet$ induced by $\Delta_\bullet$ by means of

$$\bar{\Delta}_n \circ \mathrm{coker}(\eta_C) = \mathrm{coker}(\eta_C)^{\otimes n} \circ \Delta_n,$$

for all $n \in \mathbb{N}_0$, where $\mathrm{coker}(\eta_C) : C \to J_C$ is the cokernel morphism. We remark that $\bar{\Delta}_n$ is well-defined and unique for $\mathrm{coker}(\eta_C)^{\otimes n} \circ \Delta_n \circ \eta_C$ vanishes for all $n \in \mathbb{N}_0$. A similar expression gives the action of the functor on the morphisms. It is clear that this functor is an equivalence of categories, whose inverse sends the noncounitary curved A_∞-coalgebra $(\bar{C}, \bar{\Delta}_\bullet)$ to the coaugmented curved A_∞-coalgebra structure over $C = \bar{C} \oplus k$ with counit ϵ_C given by the canonical projection on k, coaugmentation η_C given by the canonical inclusion of k, $\Delta_n : C \to C^{\otimes n}$ is the composition of the canonical projection $C \to \bar{C}$, $\bar{\Delta}_n$ and the canonical inclusion $\bar{C}^{\otimes n} \to C^{\otimes n}$ if $n \neq 2$, and $\Delta_2 : C \to C^{\otimes 2}$ satisfies that $\Delta_2(1) = 1 \otimes 1$ and $\Delta_2(\bar{c}) = 1 \otimes \bar{c} + \bar{c} \otimes 1 + \bar{\Delta}_2(\bar{c})$, for all $\bar{c} \in \bar{C}$, where $1 \in k$. A coaugmented curved A_∞-coalgebra $(C, \Delta_\bullet, \epsilon_C, \eta_C)$ is said to be *cocomplete* (or *conilpotent*) if the corresponding noncounitary curved A_∞-coalgebra structure on J_C is cocomplete. Moreover, if $(C, \Delta_\bullet, \epsilon_C, \eta_C)$ is a coaugmented curved A_∞-coalgebra, its *(coaugmented curved) cobar construction* $\Omega^+(C)$ is defined as $\Omega_{nc}(J_C)$.

All the previous definitions of this section make also perfect sense if we drop the adjective ‘curved’, by which we mean that the curvature terms Δ_0 and f_0 vanish.

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We also recall that given any unitary dg algebra A, there exists a canonical curved dg coalgebra $B^+(A)$ associated to A, called the *(curved) bar construction* of A, and it is constructed as follows. Let $v : A \to k$ be a linear map satisfying that $v \circ \eta_A = \mathrm{id}_k$, and let $V = \mathrm{Ker}(v)$. Define μ_V (resp., μ_k) as the composition of $\mu_A|_{V\otimes V}$ and the projection $\mathrm{id}_A - \eta_A \circ v : A \to V$ (resp., and v), and d_V (resp., d_k) as the composition of $d_A|_V$ and the projection $\mathrm{id}_A - \eta_A \circ v : A \to V$ (resp., and v). Let $B_v(A) = T(V[1])$ be the (cocomplete) graded tensor coalgebra cogenerated by $V[1]$. An element $s(v_1) \otimes \cdots \otimes s(v_n)$ is typically denoted by $[v_1|\dots|v_n]$, where $v_1, \dots, v_n \in V$. Let B be the unique coderivation of $T(V[1])$ whose composition with the canonical projection onto $V[1]$ is the map $b : T(V[1]) \to V[1]$ sending $[v_1|\dots|v_n]$ to zero if $n \in \mathbb{N}_{\geq 2} \cup \{0\}$, $[v_1|v_2] \mapsto (-1)^{|v_1|+1}[\mu_V(v_1, v_2)]$ and $[v] \mapsto -[d_V(v)]$. Moreover, set $h : T(V[1]) \to k$ by $h([v_1|\dots|v_n]) = 0$ if $n \in \mathbb{N}_{\geq 2} \cup \{0\}$, $h([v_1|v_2]) = (-1)^{|v_1|+1}\mu_k(v_1, v_2)$ and $h([v]) = -d_k(v)$. Then, $B_v(A) = T(V[1])$ provided with the deconcatenation coproduct, the coderivation B and $\Delta_0 = h$ is a curved dg coalgebra. It is clearly counitary for the counit $\epsilon_{B_v(A)}$ given by the canonical projection $T(V[1]) \to k$ and even coaugmented for the coaugmentation $\eta_{B_v(A)}$ defined as the canonical inclusion $k \to T(V[1])$. Given another linear map $v' : A \to k$ satisfying that $v \circ \eta_A = \mathrm{id}_k$, the coaugmented curved dg coalgebra $B_{v'}(A)$ is isomorphic to $B_v(A)$, as coaugmented curved dg coalgebras. Indeed, let $g : V' \to V$ be the linear isomorphism $\mathrm{id}_{V'} - \eta_A \circ v|_{V'}$ (with inverse $\mathrm{id}_V - \eta_A \circ v'|_V$), and $f_1 = \sum_{n\in\mathbb{N}_0} (g[1])^{\otimes n} : T(V'[1]) \to T(V[1])$. Let $f_0 : T(V'[1]) \to k$ be the composition of the canonical projection onto $V'[1]$, $s_{V'}^{-1}$ and $v|_{V'}$. Then, (f_0, f_1) determines an isomorphism of coaugmented curved dg coalgebras from $B_{v'}(A)$ to $B_v(A)$. The inverse is obtained from interchanging v and v' in the previous expressions. Hence, we may drop the explicit dependence on the retraction v and will denote $B_v(A)$ simply by $B^+(A)$. It will be called the *(curved) bar construction* of A, or also, the *Koszul codual* coalgebra of A.

Finally, it is clear that, given a morphism $f : A \to A'$ of unitary algebras, it induces a strict morphism $B^+(f) : B^+(A) \to B^+(A')$ of coaugmented curved dg coalgebras. Indeed, given $v' : A' \to k$ satisfying that $v' \circ \eta_{A'} = \mathrm{id}_k$, define $v = v' \circ f$. Then, $v \circ \eta_A = \mathrm{id}_k$, and f sends $V = \mathrm{Ker}(v)$ to $V' = \mathrm{Ker}(v')$. The mentioned strict morphism $B_v(A) \to B_{v'}(A')$ of coaugmented curved dg coalgebras is finally given by $\sum_{n\in\mathbb{N}_0} (f|_V[1])^{\otimes n}$.

2.2. A particular result. The next result follows directly from the definition of coaugmented curved A_∞-coalgebra.

PROPOSITION 2.1. *Let $(C, \Delta_C, \epsilon_C)$ be a counitary (coassociative) graded coalgebra concentrated in even degrees and (M, ρ) be a (counitary) graded bicomodule over C concentrated in odd degrees. We consider the counitary graded coalgebra $D = C[M]$ defined before, with coproduct Δ_D and counit ϵ_D. Assume we are given linear maps*

$$\Delta'_p : C \to M^{\otimes p}, \tag{2}$$

for all $p \in \mathbb{N}_0$, of degree $p - 2$, and a bicoderivation

$$\partial : M \to C, \tag{3}$$

of degree -1, i.e., $\Delta_C \circ \partial = (\mathrm{id}_C \otimes \partial) \circ \rho_\ell + (\partial \otimes \mathrm{id}_C) \circ \rho_r$. For $p \in \mathbb{N}_0$, set $\Delta_p : D \to D^{\otimes p}$ by $\Delta_p = \Delta'_p \circ \pi_C + \delta_{p,1}\partial \circ \pi_M + \delta_{p,2}\Delta_D$. Then, $(D, \Delta_\bullet)_{\bullet\in\mathbb{N}_0}$ is a curved

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A_∞-coalgebra if and only if

$$\Delta_0' \circ \partial = 0, \quad \partial \circ \Delta_1' = (\Delta_0' \otimes \mathrm{id}_C) \circ \Delta_C - (\mathrm{id}_C \otimes \Delta_0') \circ \Delta_C, \tag{4}$$

$$\Delta_{p+1}' \circ \partial = (\Delta_p' \otimes \mathrm{id}_M) \circ \rho_\ell - (-1)^p (\mathrm{id}_M \otimes \Delta_p') \circ \rho_r, \tag{5}$$

$$(\rho_\ell \otimes \mathrm{id}_M^{\otimes(q-1)}) \circ \Delta_q' = (\mathrm{id}_C \otimes \Delta_q') \circ \Delta_C + (\partial \otimes \mathrm{id}_M^{\otimes q}) \circ \Delta_{q+1}', \tag{6}$$

$$(\mathrm{id}_M^{\otimes j} \otimes \rho_\ell \otimes \mathrm{id}_M^{\otimes(q-j-1)}) \circ \Delta_q' = (\mathrm{id}_M^{\otimes(j-1)} \otimes \rho_r \otimes \mathrm{id}_M^{\otimes(q-j)}) \circ \Delta_q' + (-1)^j (\mathrm{id}_M^{\otimes j} \otimes \partial \otimes \mathrm{id}_M^{\otimes(q-j)}) \circ \Delta_{q+1}', \tag{7}$$

$$(\mathrm{id}_M^{\otimes(q-1)} \otimes \rho_r) \circ \Delta_q' = (\Delta_q' \otimes \mathrm{id}_C) \circ \Delta_C + (-1)^{q+1} (\mathrm{id}_M^{\otimes q} \otimes \partial) \circ \Delta_{q+1}', \tag{8}$$

for all $p \in \mathbb{N}_0$, $q \in \mathbb{N}$ and $j \in \{1, \dots, q-1\}$, where the latter set (so equation (7)) is empty if $q = 1$. It is strictly counital with counit ϵ_D if and only if we further have $\epsilon_C \circ \partial = 0$. Moreover, assuming that the previous conditions hold, η_D is a (strict) coaugmentation of $(D, \Delta_\bullet)$ if and only if $\Delta_p' \circ \eta_C = 0$, for all $p \in \mathbb{N}_0$.

Proof. Note that SI(0) coincides with the first equation in (4). Moreover, using that C is a graded coalgebra and M is a graded C-bicomodule, and some basic computations, we see that

(i) $\mathrm{SI}(1)|_C$ is equivalent to the second equation in (4);
(ii) $\mathrm{SI}(p+1)|_M$ is equivalent to the third equation (5), where we have used that ∂ is a bicoderivation for $p = 1$ and that ρ is a bicoaction for $p = 2$;
(iii) $\mathrm{SI}(2)|_C$ is equivalent to the equations (6) and (8) for $q = 1$, respectively;
(iv) $\pi_0 \circ \mathrm{SI}(q+1)|_C$ (resp., $\pi_j \circ \mathrm{SI}(q+1)|_C$, $\pi_q \circ \mathrm{SI}(q+1)|_C$) is equivalent to the antepenultimate (resp., penultimate, last) equation (6) (resp., (7), (8)), for $q \geq 2$, where $\pi_j : D^{\otimes(q+1)} \to M^{\otimes j} \otimes C \otimes M^{\otimes(q-j)}$ is the canonical projection.

On the other hand, D is strictly counital if and only if $\epsilon_D \circ \Delta_1 = 0$, for the other equations are automatically satisfied. The former equation is tantamount to $\epsilon_C \circ \partial = 0$. Finally, η_D is a strict coaugmentation if and only if $\Delta_2 \circ \eta_D = (\eta_D \otimes \eta_D) \circ \Delta_k$, where $\Delta_k : k \to k \otimes k$ is the obvious isomorphism, and $\Delta_p \circ \eta_D = 0$, for all $p \in \mathbb{N}_0 \setminus \{2\}$. Since Δ_D is a coaugmentation of $(D, \Delta_D, \epsilon_D)$, the first of the previous identities is tantamount to $\Delta_2' \circ \eta_C = 0$, whereas, the second collection of identities is equivalent to $\Delta_p' \circ \eta_C = 0$, for all $p \in \mathbb{N}_0 \setminus \{2\}$. The proposition follows. □

3. Koszul algebras. In this section, we recall the basic results we will need on homogeneous generalised Koszul algebras. Let V be a vector space over k, and let $A = TV/(R)$ be an N-homogeneous algebra for $N \in \mathbb{N}_{\geq 2}$, i.e., $R \subseteq V^{\otimes N}$. We say that A is *generalised Koszul* (or *N-Koszul*, if we want to emphasise the degree of R) if the minimal projective resolution $P_\bullet$ of (either left or right) A-module k satisfies that P_n is (a graded free module) generated in degree $\phi_N(n)$, for all $n \in \mathbb{N}_0$, where $\phi_N(2m) = Nm$ and $\phi_N(2m+1) = Nm + 1$, for all $m \in \mathbb{N}_0$. We recall that, if A is N-Koszul, then

$$(V^{\otimes j} \otimes R) \cap \left(\sum_{\ell=0}^{j-1} V^{\otimes \ell} \otimes R \otimes V^{\otimes(j-\ell)} \right) \subseteq V^{\otimes(j-1)} \otimes R \otimes V, \tag{9}$$

for all $j = 2, \dots, N-1$ [**1**, Theorem 2.11 and Proposition 2.5].

We now recall the (strictly) coaugmented A_∞-coalgebra structure on $\mathrm{Tor}_\bullet^A(k, k)$, for an N-Koszul algebra A. The Koszul property of A implies that $D = \mathrm{Tor}_\bullet^A(k, k)$

satisfies that

$$D_p = \mathrm{Tor}_p^A(k,k) = \bigcap_{i=0}^{\phi_N(p-2)} V^{\otimes i} \otimes R \otimes V^{\otimes\left(\phi_N(p-2)-i\right)}, \tag{10}$$

for $p \geq 2$, together with $D_0 = \mathrm{Tor}_0^A(k,k) = k$ and $D_1 = \mathrm{Tor}_1^A(k,k) = V$ [**1**, equation (2.5)]. Moreover, D has the following A_∞-coalgebra structure. We will denote the corresponding comultiplications by $\bar{\Delta}_n$, for $n \in \mathbb{N}$. If $N = 2$, all the higher coproducts vanish, as well as $\bar{\Delta}_1$, and $\bar{\Delta}_2 = \Delta_D$ is given by the usual deconcatenation formula (see (i) below). Suppose, else that $N > 2$. There are only two nonvanishing comultiplications, $\bar{\Delta}_2 = \Delta_D$ and $\bar{\Delta}_N$, which satisfy that

(i) $(\mathfrak{p}_{p_1} \otimes \mathfrak{p}_{p_2}) \circ \bar{\Delta}_2|_{D_p}$ is the canonical inclusion if $p_1 + p_2 = p$ and $\phi_N(p_1) + \phi_N(p_2) = \phi_N(p)$ for $p_1, p_2, p \in \mathbb{N}_0$, and zero else;
(ii) $(\mathfrak{p}_{p_1} \otimes \cdots \otimes \mathfrak{p}_{p_N}) \circ \bar{\Delta}_N|_{D_p}$ is the canonical inclusion if $p_1 + \cdots + p_N = p + N - 2$ and $\phi_N(p_1) + \cdots + \phi_N(p_N) = \phi_N(p)$ for $p_1, \ldots, p_N, p \in \mathbb{N}$, and zero otherwise;

where $\mathfrak{p}_m : D \to D_m$ denotes the canonical projection. Note that the nonvanishing statement of item (i) implies that either p_1 or p_2 is even, whereas in the case of item (ii) it implies that $p_1, \ldots, p_N$ are odd (and p even). It is easy to verify that $(D, \bar{\Delta}_2, \bar{\Delta}_N, \epsilon_D, \eta_D)$ is a coaugmented A_∞-coalgebra for any $N \geq 2$, where the canonical projection $\mathfrak{p}_0 : D \to D_0 = k$ gives the strict counit ϵ_D of D and the canonical inclusion $k = D_0 \to D$ is the coaugmentation (this can be seen as an application of Proposition 2.1 to this particular case). Moreover, by a result by B. Keller (see Theorem 5.2), it is a model for $\mathrm{Tor}_\bullet^A(k,k)$, i.e., it is quasi-equivalent to the coaugmented dg coalgebra $B^+(A)$.

REMARK 3.1. It is easy to see that the previous comultiplication, counit and coaugmentation of D respect a second grading on that space, induced by the internal (or Adams) grading of A, given by regarding V to be concentrated in degree 1. We remark that the Adams grading of a nonzero element in D_p is $\phi_N(p)$, whereas, the homological degree is p. This can be compactly reformulated as saying that $(D, \bar{\Delta}_2, \bar{\Delta}_N, \epsilon_D, \eta_D)$ is an Adams graded coaugmented A_∞-coalgebra.

4. PBW deformations. In this section, we will recollect the basic results on PBW deformations of homogeneous algebras. We first recall that a *filtered k-algebra B* is a k-algebra provided with an increasing sequence $\{F^\bullet B\}_{\bullet \in \mathbb{N}_0}$ of subspaces of B such that $F^m B.F^n B \subseteq F^{m+n} B$, for all $m, n \in \mathbb{N}_0$ and $1_B \in F^0 B$. As usual, such filtrations may also be seen to be indexed over $\mathbb{Z}$, where the negatively indexed terms vanish. Given a vector space V, the tensor algebra TV has a filtration $\{F^\bullet\}_{\bullet \in \mathbb{N}_0}$ defined by $F^i = \oplus_{j=0}^i V^{\otimes j}$. Now, given $P \subseteq F^N$, we shall consider the algebra $U = TV/\langle P \rangle$, with the filtration $\{F^\bullet U\}_{\bullet \in \mathbb{N}_0}$ induced by the filtration of the tensor algebra, i.e., $F^\bullet U = \pi(F^\bullet)$, where π denotes the canonical projection from TV onto U. Of course, π is a morphism of filtered algebras. The filtration can be described more concretely as follows: if $\langle P \rangle^i = F^i \cap \langle P \rangle$, then $F^i U = F^i / \langle P \rangle^i$, for $i \in \mathbb{N}_0$. If $\pi_i : TV \to V^{\otimes i}$ is the canonical projection, let us denote $R = \pi_N(P)$ and define the N-homogeneous algebra $A = TV/\langle R \rangle$.

Since $\pi : TV \to U$ is a morphism of filtered algebras, it induces a morphism of graded algebras $\mathrm{gr}(\pi) : \mathrm{gr}(TV) \to \mathrm{gr}(U)$. Moreover, the filtration of U is induced by the filtration of TV, so $\mathrm{gr}(\pi)$ is surjective. On the other hand, since the filtration of TV comes from a grading on the tensor algebra, we see that there exists a canonical isomorphism $\iota : TV \simeq \mathrm{gr}(TV)$. So, we may consider the surjective morphism of graded

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k-algebras given by the composition $\mathrm{gr}(\pi) \circ \iota : TV \to \mathrm{gr}(U)$, which we shall call Π. It is easy to see that $\Pi(R) = 0$, since $\iota(R) = P/F^{N-1}$. Hence, Π induces a surjective morphism of graded k-algebras $p : A \to \mathrm{gr}(U)$. We say that U satisfies the *PBW property* or that U is a *PBW-deformation* of A if p is an isomorphism.

As noticed by R. Berger and V. Ginzburg [**2**, Proposition 3.2], the filtered algebra U satisfies the PBW property if and only if $\langle P\rangle^n = \sum_{i+j\leq n-N} V^{\otimes i}PV^{\otimes j}$, for all $n \in \mathbb{N}_0$ (in fact, it is sufficient to prove the equality for $n \geq N-1$). Moreover, if we denote $J_n = \sum_{i+j\leq n-N} V^{\otimes i}PV^{\otimes j}$, for $n \in \mathbb{N}_0$, Proposition 3.3. in [**2**] states that U satisfies the PBW property if and only if $J_n \cap F^{n-1} = J_{n-1}$, for all $n \in \mathbb{N}_0$ (or just $n \geq N$). The identity $J_N \cap F^{N-1} = J_{N-1}$ is simply

$$P \cap F^{N-1} = 0, \tag{11}$$

whereas, $J_{N+1} \cap F^N = J_N$ is easily equivalent to

$$(V \otimes P + P \otimes V) \cap F^N \subseteq P. \tag{12}$$

From now on, we shall suppose that identity (11) holds, which implies that the map $\pi_N : F^N \to V^{\otimes N}$ gives an isomorphism between P and $R = \pi_N(P)$. Then, there exists a linear map $\varphi : R \to F^{N-1}$ such that $\mathrm{id} - \varphi$ is the inverse of $\pi_N|_P$, i.e., $P = \{r - \varphi(r) : r \in R\}$. We further write, $\varphi = \sum_{j=0}^{N-1} \varphi_j$, where $\varphi_j : R \to V^{\otimes j}$ is the composition of φ with the canonical morphism $F^{N-1} \to V^{\otimes j}$. Let $R_{N+1} = (R \otimes V) \cap (V \otimes R) \subseteq V^{\otimes(N+1)}$. Then, it is easy to see that identity (12) is equivalent to [**2**, Proposition 3.5]

$$(\varphi \otimes \mathrm{id}_V - \mathrm{id}_V \otimes \varphi)(R_{N+1}) \subseteq P,$$

or equivalently [**2**, Proposition 3.6]

$$(\varphi_{N-1} \otimes \mathrm{id}_V - \mathrm{id}_V \otimes \varphi_{N-1})(R_{N+1}) \subseteq R, \tag{13}$$

$$\varphi_0 \circ (\varphi_{N-1} \otimes \mathrm{id}_V - \mathrm{id}_V \otimes \varphi_{N-1})(R_{N+1}) = 0, \tag{14}$$

$$\big(\varphi_j \circ (\varphi_{N-1} \otimes \mathrm{id}_V - \mathrm{id}_V \otimes \varphi_{N-1}) + (\varphi_{j-1} \otimes \mathrm{id}_V - \mathrm{id}_V \otimes \varphi_{j-1})\big)(R_{N+1}) = 0, \tag{15}$$

for $0 < j < N$.

DEFINITION 4.1. Given a filtered algebra $U = TV/\langle P\rangle$, where $P \subseteq F^N$, we say that U is a *weak PBW-deformation* of $A = TV/\langle R\rangle$, where $R = \pi_N(P)$, if (11) and (13)–(15) hold.

One very interesting property of Koszul algebras is the following result, called the *Koszul deformation principle* (see for instance [**2**, Theorems 1.2 and 3.4], [**3**, Theorem 1.1], and [**6**, Theorem 3.5]).

THEOREM 4.2. *Let A be an N-homogeneous algebra satisfying that* $\mathrm{Tor}_3^A(k,k)$ *is concentrated in degree $N+1$, and let $U = TV/\langle P\rangle$ be a weak PBW-deformation of A. Then, U satisfies the PBW property.*

5. Curved A_∞-coalgebra of the Koszul codual of a filtered dg algebra.

5.1. Twisting cochains and twisted tensor products. In this section, we will present an extension of a result that was announced by B. Keller at the X ICRA of Toronto,

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Canada, in 2002. For the definitions and notation used we refer the reader to [**5**, Theorem 4.2].

We first remark that all the definitions given in Section 2.1 can be done for the category of (homological) graded vector spaces V provided with further increasing (nonnegative) filtrations $\{F^\bullet V\}_{\bullet\in\mathbb{N}_0}$ of graded vector subspaces and all the maps preserve the filtrations. We will assume that the filtrations are *exhaustive*, i.e., $\cup_{n\in\mathbb{N}_0} F^n V = V$. We will talk in that case of *filtered noncounitary (resp., counitary, coaugmented) curved A_∞-coalgebras*, *morphisms of filtered noncounitary (resp., counitary, coaugmented) curved A_∞-coalgebras*, etc. We recall that k is provided with the trivial filtration $F^n k = k$, for all $n \in \mathbb{N}_0$. In the case of a coaugmented curved A_∞-coalgebra C, one further imposes $F^0 C$ to be the image of the coaugmentation η_C of C. As a consequence, $\mathrm{Gr}_{F^\bullet C}(C)$ has zero curvature, so it is in fact a coaugmented A_∞-coalgebra. Moreover, we recall that a morphism $f_\bullet : C \to C'$ of filtered coaugmented curved A_∞-coalgebras is called a *filtered quasi-equivalence* if the associated morphism $\mathrm{gr}(f_\bullet)$ is a quasi-equivalence of coaugmented A_∞-coalgebras.

Let C be a coaugmented curved A_∞-coalgebra and A be a unitary dg algebra. We recall that a *twisting cochain* from C to A is a linear map $\tau : C \to A$ of cohomological degree 1 such that $\tau \circ \eta_C$ vanishes and that it satisfies the *Maurer–Cartan equation*

$$d_A \circ \tau + \sum_{i\in\mathbb{N}_0} (-1)^{i(i+1)/2+1} \mu_A^{(i)} \circ \tau^{\otimes i} \circ \Delta_i = 0, \tag{16}$$

where $\mu_A^{(i)} : A^{\otimes i} \to A$ is the iterative application of the product of A if $i \geq 2$, the identity map of A if $i = 1$, and the unit η_A of A if $i = 0$. Note that the sum in (16) is well-defined by the local finiteness assumption on the higher comultiplications of C. If $\mathrm{Tw}(C, A)$ denotes the set of twisting cochains from C to A, we have a canonical map

$$\mathrm{Hom}_{\text{u-dg-alg}}(\Omega^+(C), A) \to \mathrm{Tw}(C, A) \tag{17}$$

given by $g \mapsto g \circ \tau^C$, where $\tau^C : C \to \Omega^+(C)$ is the composition of the canonical projection $C \to J_C$, $s^{-1}_{J_C[-1]}$ and the canonical inclusion of $J_C[-1]$ inside $\Omega^+(C)$, where J_C is the cokernel of the coaugmentation η_C of C. It is clear that the map (17) is a bijection, and we will denote the image of a twisting cochain τ under its inverse map by F_τ. Furthermore, by means of the previous morphism, we can define the *composition twisting cochain* of a morphism of coaugmented curved A_∞-coalgebras $f_\bullet : C' \to C$ with a twisting cochain τ from C to A. Indeed, if $F_\tau \in \mathrm{Hom}_{\text{u-dg-alg}}(\Omega^+(C), A)$ is the morphism such that $F_\tau \circ \tau^C = \tau$, and $\Omega^+(f_\bullet)$ is the morphism of unitary dg algebras from $\Omega^+(C')$ to $\Omega^+(C)$, the composition twisting cochain $\tau \circ f_\bullet$ is defined as $F_\tau \circ \Omega^+(f_\bullet) \circ \tau^{C'}$.

Given a coaugmented curved A_∞-coalgebra C, a unitary dg algebra A and a (unitary) dg A-bimodule M with biaction $\sigma : A \otimes M \otimes A \to M$, the *twisted tensor product* $M \otimes_\tau C$ is the complex whose underlined graded module is the usual tensor product $M \otimes C$ and the differential is

$$\begin{aligned} &d_M \otimes \mathrm{id}_C + \mathrm{id}_A \otimes \Delta_1 \\ &\quad + \sum_{\substack{i,j\in\mathbb{N}_0 \\ i+j>0}} (-1)^{\epsilon_{i,j}} (\sigma^{(i,j)} \otimes \mathrm{id}_C) \circ (\tau^{\otimes i} \otimes \mathrm{id}_M \otimes \tau^{\otimes j} \otimes \mathrm{id}_C) \circ \varsigma_{i,j} \circ (\mathrm{id}_M \otimes \Delta_{i+j+1}), \end{aligned} \tag{18}$$

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where $\sigma^{(i,j)} : A^{\otimes i} \otimes M \otimes A^{\otimes j} \to M$ is obtained from a successive application of the biaction of M, $\varsigma_{i,j} : M \otimes C^{\otimes(i+j+1)} \to C^{\otimes i} \otimes M \otimes C^{\otimes(j+1)}$ is the cyclic permutation sending $m \otimes c_1 \otimes \cdots \otimes c_{i+j+1}$ to $c_{j+2} \otimes \cdots \otimes c_{i+j+1} \otimes m \otimes c_1 \otimes \cdots \otimes c_{j+1}$ (without sign), and the sign $\epsilon_{i,j}$ is obtained from [**5**, equations (5), (6) and (13)].[1] If A is augmented by means of ϵ_A and M is a right dg A-module, then ${}_{\epsilon_A}M$ denotes the dg A-bimodule with the same right action as before and the left action $a \cdot m = \epsilon_A(a)m$, for $m \in M$ and $a \in A$.

5.2. The main results. We now present the following preparatory result, which is an extension of a theorem announced by Keller at the X ICRA of Toronto, Canada, in 2002 (see [**8**]). The proof given in [**4**, Theorem 4.7], can be extended straightforward to the following case.

PROPOSITION 5.1. *Let C be a coaugmented (minimal) A_∞-coalgebra and A be an augmented dg algebra over a field k such that its augmentation ideal $I_A = \mathrm{Ker}(\epsilon_A)$ and the cokernel* $\mathrm{Coker}(\eta_C)$ *are provided with another grading (besides the cohomological one), called* internal *or* Adams*, concentrated in (strictly) positive degrees, and the product $\mu_A : A^e \to A$ of A and all the comultiplication maps Δ_n of C preserve this degree. Let $\tau : C \to A$ be a twisting cochain preserving the Adams degree. Then, the following are equivalent:*

(*i*) *the associated morphism of unitary dg algebras $F_\tau : \Omega^+(C) \to A$ is a quasi-isomorphism;*
(*ii*) *the twisted tensor product ${}_{\epsilon_A}A \otimes_\tau C$ is quasi-isomorphic to the trivial left dg module k (via $\epsilon_A \otimes \epsilon_C$);*
(*iii*) *the twisted tensor product $A^e \otimes_\tau C$ is quasi-isomorphic to the standard dg A-bimodule A via $\mu_A \otimes \epsilon_C$.*

Proof. Note that the augmented dg algebra $\Omega^+(C)$ is cofibrant, due to the grading assumption on C (see [**7**, 2.2.3]), so C is cocomplete. Moreover, as in the case of conilpotent coaugmented dg coalgebras, where one utilises the usual homotopy of the standard bar resolution of the trivial left dg C-comodule k constructed from the counit of C, the complex ${}_{\epsilon_{\Omega^+(C)}}\Omega^+(C) \otimes_{\tau^C} C$ is quasi-isomorphic to k via $\epsilon_{\Omega^+(C)} \otimes \epsilon_C$, where $\tau^C : C \to \Omega^+(C)$ is the universal twisting cochain of C. Else, the result follows from dualizing [**10**, Theorem 3.25]. The same holds for $\mu_{\Omega^+(C)} \otimes \epsilon_C : \Omega^+(C)^e \otimes_{\tau^C} C \to \Omega^+(C)$.

On the other hand, let $\tau' : C \to A'$ and $\tau : C \to A$ be two twisting cochains, where C is a coaugmented A_∞-coalgebra and A and A' are augmented dg algebras satisfying the extra conditions about the grading. Let M and M' be dg bimodules provided with a compatible extra degree over A and A', respectively, $f : A' \to A$ a morphism of augmented dg algebras and $g : M' \to M$ a morphism of dg bimodules over A', both preserving the extra degree, where M is a dg A'-bimodule via f. We assume further that $f \circ \tau = \tau'$. Then, if g is a quasi-isomorphism, $g \otimes \mathrm{id}_C$ induces a quasi-isomorphism from $M' \otimes_{\tau'} C$ to $M \otimes_\tau C$. Indeed, since the components of fixed internal degree of $M' \otimes_\tau C$ (resp., $M \otimes_\tau C$) are obtained from those of C and M' (resp., C and M) as the result of a finite number of steps involving taking tensor product,

[1] We incidentally correct a typo in [**5**], equation (6), where the third term of the right member should include $(\sum_{i=1}^p \deg \phi_i + \sum_{j=1}^q \deg \psi_j)$ instead of $\sum_{i=1}^p \deg \phi_i$.

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cones and shifts, and those operations preserve quasi-isomorphisms, the result follows. Conversely, by reversing the steps in the previous argument, we see that, if $g \otimes \mathrm{id}_C$ is a quasi-isomorphism, then g is a quasi-isomorphism as well.

Assume now that (i) holds. Since $F_\tau : \Omega^+(C) \to A$ is a quasi-isomorphism, the comments in the previous paragraph tell us that $F_\tau \otimes \mathrm{id}_C : {}_{\epsilon_{\Omega^+(C)}}\Omega^+(C) \otimes_{\tau^c} C \to {}_{\epsilon_A}A \otimes_\tau C$ is a quasi-isomorphism, and it clearly satisfies $(\epsilon_A \otimes \epsilon_C) \circ (F_\tau \otimes \mathrm{id}_C) = \epsilon_{\Omega^+(C)} \otimes \epsilon_C$. By the 2-out-of-3 property of quasi-isomorphisms, we conclude that $\epsilon_A \otimes \epsilon_C$ is a quasi-isomorphism, which proves (ii). Replacing $F_\tau : \Omega^+(C) \to A$ with $F_\tau^e : \Omega^+(C)^e \to A^e$ in the previous argument, we obtain (iii).

Finally, assume that either (ii) or (iii) holds. In the first case, since the morphism $F_\tau \otimes \mathrm{id}_C : {}_{\epsilon_{\Omega^+(C)}}\Omega^+(C) \otimes_{\tau^c} C \to {}_{\epsilon_A}A \otimes_\tau C$ satisfies that $(\epsilon_A \otimes \epsilon_C) \circ (F_\tau \otimes \mathrm{id}_C) = \epsilon_{\Omega^+(C)} \otimes \epsilon_C$, the 2-out-of-3 property of quasi-isomorphisms tells us that $F_\tau \otimes \mathrm{id}_C$ is a quasi-isomorphism. Using the converse result in the second paragraph of this proof, we see that F_τ is a quasi-isomorphism of augmented dg algebras. If we assume that (iii) holds, then the same argument shows that $F_\tau^e \otimes \mathrm{id}_C : \Omega^+(C)^e \otimes_{\tau^c} C \to A^e \otimes_\tau C$ is a quasi-isomorphism. From the converse result in the second paragraph of this proof we see that F_τ^e is a quasi-isomorphism, which in turn implies that F_τ is a quasi-isomorphism as well, by the Künneth theorem. This proves (i). $\square$

We now prove the first main result of this article.

THEOREM 5.2. *Let D be a filtered coaugmented curved A_∞-coalgebra and U be a nonnegatively (Adams) filtered unitary dg algebra over a field k, whose associated graded algebra $A = \mathrm{Gr}_{F^\bullet U}(U)$ is an Adams nonnegatively graded connected dg algebra. Then, the following are equivalent:*

(*i*) *there is a filtered quasi-equivalence of coaugmented curved A_∞-coalgebras*

$$F : D \to B^+(U); \tag{19}$$

(*ii*) *there is a filtered twisting cochain $\tau : D \to U$ such that one of the following equivalent conditions holds:*
- (*a*) *the associated graded of the filtered morphism of unitary dg algebras $F_\tau : \Omega^+(D) \to U$ is a quasi-isomorphism;*
- (*b*) *the associated graded map of the filtered morphism $\mu_U \otimes \epsilon_D : U^e \otimes_\tau D \to U$ to the standard dg U-bimodule U is a quasi-isomorphism, where $\mu_U : U \otimes U \to U$ is the product of U.*

Proof. Given a morphism F satisfying (19), consider the twisting cochain $\tau : D \to U$ such that $F_\tau = \beta_U \circ \Omega^+(F)$, where $\beta_U : \Omega^+(B^+(U)) \to U$ is the usual adjunction morphism. It is clear that $\mathrm{Gr}(B^+(U)) \simeq B^+(\mathrm{Gr}_{F^\bullet U}(U)) = B^+(A)$, where the grading of $B^+(U)$ is the one induced from that of U following the usual tensor constructions, and analogously $\mathrm{Gr}(\Omega^+(B^+(U))) \simeq \Omega^+(\mathrm{Gr}(B^+(U))) \simeq \Omega^+(B^+(A))$. Furthermore, β_U is a quasi-isomorphism (see for instance the proof of [**9**, Theorem 6.10]), and $\mathrm{gr}(\beta_U)$ coincides with β_A under the previous identifications. Moreover, τ and F_τ preserve the corresponding Adams filtrations. Conversely, given a twisting cochain $\tau : D \to U$ preserving the corresponding Adams filtrations, consider the associated morphism $F_\tau : \Omega^+(D) \to U$ of unitary dg algebras. Recall that there is a canonical filtered quasi-equivalence $B^+(\Omega^+(D)) \to D$ whose cobar construction is $\beta_{\Omega^+(D)}$. Let $j : D \to B^+(\Omega^+(D))$ be any quasi-inverse (preserving the Adams filtration), which exists by the usual homotopic argument using the model structure on the category of unitary

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dg algebras. Finally, define $F : D \to B^+(U)$ as the composition of j and $B^+(F_\tau)$. Let $C = \mathrm{Gr}_{F^\bullet D}(D)$. By the previous comments, it is clear that $\mathrm{gr}(F_\tau)$ is a quasi-isomorphism if and only if $\mathrm{gr}(F)$ is so.

Moreover, it is clear that, if C and C' are coaugmented A_∞-coalgebras provided with an extra degree satisfying the conditions of Proposition 5.1 and $f_\bullet : C \to C'$ is a morphism between them such that f_1 is a quasi-isomorphism, then $\Omega^+(f_\bullet)$ is quasi-isomorphisms of augmented dg algebras. Indeed, this follows from the fact that the components of fixed internal degree of $\Omega^+(C)$ (resp., $\Omega^+(C')$) are obtained from those of C (resp., C') as the result of a finite number of steps involving taking tensor product, cones and shifts, and those operations preserve quasi-isomorphisms. As a consequence, a morphism of coaugmented A_∞-coalgebras preserving the internal degree is a quasi-equivalence if and only if it is a quasi-isomorphism. This proves the equivalence between (i) and (ii), (a).

It remains to prove that the conditions (a) and (b) in (ii) are equivalent. In order to prove it, we first note that if $\tau : D \to U$ is a filtered twisting cochain, then the associated graded morphism $\mathrm{gr}(\tau)$ of τ is a twisting cochain from C to A and it preserves the Adams degree. Moreover, the complex $U^e \otimes_\tau D$ is canonically provided with the filtration on a tensor product induced from those of the factors and its associated graded space is precisely $A^e \otimes_{\mathrm{gr}(\tau)} C$. Hence, Proposition 5.1 tells us that (a) and (b) are equivalent. $\square$

6. Application: Curved A_∞-coalgebras from PBW deformations.

6.1. The conventions, basic facts and the main result. Let A be an N-homogeneous algebra that is generalised Koszul, and let $U = TV/\langle P\rangle$ be a PBW-deformation of A. We use the notation of the previous sections. Since the case $N = 2$ is well-known in the literature, we will focus on $N > 2$.

Let $C = \oplus_{2p\in 2\mathbb{N}_0} \mathrm{Tor}^A_{2p}(k, k)$ and $M = \oplus_{2p+1\in 2\mathbb{N}_0+1} \mathrm{Tor}^A_{2p+1}(k, k)$. By the comments in Section 3, C is a coaugmented counitary graded coalgebra (concentrated in even homological degrees) with coproduct Δ_C, counit ϵ_C and coaugmentation η_C, and M is a counitary graded bicomodule over C with bicoaction ρ, concentrated in odd homological degrees. We will consider the increasing filtrations $\{F^\bullet C\}_{\bullet\in\mathbb{N}_0}$ and $\{F^\bullet M\}_{\bullet\in\mathbb{N}_0}$ of C and M associated with the corresponding Adams gradings recalled in Remark 3.1. We will call them *Adams filtrations*. Regarding both C and M inside of TV, the previous filtrations are just the induced ones from the standard filtration $\{F^\bullet\}_{\bullet\in\mathbb{N}_0}$ of TV seen in Section 4. Since the coproduct Δ_C, the counit ϵ_C and the coaugmentation η_C preserve the Adams degree, as mentioned in Remark 3.1, they preserve *a fortiori* the corresponding filtrations, where k has the trivial filtration $F^\bullet k = k$ if $\bullet \in \mathbb{N}_0$, and the filtration of a tensor product is given by the usual formula. The same argument implies that the bicoaction of M preserves the corresponding filtrations.

REMARK 6.1. The reason for considering the previous filtrations is the following: although the previous structures even preserve the mentioned Adams degree, we will soon complete them with several maps involved in a curved A_∞-coalgebra structure on $D = C[M]$ that will preserve the corresponding filtration but not the grading.

The main results we will prove in this section are the following.

THEOREM 6.2. *Let A be an N-homogeneous algebra that is generalised Koszul, and let $U = TV/\langle P\rangle$ be a PBW-deformation of A. Let $D = \mathrm{Tor}^A_\bullet(k,k)$ be the coaugmented A_∞-coalgebra described in Section 3, with nonvanishing comultiplications $\bar\Delta_2$ and $\bar\Delta_N$ given by (i) and (ii), counit $\epsilon_D : D \to k$ given by the canonical projection and coaugmentation $\eta_D : k \to D$ given by the canonical inclusion. Recall that $D = C \oplus M$, where C is the direct sum of the even homogeneous components, whereas, M is the direct sum of the odd homogeneous components, and let $\partial : M \to C$ be the coderivation given in Lemma 6.7. We provide D with the Adams filtration. Then, there is a unique structure $\{\Delta_\bullet\}_{\bullet\in\mathbb{N}_0}$ of filtered curved A_∞-coalgebra on D such that $\Delta_N = \bar\Delta_N$, $\Delta_p = \Delta'_p \circ \pi_C + \delta_{p,1}\partial \circ \pi_M + \delta_{p,2}\bar\Delta_2$ for $p \in \{0,\dots,N-1\}$, with $\Delta'_p = \Delta'_p|_C^{M^{\otimes p}}$ and $\Delta'_p|_R = -\varphi_p$ for all $p \in \{0,\dots,N-1\}$, and $\Delta_\bullet = 0$ for all $\bullet > N$. The previous structure is also counitary for ϵ_D and coaugmented for η_D.*

Proof. The fact that we obtain a structure of a counitary curved A_∞-coalgebra on D is a direct consequence of the Lemmas 6.7, 6.14, 6.16, 6.19–6.21, which will be proved in the forthcoming sections, together with Proposition 2.1. As noted in the first and last paragraphs of Section 6.4, the morphisms Δ'_p preserve the Adams filtrations for all $p \in \mathbb{N}_0$, as well as ϵ_D and η_D. The latter condition implies in particular that $\Delta'_p \circ \eta_C = 0$, for all $p \in \mathbb{N}_0$. By Proposition 2.1, we conclude that D is a filtered coaugmented curved A_∞-coalgebra. The uniqueness follows from the fact that the definition of Δ'_p for $p \in \{0,\dots,N-1\}$ given via (33) and (34) is tantamount to some special cases of identities (6) and (7). □

REMARK 6.3. As noted in **[3]** for the particular case that φ_0 vanishes, the converse of the previous result holds: consider a coaugmented curved A_∞-coalgebra structure on $D = \mathrm{Tor}^A_\bullet(k,k)$ that is compatible with the Adams filtrations and of the form given in Proposition 2.1 for the coaugmented coalgebra structure on D given by $\bar\Delta_2$ in Section 3, $\Delta_p = 0$ for $p > N$, and Δ_N is given by $\bar\Delta_N$ in Section 3. It determines a PBW deformation of A by means of $\varphi_p = -\Delta'_p|_R$ for all $p \in \{0,\dots,N-1\}$. Indeed, we first note that the bicoderivation ∂ is uniquely determined by $\mathfrak{p}_2 \circ \partial$, which in turn satisfies that $\mathfrak{p}_2 \circ \partial|_{M_{2k+1}} = 0$ for $k > 1$ and $\mathfrak{p}_2 \circ \partial|_{M_1} = 0$ for it has homological degree -1. The expression for $\mathfrak{p}_2 \circ \partial|_{M_3}$ given in Section 6.3 follows now from (5) for $p = N-1$. Finally, we see that (14) is equivalent to the vanishing of the restriction $\Delta'_0 \circ \partial|_{M_3}$; if $1 \leq j \leq N-1$, (15) is precisely the composition of (5) for $p = j-1$ with $\tilde\pi^M_{\bar\imath}$, where $\bar\imath = (1,\dots,1) \in \mathbb{N}^{p+1}$, whereas (13) is the previous composition for $p = N-1$. This induces a bijection between the set of filtered coaugmented curved A_∞-coalgebra structure on D satisfying the previous properties and the one of PBW deformations of A.

THEOREM 6.4. *Let A be an N-homogeneous algebra that is generalised Koszul, and let $U = TV/\langle P\rangle$ be a PBW-deformation of A. Let $D = \mathrm{Tor}^A_\bullet(k,k)$ be provided with the coaugmented curved A_∞-coalgebra described in Theorem 6.2. Then, there is a filtered quasi-equivalence between D and $B^+(U)$.*

Proof. It is easy to check that $\tau : D \to U$ given by the composition of minus the canonical projection $D \to V$ and the canonical inclusion $V \to U$ is a twisting cochain that preserves the Adams filtrations. By Theorem 5.2, it suffices to show that the associated graded to the filtered complex $U^e \otimes_\tau D$ is quasi-isomorphic to the standard A-bimodule A. This is indeed true, since the associated graded complex is just $A^e \otimes_{\mathrm{gr}(\tau)} \mathrm{Gr}(D)$, where $\mathrm{Gr}(D)$ is precisely the coaugmented A_∞-coalgebra described

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in Section 3, and $A^e \otimes_\tau \mathrm{Gr}(D)$ coincides with the Koszul bimodule complex of the standard dg A-bimodule A. □

REMARK 6.5. Note that the proof of the previous result also gives a description of a 'small' projective resolution of the standard U-bimodule U, which extends the Koszul bimodule complex given by R. Berger and V. Ginzburg for the particular case where $\varphi = \varphi_0$ (see [**2**, Section 5]).

6.2. Some larger spaces. The rest of the paper is dedicated to prove Theorem 6.2. We will essentially follow and complete the (duals of the) ideas in [**3**, Section 3]. However, as indicated in the introduction, there were missing steps in that article, e.g., the identities $m_1 \circ d = 0 = d \circ m_1$ were not proved there (cf. Lemma 6.20). Furthermore, some of our proofs are shorter and clearer (cf. [**3**, Lemmas 3.6 and 3.7] and our Lemmas 6.19 and 6.21, resp.), and mostly with less signs (cf. [**3**, Lemma 3.5] and our Fact 6.18).

Define the graded vector spaces $TR = \oplus_{j\in\mathbb{N}_0} R^{\otimes j}$ and $T(V^{\otimes N}) = \oplus_{j\in\mathbb{N}_0} V^{\otimes Nj}$, where $R^{\otimes j}$ and $V^{\otimes Nj}$ are concentrated in homological degree $2j$, for all $j \in \mathbb{N}_0$. We provide TR (resp., $T(V^{\otimes N})$) with the cofree cocomplete coaugmented graded coalgebra structure cogenerated by R (resp., $V^{\otimes N}$), and that we will denote by $\tilde{C}$ (resp., $\hat{C}$), i.e., the coproduct $\Delta_{\tilde{C}}$ (resp., $\Delta_{\hat{C}}$) is given by deconcatenation, the counit $\epsilon_{\tilde{C}} : \tilde{C} \to k$ (resp., $\epsilon_{\hat{C}} : \hat{C} \to k$) is the canonical projection, and the coaugmentation $\eta_{\tilde{C}} : k \to \tilde{C}$ (resp., $\eta_{\hat{C}} : k \to \hat{C}$) is the canonical inclusion. They are also provided with the induced Adams grading from that of A, and thus the associated filtration, which also coincides with the induced filtration from that of the tensor algebra TV where TR and $T(V^{\otimes N})$ are canonically included. We will also call them the *Adams filtration* of $\tilde{C}$ and $\hat{C}$, and will denote them by $\{F^\bullet \tilde{C}\}_{\bullet\in\mathbb{N}_0}$ and $\{F^\bullet \hat{C}\}_{\bullet\in\mathbb{N}_0}$, respectively. It is easy to see that the canonical inclusions $\tilde{\iota}_C : C \to \tilde{C}$ and $\hat{\iota}_{\tilde{C}} : \tilde{C} \to \hat{C}$ as well as their composition $\hat{\iota}_C = \hat{\iota}_{\tilde{C}} \circ \tilde{\iota}_C$ are morphisms of counitary graded coalgebras that preserve the Adams degree, so *a fortiori* they preserve the corresponding Adams filtrations. As a consequence, M is canonically a counitary graded bicomodule over $\tilde{C}$ and $\hat{C}$.

Let us also define the graded vector spaces $\tilde{M} = \oplus_{j\in\mathbb{N}_0} V^{\otimes Nj+1}$, where we set $V^{\otimes Nj+1}$ to have homological degree $2j+1$. We provide $\tilde{M}$ with a bicomodule structure $\tilde{\rho}$ over $\hat{C}$, whose bicoaction $\tilde{\rho}$ is defined by

$$\begin{aligned} &\tilde{\rho}(v_1 \dots v_{Nj+1}) \\ &\quad = \sum_{i_1=0}^{j} \sum_{i_2=0}^{j-i_1} v_1 \dots v_{Ni_1} \otimes v_{Ni_1+1} \dots v_{N(i_1+i_2)+1} \otimes v_{N(i_1+i_2)+2} \dots v_{Nj+1}, \end{aligned} \tag{20}$$

where $v_1, \dots, v_{Nj+1} \in V$, we omit the tensor symbols for the elements of $\tilde{M}$, $v_1 \dots v_{Ni_1} = 1$ if $i_1 = 0$ and $v_{N(i_1+i_2)+2} \dots v_{Nj+1} = 1$ if $i_1 + i_2 = j$. In particular, this induces a left (resp., right) coaction $\tilde{\rho}_\ell$ (resp., $\tilde{\rho}_r$) of $\tilde{M}$ over $\hat{C}$. Moreover, let $\tilde{\iota}_M : M \to \tilde{M}$ be the canonical inclusion. It is clearly a morphism of $\hat{C}$-bicomodules, so,

$$\tilde{\rho}_\ell \circ \tilde{\iota}_M = (\mathrm{id}_{\hat{C}} \otimes \tilde{\iota}_M) \circ \tilde{\rho}_\ell, \quad \text{and} \quad \tilde{\rho}_r \circ \tilde{\iota}_M = (\tilde{\iota}_M \otimes \mathrm{id}_{\hat{C}}) \circ \tilde{\rho}_r, \tag{21}$$

where we denote the left (resp., right) coaction on M over $\hat{C}$ also by $\tilde{\rho}_\ell$ (resp., $\tilde{\rho}_r$). As usual, $\tilde{M}$ has the Adams filtration $\{F^\bullet \tilde{M}\}_{\bullet\in\mathbb{N}_0}$ induced from the canonical filtration of TV by regarding its canonical contention of $\tilde{M}$. We will also consider the map

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$$\hat{\Delta}_N : \hat{C} \to \tilde{M}^{\otimes N} \tag{22}$$

defined as the linear map sending $1 \in k$ to zero, and $v_1 \ldots v_{Nj} \in \hat{C}_{2j} = V^{\otimes(Nj)}$ to

$$\sum_{\substack{\bar{\imath} \in \mathbb{N}_0^N \\ |\bar{\imath}| = j-1}} V_1^{\bar{\imath}} \otimes \cdots \otimes V_N^{\bar{\imath}} \in \tilde{M}^{\otimes N}, \tag{23}$$

where $v_1, \ldots, v_{Nj} \in V$, and $V_j^{\bar{\imath}} \in \tilde{M}_{2i_j+1} = V^{\otimes(Ni_j+1)}$ is given by

$$V_j^{\bar{\imath}} = v_{N(i_1+\cdots+i_{j-1})+j} \ldots v_{N(i_1+\cdots+i_j)+j}, \tag{24}$$

for all $j = 1, \ldots, N$. Note that $\hat{\Delta}_N \circ \hat{\iota}_C = \tilde{\iota}_M^{\otimes N} \circ \bar{\Delta}_N|_C$.

We will use the following notation. For $p \in \mathbb{N}$ and $\bar{\imath} = (i_1, \ldots, i_p) \in \mathbb{N}^p$, set $\tilde{\pi}_{\bar{\imath}} = \tilde{\pi}_{i_1} \otimes \cdots \otimes \tilde{\pi}_{i_p}$, with $\tilde{\pi}_i : TV \to V^{\otimes \phi_N(i)}$ the canonical projection. For the following, recall the canonical inclusion $\iota_D : D \to TV$. We remark that $\tilde{\pi}_i \circ \iota_D|_C$ vanishes if i is odd and it coincides with the composition of the canonical projection $\mathfrak{p}_i : C \to C_i$ together with the inclusion $C_i \to V^{\otimes \phi_N(i)}$, if i is even. Analogously, $\tilde{\pi}_i \circ \iota_D|_M$ vanishes if i is even and it coincides with the composition of the canonical projection $\mathfrak{p}_i : M \to M_i$ together with the inclusion $M_i \to V^{\otimes \phi_N(i)}$, if i is odd. We will denote $\tilde{\pi}_i \circ \iota_D|_M$ by $\tilde{\pi}_i^M$, and the tensor product $\tilde{\pi}_{i_1}^M \otimes \cdots \otimes \tilde{\pi}_{i_p}^M$ by $\tilde{\pi}_{\bar{\imath}}^M$. The analogous notation will be used for C, $\hat{C}$ and $\tilde{M}$.

FACT 6.6. *For $i, j \in \mathbb{N}_0$ such that i is even and j is odd, we have $\tilde{\pi}_{i+j} \circ \iota_{\tilde{M}} = (\tilde{\pi}_i^{\hat{C}} \otimes \tilde{\pi}_j^{\tilde{M}}) \circ \tilde{\rho}_\ell$ and $\tilde{\pi}_{i+j} \circ \iota_{\tilde{M}} = (\tilde{\pi}_j^{\tilde{M}} \otimes \tilde{\pi}_i^{\hat{C}}) \circ \tilde{\rho}_r$, by definition of the coactions of $\tilde{M}$, where $\iota_{\tilde{M}} : \tilde{M} \to TV$ is the canonical inclusion. Analogously, the same holds for the coactions ρ_ℓ and ρ_r of M.*

6.3. The definition of the morphism ∂.

Define

$$\tilde{\partial} : M \to \tilde{C} \tag{25}$$

as the unique bicoderivation satisfying that $\tilde{\mathfrak{p}}_2 \circ \tilde{\partial} = (\mathrm{id}_V \otimes \varphi_{N-1} - \varphi_{N-1} \otimes \mathrm{id}_V) \circ \mathfrak{p}_3$, where $\tilde{\mathfrak{p}}_2 : \tilde{C} \to R$ and $\mathfrak{p}_3 : M \to M_3 = (R \otimes V) \cap (V \otimes R)$ are the canonical projections. This is well-defined by (13). It is clear that $\tilde{\partial}|_V$ vanishes, where $V = \mathrm{Tor}_1^A(k, k) \subseteq M$, as well as $\epsilon_{\tilde{C}} \circ \tilde{\partial}$. Moreover, if $\alpha \in \mathrm{Tor}_{2p+1}^A(k, k) \subseteq M$, where $p \in \mathbb{N}$, we have *a fortiori*

$$\alpha \in \bigcap_{i=0}^{p-1} R^{\otimes i} \otimes \big((R \otimes V) \cap (V \otimes R)\big) \otimes R^{\otimes(p-i-1)}.$$

So, for every $i \in \{0, \ldots, p-1\}$, we can write $\alpha = \sum_{(a_i)} \bar{r}_{a_i} \otimes \bar{\alpha}_{a_i} \otimes \bar{r}'_{a_i}$, where $\bar{r}_{a_i} \in R^{\otimes i}$, $\bar{\alpha}_{a_i} \in R_{N+1}$ and $\bar{r}'_{a_i} \in R^{\otimes(p-i-1)}$. Then,

$$\tilde{\partial}(\alpha) = \sum_{i=0}^{p-1} \sum_{(a_i)} \bar{r}_{a_i} \otimes \big(\mathrm{id}_V \otimes \varphi_{N-1} - \varphi_{N-1} \otimes \mathrm{id}_V\big)(\bar{\alpha}_{a_i}) \otimes \bar{r}'_{a_i}. \tag{26}$$

Finally, we remark that $\tilde{\partial}$ preserves the corresponding filtrations (but it does not preserve in general the Adams degree).

The first result is the following.

LEMMA 6.7. *The bicoderivation* $\tilde{\partial}$ *defined in* (25) *factors through the canonical inclusion* $C \to \tilde{C}$, *so it induces a bicoderivation* $\partial : M \to C$ *satisfying that* $\epsilon_C \circ \partial = 0$. *Moreover,* ∂ *preserves the Adams filtrations.*

Proof. By definition of $\tilde{\partial}$, the image of $\tilde{\partial}|_{D_3}$ is included in $R \subseteq C$, so $\tilde{\partial}(D_{2p+1}) \subseteq C$ for $p = 0$ and $p = 1$. Let $p \geq 2$. It is clear that it suffices to show that

$$\tilde{\partial}(D_{2p+1}) \subseteq R^{\otimes h} \otimes V^{\otimes j} \otimes R \otimes V^{\otimes (N-j)} \otimes R^{\otimes (p-h-2)}, \tag{27}$$

for every $j \in \{0, \dots, N\}$ and every $h \in \{0, \dots, p-2\}$. The cases $j = 0$ and $j = N$ (and arbitrary h) follow immediately from the definition of $\tilde{\partial}$. Moreover, since

$$\begin{aligned} D_{2p+1} \subseteq & \left(\left(\bigcap_{i=0}^{h-1} R^{\otimes i} \otimes \big((R \otimes V) \cap (V \otimes R)\big) \otimes R^{\otimes (h-i-1)} \right) \otimes D_4 \otimes R^{\otimes (p-h-2)} \right) \\ & \cap \left(R^{\otimes h} \otimes D_4 \otimes \left(\bigcap_{i=0}^{p-h-3} R^{\otimes i} \otimes \big((R \otimes V) \cap (V \otimes R)\big) \otimes R^{\otimes (p-h-i-3)} \right) \right), \end{aligned}$$

for every $h \in \{0, \dots, p-2\}$ and $p \geq 3$, (26) tells us that it suffices to prove (27) for $p = 2$. Let us first prove the case $j = N - 1$. Furthermore, since

$$\tilde{\partial}|_{D_3} = (\mathrm{id}_V \otimes \varphi_{N-1} - \varphi_{N-1} \otimes \mathrm{id}_V),$$

a telescopic argument implies that

$$\begin{aligned} & \sum_{\ell=0}^{N} \mathrm{id}_V^{\otimes (Nh+\ell)} \otimes \tilde{\partial}|_{D_3} \otimes \mathrm{id}_V^{\otimes \left(N-\ell+N(p-h-2)\right)} \\ & \quad = \mathrm{id}_V^{\otimes (Nh)} \otimes (\mathrm{id}_V^{\otimes (N+1)} \otimes \varphi_{N-1} - \varphi_{N-1} \otimes \mathrm{id}_V^{\otimes (N+1)}) \otimes \mathrm{id}_V^{\otimes \left(N(p-h-2)\right)}, \end{aligned}$$

for $p \geq 2$ and $h \in \{0, \dots, p-2\}$. If $p = 2$, this is equivalent to

$$\begin{aligned} & \tilde{\partial}|_{D_3} \otimes \mathrm{id}_V^{\otimes N} + \mathrm{id}_V^{\otimes N} \otimes \tilde{\partial}|_{D_3} \\ & = \left(\mathrm{id}_V^{\otimes (N+1)} \otimes \varphi_{N-1} - \varphi_{N-1} \otimes \mathrm{id}_V^{\otimes (N+1)}\right) - \sum_{j=1}^{N-1} \mathrm{id}_V^{\otimes j} \otimes \tilde{\partial}|_{D_3} \otimes \mathrm{id}_V^{\otimes (N-j)}. \end{aligned} \tag{28}$$

Every term on the right member of (28) clearly sends D_5 to $\sum_{\ell=1}^{N-1} V^{\otimes \ell} \otimes R \otimes V^{\otimes (N-\ell)}$, so the same is true for the left member. By (26), the left member of (28) is precisely $\tilde{\partial}|_{D_5}$. Moreover, the tensor product of the inclusion (9) for $j = N - 1$ with V on the left tells us that

$$(V^{\otimes N} \otimes R) \cap \left(\sum_{\ell=1}^{N-1} V^{\otimes \ell} \otimes R \otimes V^{\otimes (N-\ell)} \right) \subseteq V^{\otimes (N-1)} \otimes R \otimes V,$$

which, combined with the previous arguments, yields (27) for $j = N - 1$ (and $p = 2$). Assume that (27) holds for all $j = j_0 + 1, \dots, N$ (and $p = 2$), with $1 \leq j_0 \leq N - 2$.

Then, by tensoring (9) for $j = j_0 + 1$ on the right with $V^{\otimes(N-j_0-1)}$, we get

$$(V^{\otimes(j_0+1)} \otimes R \otimes V^{\otimes(N-j_0-1)}) \cap \left(\sum_{\ell=0}^{j_0} V^{\otimes \ell} \otimes R \otimes V^{\otimes(N-\ell)} \right) \subseteq V^{\otimes j_0} \otimes R \otimes V^{\otimes(N-j_0)},$$

which, together with (27) for $p = 2$ and $j = 0$, implies that (27) holds for all $j = j_0$, by the inductive hypothesis. Finally, $\epsilon_C \circ \partial = 0$ follows from $\epsilon_{\tilde{C}} \circ \tilde{\partial} = 0$. The lemma is thus proved. □

In what follows we will denote by $\hat{\partial} : M \to \hat{C}$ the composition of ∂ and $\hat{\iota}_C$.

6.4. The definition of the morphisms Δ'_p. For $p \in \mathbb{N}_0$, we will proceed to define linear maps

$$\Delta'_p : C \to M^{\otimes p} \tag{29}$$

that preserve the Adams filtrations. If $p = 0$, set Δ'_0 as the composition of the canonical projection $C \to R$ together with $-\varphi_0$. Moreover, Δ'_p is the zero map if $p > N$ and the map $\bar{\Delta}_N$, given in Section 3, if $p = N$. It is clear that these maps also preserve the corresponding filtrations.

FACT 6.8. *The first equation in* (4) *holds, i.e.,* $\Delta'_0 \circ \partial = 0$, *where* ∂ *is given in Lemma 6.7.*

Proof. By definition of ∂ and of Δ'_0, we see that $\Delta'_0 \circ \partial|_{M_{2p+1}}$ trivially vanishes for all $p \neq 1$. Furthermore, $\Delta'_0 \circ \partial|_{M_3} = \varphi_0 \circ (\varphi_{N-1} \otimes \mathrm{id}_V - \mathrm{id}_V \otimes \varphi_{N-1})$, which vanishes by (14). □

FACT 6.9. *Equations* (5)–(8) *hold for all* $p, q \geq N$, *where* ∂ *is given in Lemma 6.7.*

Proof. Indeed, the mentioned identities are precisely the Stasheff equations SI(m) with $m > N$ for the coaugmented A_∞-coalgebra structure on D considered in Section 3. □

Note that $\Delta'_0 \circ \eta_C = 0$ and

$$\Delta'_p \circ \eta_C = 0 \tag{30}$$

hold for all $p \geq N$, because $(D, \bar{\Delta}_2, \bar{\Delta}_N, \epsilon_D, \eta_D)$ is a coaugmented A_∞-coalgebra.

DEFINITION 6.10. Let $\ell \in \{0, \ldots, N-1\}$. Recall that Δ'_p is defined for all $p \in \{0\} \cup \mathbb{N}_{\geq N}$ in (29). Assume that $\Delta'_{\ell+1}, \ldots, \Delta'_{N-1}$ are also defined, they preserve the Adams filtrations, and that the equations (5), (6), (7) and (8) hold for all $p, q > \ell$, where ∂ is given in Lemma 6.7. We will call this hypothesis the *assumption* $\mathcal{A}_\ell$ *of degree* ℓ. By Fact 6.9, the assumption $\mathcal{A}_{N-1}$ holds.

Assume $\mathcal{A}_p$ holds for some $p \in \{1, \ldots, N-1\}$. For $q \in \mathbb{N}_{>p} \cup \{0\}$, define the linear map

$$\tilde{\Delta}_q : C \to \tilde{M}^{\otimes q} \tag{31}$$

as the composition of Δ'_q and $\tilde{\iota}_M^{\otimes q}$, where $\tilde{\iota}_M : M \to \tilde{M}$ is the canonical inclusion (defined in the third paragraph of Section 6.2). Moreover, set $\tilde{\Delta}_p^{[0]} : C \to V^{\otimes p}$ as the

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composition of the canonical projection $\mathfrak{p}_2 : C \to R$ together with $-\varphi_p$. Consider the finite increasing filtration $\{G^{\bullet}(\tilde{M}^{\otimes p})\}_{\bullet \in \{0,\dots,p\}}$ given by $G^{\ell}(\tilde{M}^{\otimes p}) = \tilde{M}^{\otimes \ell} \otimes V^{\otimes p-\ell}$. This is clearly different from the Adams filtration on $\tilde{M}^{\otimes p}$. Note that the image of $\tilde{\Delta}_p^{[0]}$ is included in $G^0(\tilde{M}^{\otimes p})$, which is also included in $M^{\otimes p}$. Suppose further that there is $j_0 \in \{0, \dots, p-1\}$ such that $\tilde{\Delta}_p^{[j]} : C \to G^j(\tilde{M}^{\otimes p})$ is defined for all $j \in \{0, \dots, j_0\}$ and the composition of $\tilde{\Delta}_p^{[j]}$ with the map $\mathrm{id}_{\tilde{M}^{\otimes (j-1)}} \otimes \tilde{\mathfrak{p}}_1 \otimes \mathrm{id}_V^{\otimes (p-j)}$ (whose image is $G^{(j-1)}(\tilde{M}^{\otimes p})$) is $\tilde{\Delta}_p^{[j-1]}$ for all $j \in \{1, \dots, j_0\}$, where $\tilde{\mathfrak{p}}_1 : \tilde{M} \to V$ the canonical projection. Let $\bar{\imath} = (i_1, \dots, i_p) \in (2\mathbb{N}_0 + 1)^p$ be such that $i_{j_0+2} = \cdots = i_p = 1$ and $i_{j_0+1} > 1$. Then, define

$$\tilde{\Delta}_p^{\bar{\imath}} : C \to \tilde{M}_{i_1} \otimes \cdots \otimes \tilde{M}_{i_p} \tag{32}$$

by

$$\big(\tilde{\pi}^C_{i_1-1} \otimes (\tilde{\pi}^{\tilde{M}}_{(1,i_2,\dots,i_p)} \circ \tilde{\Delta}_p^{[j_0]})\big) \circ \Delta_C + \big((\tilde{\pi}^C_{i_1-1} \circ \partial) \otimes \tilde{\pi}^M_{(1,i_2,\dots,i_p)}\big) \circ \Delta'_{p+1}, \tag{33}$$

if $p = 1$, or $p \geq 2$ and $j_0 = 0$, and by

$$\begin{aligned} &\tilde{\pi}^{\tilde{M}}_{(i_1,\dots,i_{j_0-1},i_{j_0}+i_{j_0+1}-1,1,i_{j_0+2},\dots,i_p)} \circ \tilde{\Delta}_p^{[j_0]} \\ &\quad + (-1)^{j_0}\big(\tilde{\pi}^M_{(i_1,\dots,i_{j_0})} \otimes (\tilde{\pi}^C_{i_{j_0+1}-1} \circ \partial) \otimes \tilde{\pi}^M_{(1,i_{j_0+2},\dots,i_p)}\big) \circ \Delta'_{p+1}, \end{aligned} \tag{34}$$

if $p \geq 2$ and $j_0 > 0$. It is clearly well-defined, by the assumption $\mathcal{A}_p$. Moreover, the image of $\tilde{\Delta}_p^{\bar{\imath}}$ is even included in $\tilde{M}^{\otimes (j_0+1)} \otimes M^{\otimes (p-j_0-1)}$. Note that $\tilde{\Delta}_p^{\bar{\imath}}$ trivially preserves the Adams filtrations. Finally, define $\tilde{\Delta}_p^{[j_0+1]} : C \to G^{j_0+1}(\tilde{M}^{\otimes p})$ as the unique linear map satisfying that $\tilde{\pi}^{\tilde{M}}_{\bar{\imath}} \circ \tilde{\Delta}_p^{[j_0+1]} = \tilde{\Delta}_p^{\bar{\imath}}$, for all $\bar{\imath} \in \mathbb{N}^{q+1}$ such that $i_{j_0+2} = \cdots = i_p = 1$. The fact that the Adams filtration is locally finite dimensional, i.e., each subspace of the filtration is finite dimensional, implies that $\tilde{\Delta}_p^{[j_0+1]}$ is well-defined, for all $j_0 \in \{0, \dots, p-1\}$. Finally, set $\tilde{\Delta}_p = \tilde{\Delta}_p^{[p]}$.

FACT 6.11. *Assume the hypothesis $\mathcal{A}_q$ for some $q \in \{1, \dots, N-1\}$. Then, the map $\tilde{\Delta}_q$ preserves the corresponding Adams filtrations.*

6.5. The properties satisfied by the morphisms Δ'_p.

FACT 6.12. *Assume the hypothesis $\mathcal{A}_q$ for some $q \in \{1, \dots, N-1\}$. The identity*

$$\bar{\pi}_{\bar{\imath}} \circ (\tilde{\rho}_{\ell} \otimes \mathrm{id}_{\tilde{M}}^{\otimes (q-1)}) \circ \tilde{\Delta}_q = \bar{\pi}_{\bar{\imath}} \circ (\mathrm{id}_C \otimes \tilde{\Delta}_q) \circ \Delta_C + \bar{\pi}_{\bar{\imath}} \circ (\partial \otimes \tilde{\imath}_M^{\otimes q}) \circ \Delta'_{q+1} \tag{35}$$

is verified for all $\bar{\imath} \in \mathbb{N}^{q+1}$ such that $i_2 = \cdots = i_{q+1} = 1$, where $\bar{\pi}_{\bar{\imath}} = \tilde{\pi}^C_{i_1} \otimes \tilde{\pi}^{\tilde{M}}_{(i_2,\dots,i_{q+1})}$.

If $q \geq 2$, then,

$$\begin{aligned} \bar{\pi}_{\bar{\imath}} \circ (\mathrm{id}_{\tilde{M}}^{\otimes j} \otimes \tilde{\rho}_{\ell} \otimes \mathrm{id}_{\tilde{M}}^{\otimes (q-j-1)}) \circ \tilde{\Delta}_q &= \bar{\pi}_{\bar{\imath}} \circ (\mathrm{id}_{\tilde{M}}^{\otimes (j-1)} \otimes \tilde{\rho}_r \otimes \mathrm{id}_{\tilde{M}}^{\otimes (q-j)}) \circ \tilde{\Delta}_q \\ &\quad + (-1)^j \bar{\pi}_{\bar{\imath}} \circ (\tilde{\imath}_M^{\otimes j} \otimes \partial \otimes \tilde{\imath}_M^{\otimes (q-j)}) \circ \Delta'_{q+1} \end{aligned} \tag{36}$$

holds for all $j \in \{1, \dots, q-1\}$, and all $\bar{\imath} \in \mathbb{N}^{q+1}$ such that $i_{j+2} = \cdots = i_{q+1} = 1$, where $\bar{\pi}_{\bar{\imath}} = \tilde{\pi}^{\tilde{M}}_{(i_1,\dots,i_j)} \otimes \tilde{\pi}^{\hat{C}}_{i_{j+1}} \otimes \tilde{\pi}^{\tilde{M}}_{(i_{j+2},\dots,i_{q+1})}$.

Proof. Note that (35) trivially holds if $\bar{\imath}$ does not satisfy that i_1 is even, and (36) is directly fulfilled if $\bar{\imath}$ does not satisfy that i_{j+1} is even and $i_1, \dots, i_j$ are odd. It thus suffices to prove them when these conditions are met. Then, (33) for $p = 1$ is equivalent to (35) for $q = 1$ and $\bar{\imath} = (i-1, 1)$, where $i \in \mathbb{N}_{>1}$, and we have used Fact 6.6 to deal with the first member of (35). Analogously, if $q \geq 2$, (33) is tantamount to (35) for $q = p$ and $(i_1 - 1, 1, i_2, \dots, i_p)$, where $\bar{\imath} \in \mathbb{N}^p$ satisfies that $i_2 = \cdots = i_p = 1$ and $i_1 > 1$. The first part of the statement thus follows. The second part is proved from an easy inductive argument based on the fact that (34) for $p \geq 2$ is equivalent to (36) for $q = p$, $j = j_0$ and $(i_1, \dots, i_{j_0}, i_{j_0+1} - 1, 1, i_{j_0+2}, \dots, i_p)$, where $\bar{\imath} \in \mathbb{N}^p$ satisfies that $i_{j_0+2} = \cdots = i_p = 1$ and $i_{j_0+1} > 1$. □

REMARK 6.13. If the assumption $\mathcal{A}_p$ holds for some $p \in \{1, \dots, N-1\}$ and let $q > p$, then (35) and (36) are a particular case of (6) and (7), respectively.

LEMMA 6.14. *Assume the hypothesis $\mathcal{A}_q$ for some $q \in \{1, \dots, N-1\}$. Identity* (35) *(resp.,* (36)*) holds for all $\bar{\imath} \in \mathbb{N}^{q+1}$ (resp., for all $j \in \{1, \dots, q-1\}$ and all $\bar{\imath} \in \mathbb{N}^{q+1}$). In other words,*

$$(\tilde{\rho}_\ell \otimes \mathrm{id}_{\tilde{M}}^{\otimes(q-1)}) \circ \tilde{\Delta}_q = (\hat{\iota}_C \otimes \tilde{\Delta}_q) \circ \Delta_C + (\hat{\partial} \otimes \tilde{\iota}_M^{\otimes q}) \circ \Delta'_{q+1} \tag{37}$$

holds, and

$$\begin{aligned}(\mathrm{id}_{\tilde{M}}^{\otimes j} \otimes \tilde{\rho}_\ell \otimes \mathrm{id}_{\tilde{M}}^{\otimes(q-j-1)}) \circ \tilde{\Delta}_q &= (\mathrm{id}_{\tilde{M}}^{\otimes(j-1)} \otimes \tilde{\rho}_r \otimes \mathrm{id}_{\tilde{M}}^{\otimes(q-j)}) \circ \tilde{\Delta}_q \\ &\quad + (-1)^j (\tilde{\iota}_M^{\otimes j} \otimes \hat{\partial} \otimes \tilde{\iota}_M^{\otimes(q-j)}) \circ \Delta'_{q+1}\end{aligned} \tag{38}$$

holds if $q \geq 2$ and $j \in \{1, \dots, q-1\}$.

Proof. Note again that (35) trivially holds if $\bar{\imath}$ does not satisfy that i_1 is even and for all $j \in \{2, \dots, q+1\}$, i_j is odd, and (36) is directly fulfilled if $\bar{\imath}$ does not satisfy that i_{j+1} is even and the other indices $i_1, \dots, i_j, i_{j+2}, \dots, i_{q+1}$ are odd. It thus suffices to prove them when these conditions are met. For simplicity we shall treat the case of (35), but the proof for (36) follows the same pattern.

We will proceed by induction. Assume that there is $\ell_0 \in \{1, \dots, q-1\}$ such that (35) holds for all $\bar{\imath} \in \mathbb{N}^{q+1}$ such that $i_{\ell_0+1} = \cdots = i_{q+1} = 1$. By Fact 6.12, this hypothesis is fulfilled if $\ell_0 = 1$. We will prove it holds for $\ell_0 + 1$, provided it does for ℓ_0. Assume $\ell_0 \geq 2$. Let $\bar{\imath} \in \mathbb{N}^{q+1}$ be such that $i_{\ell_0+2} = \cdots = i_{q+1} = 1$ and $i_{\ell_0+1} > 1$. We recall that we are also assuming that i_1 is even and the other indices $i_2, \dots, i_{q+1}$ are odd. Using definition (34), the left member of (35) is just

$$\begin{aligned}&\tilde{\pi}^{\tilde{M}}_{(i_1+i_2, i_3, \dots, i_{\ell_0-1}, i_{\ell_0}+i_{\ell_0+1}-1, 1, i_{\ell_0+2}, \dots, i_{q+1})} \circ \tilde{\Delta}_q \\ &\quad + (-1)^{\ell_0-1} \big(\tilde{\pi}^M_{(i_1+i_2, i_3, \dots, i_{\ell_0})} \otimes (\tilde{\pi}^C_{i_{\ell_0+1}-1} \circ \partial) \otimes \tilde{\pi}^M_{(1, i_{\ell_0+2}, \dots, i_{q+1})}\big) \circ \Delta'_{q+1},\end{aligned} \tag{39}$$

whereas, the first term of the right member of (35) is given by

$$\begin{aligned}&\big(\tilde{\pi}^C_{i_1} \otimes (\tilde{\pi}^{\tilde{M}}_{(i_2, \dots, i_{\ell_0-1}, i_{\ell_0}+i_{\ell_0+1}-1, 1, i_{\ell_0+2}, \dots, i_{q+1})} \circ \tilde{\Delta}_q)\big) \circ \Delta_C \\ &\quad + (-1)^{\ell_0+1} \Big(\tilde{\pi}^C_{i_1} \otimes \big(\tilde{\pi}^M_{(i_2, \dots, i_{\ell_0})} \otimes (\tilde{\pi}^C_{i_{\ell_0+1}-1} \circ \partial) \otimes \tilde{\pi}^M_{(1, i_{\ell_0+2}, \dots, i_{q+1})}\big) \circ \Delta'_{q+1}\Big) \circ \Delta_C,\end{aligned} \tag{40}$$

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and the second term of the right member of (35) is

$$\begin{aligned} \big((\tilde{\pi}^C_{i_1}\circ\partial)\otimes \mathrm{id}_{M^{\otimes q}}\big)\circ\Big(&\tilde{\pi}^M_{(i_1+1,i_2,\dots,i_{\ell_0-1},i_{\ell_0}+i_{\ell_0+1}-1,1,i_{\ell_0+2},\dots,i_{q+1})}\circ\Delta'_{q+1}\\ &+(-1)^{\ell_0}\big(\tilde{\pi}^M_{(i_1+1,i_2,\dots,i_{\ell_0})}\otimes(\tilde{\pi}^C_{i_{\ell_0+1}-1}\circ\partial)\otimes\tilde{\pi}^M_{(1,i_{\ell_0+2},\dots,i_{q+1})}\big)\circ\Delta'_{q+2}\Big). \end{aligned} \tag{41}$$

By applying the inductive assumption we can rewrite the first term of (39) as the sum of the first two terms of (40) and (41), respectively. Moreover, since the statement of this lemma is a particular case of (6) and (7), which holds for $q+1$ by $\mathcal{A}_q$ (see Remark 6.13), we see that the second term of (39) is the sum of the last two terms of (40) and (41), respectively. Note that the sign difference is compensated by the Koszul sign rule applied to the commutation of the morphisms $\tilde{\pi}^C_{i_1}\circ\partial$ and $\tilde{\pi}^C_{i_{\ell_0+1}-1}\circ\partial$.

Finally, the missing inductive step for $\ell_0=1$ in the proof of (35) follows in the same fashion, using (33) instead of (34) for rewriting the left member and the first term of the right member of (35), but without rewriting the last term of the right member of (35). The first terms obtained from these reexpressions of the left member and the first term of the right member of (35) clearly coincide. The required equality, involving the 3 remaining terms, follows from the property that ∂ is a coderivation. □

FACT 6.15. *Suppose that the assumption $\mathcal{A}_p$ holds for some $p\in\{1,\dots,N-1\}$. Then, given any $j_0\in\{1,\dots,p\}$ we have that*

$$\begin{aligned} \Big(&\mathrm{id}_{\tilde{M}}^{\otimes(j_0-1)}\otimes\big((\hat{\Delta}_N\otimes\mathrm{id}_{\tilde{M}})\circ\tilde{\rho}_\ell\big)\otimes\mathrm{id}_{\tilde{M}}^{\otimes(p-j_0)}\Big)\circ\tilde{\Delta}_p\\ &=(-1)^{(j_0-1)N}(\tilde{\Delta}_N\otimes\tilde{\Delta}_p)\circ\Delta_C\\ &\quad+\sum_{j=0}^{j_0-1}(-1)^{j+(j_0-j-1)N}\big(\tilde{\iota}_M^{\otimes j}\otimes(\tilde{\Delta}_N\circ\partial)\otimes\tilde{\iota}_M^{\otimes(p-j)}\big)\circ\Delta'_{p+1}. \end{aligned} \tag{42}$$

Note that $p-j\geq p-j_0+1\geq 1$ in the last sum.

Proof. If $j_0=1$, (42) is precisely the composition of (37) for $q=p$ with $\hat{\Delta}_N\otimes\mathrm{id}_{\tilde{M}}^{\otimes p}$, where we have used that $\hat{\Delta}_N\circ\hat{\iota}_C=\tilde{\Delta}_N$. Assume that $j_0>1$, and that (42) holds for $\{1,\dots,j_0-1\}$. It is easy to see that the composition of (38) for $q=p$ and $j=j_0-1$ with $\mathrm{id}_{\tilde{M}}^{\otimes(j_0-1)}\otimes\hat{\Delta}_N\otimes\mathrm{id}_{\tilde{M}}^{\otimes(p-j_0+1)}$ gives

$$\begin{aligned} \Big(&\mathrm{id}_{\tilde{M}}^{\otimes(j_0-1)}\otimes\big((\hat{\Delta}_N\otimes\mathrm{id}_{\tilde{M}})\circ\tilde{\rho}_\ell\big)\otimes\mathrm{id}_{\tilde{M}}^{\otimes(p-j_0)}\Big)\circ\tilde{\Delta}_p\\ &=(-1)^N\Big(\mathrm{id}_{\tilde{M}}^{\otimes(j_0-2)}\otimes\big((\hat{\Delta}_N\otimes\mathrm{id}_{\tilde{M}})\circ\tilde{\rho}_\ell\big)\otimes\mathrm{id}_{\tilde{M}}^{\otimes(p+1-j_0)}\Big)\circ\tilde{\Delta}_p\\ &\quad+(-1)^{j_0-1}\big(\tilde{\iota}_M^{\otimes(j_0-1)}\otimes(\tilde{\Delta}_N\circ\partial)\otimes\tilde{\iota}_M^{\otimes(p+1-j_0)}\big)\circ\Delta'_{p+1}, \end{aligned} \tag{43}$$

where we have used the identity $(\hat{\Delta}_N\otimes\mathrm{id}_{\tilde{M}})\circ\tilde{\rho}_\ell=(-1)^N(\mathrm{id}_{\tilde{M}}\otimes\hat{\Delta}_N)\circ\tilde{\rho}_r$ (which follows from the definitions) in the first term of the right member of (43), and $\hat{\Delta}_N\circ\hat{\partial}=\tilde{\Delta}_N\circ\partial$ in the second term of the right member of (43). The result follows from the inductive hypothesis. □

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LEMMA 6.16. *Assume the hypothesis $\mathcal{A}_q$ for some $q \in \{1, \dots, N-1\}$. Then,*

$$\tilde{\Delta}_{q+1} \circ \partial = (\tilde{\Delta}_q \otimes \tilde{\iota}_M) \circ \rho_\ell - (-1)^q (\tilde{\iota}_M \otimes \tilde{\Delta}_q) \circ \rho_r \tag{44}$$

holds.

Proof. We will prove that (44) holds for $q \in \{1, \dots, N-1\}$, if $\mathcal{A}_q$ is verified. It suffices to prove that the compositions of the left and the right member of (44) with $\tilde{\pi}_{\bar{\imath}}$ coincide, for all $\bar{\imath} \in (2\mathbb{N}_0+1)^{q+1}$. If $1 \leq q < N-1$, it is clear that the composition of (44) with $\tilde{\pi}_{\bar{\imath}}^{\tilde{M}}$, where $\bar{\imath} = (1, \dots, 1) \in \mathbb{N}^{q+1}$, is precisely (15) for $j = q+1$, whereas if $q = N-1$ it gives (13). It thus suffices to prove that the compositions of the left and the right member of (44) with $\tilde{\pi}_{\bar{\imath}}^{\tilde{M}}$ coincide, for all $\bar{\imath} \in (2\mathbb{N}_0+1)^{q+1}$ such that $|\bar{\imath}| > q+1$. This is clearly tantamount to proving that the compositions of the left and the right member of (44) with

$$\Big(\mathrm{id}_{\tilde{M}}^{\otimes(j_0-1)} \otimes \big((\hat{\Delta}_N \otimes \mathrm{id}_{\tilde{M}}) \circ \tilde{\rho}_\ell\big) \otimes \mathrm{id}_{\tilde{M}}^{\otimes(q+1-j_0)}\Big) \tag{45}$$

coincide for all $j_0 \in \{1, \dots, q+1\}$. This follows from the fact that $(\hat{\Delta}_N \otimes \mathrm{id}_{\tilde{M}}) \circ \tilde{\rho}_\ell|_{\tilde{M}_\ell}$ is injective for every odd integer $\ell > 1$, which is a consequence of (23), (24) and Fact 6.6. Moreover, we will prove that the composition of (44), (45) and $\tilde{\pi}_{\bar{a}}^{\tilde{M}}$ holds for all $\bar{a} \in (2\mathbb{N}_0+1)^{q+N+1}$, by induction on $|\bar{a}|$. Assume we have proved it for all $\bar{a} \in (2\mathbb{N}_0+1)^{q+N+1}$ such that $|\bar{a}| < \ell_0$ for some $\ell_0 > q+1$, and we will now show it holds for $|\bar{a}| = \ell_0$.

Consider now the composition of ∂ and equation (42) for $p = q+1$. This tells us that the composition of the left member of (44) with (45) is precisely

$$\begin{aligned}
&(-1)^{(j_0-1)N+q+1}\big((\tilde{\Delta}_N \circ \partial) \otimes \tilde{\Delta}_{q+1}\big) \circ \rho_r + (-1)^{j+(j_0-1)N}\big(\tilde{\Delta}_N \otimes (\tilde{\Delta}_{q+1} \circ \partial)\big) \circ \rho_\ell \\
&\quad + \sum_{j=0}^{j_0-1} (-1)^{j+(j_0-j-1)N}\big(\tilde{\iota}_M^{\otimes j} \otimes (\tilde{\Delta}_N \circ \partial) \otimes \tilde{\iota}_M^{\otimes(q+1-j)}\big) \circ (\Delta'_{q+1} \otimes \mathrm{id}_M) \circ \rho_\ell \\
&\quad - \sum_{j=1}^{j_0-1} (-1)^{j+(j_0-j-1)N+q+1}\big(\tilde{\iota}_M^{\otimes j} \otimes (\tilde{\Delta}_N \circ \partial) \otimes \tilde{\iota}_M^{\otimes(q+1-j)}\big) \circ (\mathrm{id}_M \otimes \Delta'_{q+1}) \circ \rho_r \\
&\quad - (-1)^{(j_0-1)N+q+1}\big((\tilde{\Delta}_N \circ \partial) \otimes \tilde{\Delta}_{q+1}\big) \circ \rho_r,
\end{aligned} \tag{46}$$

where we have used that ∂ is a bicoderivation, i.e., $\Delta_C \circ \partial = (\partial \otimes \mathrm{id}_C) \circ \rho_r + (\mathrm{id}_C \otimes \partial) \circ \rho_\ell$, and (5) for $p = q+1$, which holds due to the assumption $\mathcal{A}_q$. It is clear that the first and last terms cancel each other. Consider now the composition of the remaining terms in (46) with $\Pi = \tilde{\pi}_{\bar{a}}^{\tilde{M}}$. By the inductive hypothesis, we see that

$$\begin{aligned}
&\Pi \circ \big(\tilde{\Delta}_N \otimes (\tilde{\Delta}_{q+1} \circ \partial)\big) \circ \rho_\ell \\
&\quad = \Pi \circ \big(\tilde{\Delta}_N \otimes (\tilde{\Delta}_q \otimes \tilde{\iota}_M) \circ \rho_\ell\big) \circ \rho_\ell - (-1)^q \Pi \circ \big(\tilde{\Delta}_N \otimes (\tilde{\iota}_M \otimes \tilde{\Delta}_q) \circ \rho_r\big) \circ \rho_\ell \\
&\quad = \Pi \circ \Big(\big((\tilde{\Delta}_N \otimes \tilde{\Delta}_q) \circ \Delta_C\big) \otimes \tilde{\iota}_M\Big) \circ \rho_\ell - (-1)^q \Pi \circ \Big(\big((\tilde{\Delta}_N \otimes \tilde{\iota}_M) \circ \rho_\ell\big) \otimes \tilde{\Delta}_q\Big) \circ \rho_r \\
&\quad = \Pi \circ \Big(\big((\tilde{\Delta}_N \otimes \tilde{\Delta}_q) \circ \Delta_C\big) \otimes \tilde{\iota}_M\Big) \circ \rho_\ell - (-1)^{q+N} \Pi \circ \Big(\big((\tilde{\iota}_M \otimes \tilde{\Delta}_N) \circ \rho_r\big) \otimes \tilde{\Delta}_q\Big) \circ \rho_r \\
&\quad = \Pi \circ \bigg(\Big(\big((\tilde{\Delta}_N \otimes \tilde{\Delta}_q) \circ \Delta_C\big) \otimes \tilde{\iota}_M\Big) \circ \rho_\ell - (-1)^{q+N}\Big(\tilde{\iota}_M \otimes \big((\tilde{\Delta}_N \otimes \tilde{\Delta}_q) \circ \Delta_C\big)\Big) \circ \rho_r\bigg),
\end{aligned} \tag{47}$$

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where we have used that M is a C-bicomodule in the second and last equality, and the identity $(\tilde{\Delta}_N \otimes \tilde{\iota}_M) \circ \rho_\ell = (-1)^N (\tilde{\iota}_M \otimes \tilde{\Delta}_N) \circ \rho_r$ in the penultimate equality, which follows from (5) for $p = N$ due to Fact 6.9. Hence, the composition of the left member of (44) with (45) and Π gives

$$\begin{aligned}
&(-1)^{(j_0-1)N} \Pi \circ \Big(\big((\tilde{\Delta}_N \otimes \tilde{\Delta}_q) \circ \Delta_C \big) \otimes \tilde{\iota}_M \Big) \circ \rho_\ell \\
&\quad - (-1)^{j_0 N + q} \Pi \circ \Big(\tilde{\iota}_M \otimes \big((\tilde{\Delta}_N \otimes \tilde{\Delta}_q) \circ \Delta_C \big) \Big) \circ \rho_r \\
&\quad + \sum_{j=0}^{j_0-1} (-1)^{j+(j_0-j-1)N} \Pi \circ \big(\tilde{\iota}_M^{\otimes j} \otimes (\tilde{\Delta}_N \circ \partial) \otimes \tilde{\iota}_M^{\otimes(q+1-j)} \big) \circ (\Delta'_{q+1} \otimes \mathrm{id}_M) \circ \rho_\ell \\
&\quad - \sum_{j=1}^{j_0-1} (-1)^{j+(j_0-j-1)N+q+1} \Pi \circ \big(\tilde{\iota}_M^{\otimes j} \otimes (\tilde{\Delta}_N \circ \partial) \otimes \tilde{\iota}_M^{\otimes(q+1-j)} \big) \circ (\mathrm{id}_M \otimes \Delta'_{q+1}) \circ \rho_r .
\end{aligned} \tag{48}$$

Fact 6.15 tells us that the first and third terms add up to precisely give the composition of the first term of the second member of (44) with (45) and Π, whereas the second and the fourth terms give precisely the composition of the second term of the second member of (44) with (45) and Π. The lemma is thus proved. $\square$

FACT 6.17. *Suppose that the assumption $\mathcal{A}_p$ holds for some $p \in \{1, \dots, N-1\}$. Then, for every $p' \in \{0, \dots, N-p-1\}$ we have*

$$\begin{aligned}
&\sum_{j=0}^{p+p'} (-1)^{j(N-p'-1)} \big(\tilde{\iota}_M^{\otimes j} \otimes (\tilde{\Delta}_{N-p'} \circ \partial) \otimes \tilde{\iota}_M^{\otimes(p+p'-j)} \big) \circ \Delta'_{p+p'+1} \\
&\quad = (\tilde{\Delta}_{N-p'-1} \otimes \tilde{\Delta}_{p+p'+1}) \circ \Delta_C - (\tilde{\Delta}_{p+p'+1} \otimes \tilde{\Delta}_{N-p'-1}) \circ \Delta_C \\
&\qquad + \sum_{j=0}^{p+p'+1} (-1)^{j(N-p')} \big(\tilde{\iota}_M^{\otimes j} \otimes (\tilde{\Delta}_{N-p'-1} \circ \partial) \otimes \tilde{\iota}_M^{\otimes(p+p'+1-j)} \big) \circ \Delta'_{p+p'+2}.
\end{aligned} \tag{49}$$

Proof. Since $N - p' - 1 \geq p$, applying (44) for $q = N - p' - 1$ in the left member of (49), we obtain that the latter is given by

$$\begin{aligned}
&\sum_{j=0}^{p+p'} (-1)^{j(N-p'-1)} (\tilde{\iota}_M^{\otimes j} \otimes \tilde{\Delta}_{N-p'-1} \otimes \tilde{\iota}_M^{\otimes(p+p'+1-j)}) \circ (\mathrm{id}_M^{\otimes j} \otimes \rho_\ell \otimes \mathrm{id}_M^{\otimes(p+p'-j)}) \circ \Delta'_{p+p'+1} \\
&\quad - \sum_{j'=0}^{p+p'} (-1)^{(j'+1)(N-p'-1)} (\tilde{\iota}_M^{\otimes j'+1} \otimes \tilde{\Delta}_{N-p'-1} \otimes \tilde{\iota}_M^{\otimes(p+p'-j')}) \circ (\mathrm{id}_M^{\otimes j'} \otimes \rho_r \otimes \mathrm{id}_M^{\otimes(p+p'-j')}) \circ \Delta'_{p+p'+1}.
\end{aligned} \tag{50}$$

Then,

(i) using identity (6) for $q = p + p' + 1$ in the term with $j = 0$ of the first sum of (50), we obtain the first term of the right member of (49) and the term with $j = 0$ in the last sum of the second member of (49);
(ii) using (7) for $q = p + p' + 1$ in the terms with $j = 1, \dots, p + p'$ of the first sum of (50), we obtain an expression consisting of a summand that precisely cancels the term with $j' = j - 1$ in the second sum of (50), and the j-th summand in the last sum of the second member of (49);

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(iii) using identity (8) for $q = p + p' + 1$ in the term with $j = p + p'$ of the second sum of (50), we obtain the second term of the right member of (49) and the term with $j = p + p' + 1$ in the last sum of the second member of (49).

The statement is thus proved. □

FACT 6.18. *Suppose that the assumption $\mathcal{A}_p$ holds for some $p \in \{1, \dots, N-1\}$. Then,*

$$(\tilde{\Delta}_p \otimes \tilde{\Delta}_N) \circ \Delta_C - (\tilde{\Delta}_N \otimes \tilde{\Delta}_p) \circ \Delta_C = \sum_{j=0}^{p} (-1)^{j(N-1)} \big(\tilde{\iota}_M^{\otimes j} \otimes (\tilde{\Delta}_N \circ \partial) \otimes \tilde{\iota}_M^{\otimes (p-j)}\big) \circ \Delta'_{p+1}. \tag{51}$$

Proof. A repeated use of identity (49) for $p' \in \{0, \dots, N-p-2\}$ tells us that

$$\begin{aligned}\sum_{j=0}^{p} &(-1)^{j(N-1)} \big(\tilde{\iota}_M^{\otimes j} \otimes (\tilde{\Delta}_N \circ \partial) \otimes \tilde{\iota}_M^{\otimes (p-j)}\big) \circ \Delta'_{p+1} \\ &= \sum_{p'=0}^{N-p-2} \big((\tilde{\Delta}_{N-p'-1} \otimes \tilde{\Delta}_{p+p'+1}) \circ \Delta_C - (\tilde{\Delta}_{p+p'+1} \otimes \tilde{\Delta}_{N-p'-1}) \circ \Delta_C\big) \\ &\quad + \sum_{j=0}^{N-1} (-1)^{jp} \big(\tilde{\iota}_M^{\otimes j} \otimes (\tilde{\Delta}_{p+1} \circ \partial) \otimes \tilde{\iota}_M^{\otimes (N-1-j)}\big) \circ \Delta'_N.\end{aligned} \tag{52}$$

Moreover, the first sum in the second member of (52) vanishes. Indeed, the first (resp., second) summand of the p'-th term of the second sum in (52) with $p' \in \{0, \dots, \lfloor (N-p-2)/2 \rfloor\}$ cancels with the second (resp., first) summand of the $(N-p-p'-2)$-th term of the second sum in (52), where $\lfloor\ \rfloor$ is the (integer) floor function. Since $p \geq 1$, another application of (49) for $p' = N-p-1$ tells us that second sum in the second member of (52) gives the first member of (51). The statement follows. □

LEMMA 6.19. *Assume the hypothesis $\mathcal{A}_q$ for some $q \in \{1, \dots, N-1\}$. Then,*

$$(\mathrm{id}_{\tilde{M}}^{\otimes(q-1)} \otimes \tilde{\rho}_r) \circ \tilde{\Delta}_q = (\tilde{\Delta}_q \otimes \hat{\iota}_C) \circ \Delta_C + (-1)^{q+1} (\tilde{\iota}_M^{\otimes q} \otimes \hat{\partial}) \circ \Delta'_{q+1} \tag{53}$$

holds.

Proof. We will prove that the identity

$$\bar{\pi}_{\bar{\imath}} \circ (\mathrm{id}_{\tilde{M}}^{\otimes(q-1)} \otimes \tilde{\rho}_r) \circ \tilde{\Delta}_q = \bar{\pi}_{\bar{\imath}} \circ (\tilde{\Delta}_q \otimes \mathrm{id}_C) \circ \Delta_C + (-1)^{q+1} \bar{\pi}_{\bar{\imath}} \circ (\tilde{\iota}_M^{\otimes q} \otimes \hat{\partial}) \circ \Delta'_{q+1} \tag{54}$$

is verified for all $\bar{\imath} \in \mathbb{N}^{q+1}$, where $\bar{\pi}_{\bar{\imath}} = \tilde{\pi}^{\tilde{M}}_{(i_1,\dots,i_q)} \otimes \tilde{\pi}^{\hat{C}}_{i_{q+1}}$. It is clear that (54) holds if $i_{q+1} = 0$, since $\epsilon_C \circ \partial$ vanishes (see Lemma 6.7). It suffices to prove the statement for $i_{q+1} > 1$ even. This is tantamount to show that the composition of (53) with $\mathrm{id}_{\tilde{M}}^{\otimes q} \otimes \hat{\Delta}_N$ holds, by (23) and (24). It is clear that the composition of the first member of (53) with $\mathrm{id}_{\tilde{M}}^{\otimes q} \otimes \hat{\Delta}_N$ gives precisely the left member of (42) for $j_0 = p = q$ multiplied by $(-1)^N$, since $(\hat{\Delta}_N \otimes \mathrm{id}_{\tilde{M}}) \circ \tilde{\rho}_\ell = (-1)^N (\mathrm{id}_{\tilde{M}} \otimes \hat{\Delta}_N) \circ \tilde{\rho}_r$ (which follows from the definitions).

By (42), we conclude that

$$\Big(\mathrm{id}_{\tilde{M}}^{\otimes(q-1)} \otimes \big((\mathrm{id}_{\tilde{M}} \otimes \hat{\Delta}_N) \circ \tilde{\rho}_r\big)\Big) \circ \tilde{\Delta}_q = (-1)^{qN}(\tilde{\Delta}_N \otimes \tilde{\Delta}_q) \circ \Delta_C \\ + \sum_{j=0}^{q-1} (-1)^{j+(q-j)N} \big(\tilde{\imath}_M^{\otimes j} \otimes (\tilde{\Delta}_N \circ \partial) \otimes \tilde{\imath}_M^{\otimes(q-j)}\big) \circ \Delta'_{q+1}. \tag{55}$$

By applying (51) for $p = q$ to reexpress the first term of the second member of (55), we obtain precisely the composition of (53) with $\mathrm{id}_{\tilde{M}}^{\otimes q} \otimes \hat{\Delta}_N$, where we have used $\hat{\Delta}_N \circ \hat{\imath}_C = \tilde{\Delta}_N$. The lemma is thus proved. □

LEMMA 6.20. *Assume that the hypothesis $\mathcal{A}_0$ holds. Then,*

$$\partial \circ \Delta'_1 = (\Delta'_0 \otimes \mathrm{id}_C) \circ \Delta_C - (\mathrm{id}_C \otimes \Delta'_0) \circ \Delta_C, \\ \tilde{\Delta}_1 \circ \partial = (\tilde{\Delta}_0 \otimes \tilde{\imath}_M) \circ \rho_\ell - (\tilde{\imath}_M \otimes \tilde{\Delta}_0) \circ \rho_r \tag{56}$$

hold.

Proof. It suffices to prove that the composition of the first (resp., second) identity in (56) with $\tilde{\pi}_i^C$ (resp., $\tilde{\pi}_i^M$) holds for every $i \in 2\mathbb{N}_0$ (resp., $i \in 2\mathbb{N}_0 + 1$). The composition of the first equation in (56) with $\tilde{\pi}_0^C$ is clearly verified, by definition of ∂, Δ'_0 and Δ'_1. Moreover, the composition of the second equation in (56) with $\tilde{\pi}_1^M$ is precisely (15) for $j = 1$. Assume now that the composition of both identities in (56) with $\tilde{\pi}_i^C$ (resp., $\tilde{\pi}_i^M$) holds for every $i \in 2\mathbb{N}_0$ (resp., $i \in 2\mathbb{N}_0 + 1$) such that $i < \ell_0$, for some $\ell_0 \in \mathbb{N}$. We will prove that the composition of both identities in (56) with $\tilde{\pi}_i^C$ (resp., $\tilde{\pi}_i^M$) holds for $i = \ell_0$.

Let us deal with the second identity in (56). We remark that all of the arguments in the proof of Lemma 6.16 up to (and including) (48) are also valid for $q = 0$. We will also follow the notation given there. In this case, the last sum in (48) is trivial, and we only use the expression for the first two terms of (48) given by the third member of (47), which tells us that the composition of the left member of (44) for $q = 0$ with (45) and Π gives

$$\Pi \circ \Big(\big((\tilde{\Delta}_N \otimes \tilde{\Delta}_0) \circ \Delta_C\big) \otimes \tilde{\imath}_M\Big) \circ \rho_\ell - \Pi \circ \Big(\big((\tilde{\Delta}_N \otimes \tilde{\imath}_M) \circ \rho_\ell\big) \otimes \tilde{\Delta}_0\Big) \circ \rho_r \\ + \Pi \circ \big((\tilde{\Delta}_N \circ \partial \circ \Delta'_1) \otimes \tilde{\imath}_M\big) \circ \rho_\ell. \tag{57}$$

By (21), it is clear that second term in (57) is precisely the composition of the second term of the second member of (44) for $q = 0$ with (45) and Π. Moreover, using the fact that M is a left C-comodule together with the inductive assumption on the first equation of (56), we conclude that the sum of the first and third terms of (57) is equal to the composition of the first term of the second member of (44) for $q = 0$ with (45) and Π.

Finally, let us prove the first identity in (56). We first note that we can consider $\ell_0 > 1$. Moreover, proving that the composition of the first identity in (56) with $\tilde{\pi}_{\ell_0}^C$ holds is tantamount to showing that the composition of the first equation in (56) with $\tilde{\pi}_{\bar{\imath}}^{\tilde{M}} \circ \tilde{\Delta}_N$ holds for all $\bar{\imath} \in (2\mathbb{N}_0 + 1)^N$ such that $|\bar{\imath}| = \ell_0$, by definition of $\tilde{\Delta}_N$. This latter composition is precisely the composition of the identity (51) for $p = 0$ with $\tilde{\pi}_{\bar{\imath}}^{\tilde{M}}$. To prove it holds, we remark first that all of the arguments in the proof of Fact 6.18 up to (and including) the vanishing of the first sum in the second member of (52) are also valid for $p = 0$. Indeed, by the inductive assumption on the second equation of (56) and Lemma 6.19, we see that the proof in Fact 6.17 also holds for the composition of (49) with $\tilde{\pi}_{\bar{\imath}}^{\tilde{M}}$ if $p = 0$. Hence, another application of (49) for $p = 0$ and $p' = N - 1$ gives exactly the composition of the identity (51) for $p = 0$ with $\tilde{\pi}_{\bar{\imath}}^{\tilde{M}}$, as was to be shown. The lemma is thus proved. □

LEMMA 6.21. *Assume that the hypothesis $\mathcal{A}_q$ holds for some $q \in \{1, \dots, N-1\}$. Then, $\tilde{\Delta}_q : C \to \tilde{M}^{\otimes q}$ factors through the canonical inclusion $\tilde{\imath}_M^{\otimes q}$, i.e., there exists (a unique map) $\Delta'_q : C \to M^{\otimes q}$ such that $\tilde{\imath}_M^{\otimes q} \circ \Delta'_q = \tilde{\Delta}_q$.*

Proof. It suffices to prove that, given any $j \in \{0, \dots, q\}$ and any $\bar{\imath}' \in (2\mathbb{N}_0 + 1)^q$, then

$$\operatorname{Im}(\tilde{\pi}_{\bar{\imath}'}^{\tilde{M}} \circ \tilde{\Delta}_q) \subseteq M_{i'_1} \otimes \cdots \otimes M_{i'_j} \otimes \tilde{M}_{i'_{(j+1)}} \otimes \cdots \otimes \tilde{M}_{i'_q}, \tag{58}$$

where for the case $j = 0$ the right member of (58) is $\tilde{M}_{i'_1} \otimes \cdots \otimes \tilde{M}_{i'_q}$, whereas, for the case $j = q$ it gives precisely $M_{i'_1} \otimes \cdots \otimes M_{i'_q}$, which is what we want to prove. In order to do it, we proceed by induction on the index j, the case $j = 0$ being obviously verified. Assume that (58) holds for all $j \in \{0, \dots, j_0 - 1\}$, with $j_0 \in \mathbb{N}_{\leq q}$. We will prove it for $j = j_0$. Note that the inductive assumption together with (35) for $\bar{\imath} = (i'_1 - 1, 1, i'_2, \dots, i'_q)$ if $j_0 = 1$, and (36) for $j = j_0 - 1$ and $\bar{\imath} = (i'_1, \dots, i'_{j_0-1}, i'_{j_0} - 1, 1, i'_{j_0+1}, \dots, i'_q)$ if $j_0 > 1$ (which hold due to Lemma 6.14) tell us precisely that

$$\operatorname{Im}(\tilde{\pi}_{\bar{\imath}}^{\tilde{M}} \circ \tilde{\Delta}_q) \subseteq M_{i'_1} \otimes \cdots \otimes M_{i'_{j_0-1}} \otimes (C_{i'_{j_0}-1} \otimes V) \otimes \tilde{M}_{i'_{j_0+1}} \otimes \cdots \otimes \tilde{M}_{i'_q}, \tag{59}$$

whereas, the inductive assumption together with (36) for $j = j_0$ and the vector $\bar{\imath} = (i'_1, \dots, i'_{j_0-1}, 1, i'_{j_0} - 1, i'_{j_0+1}, \dots, i'_q)$ if $j_0 < q$ (which holds due to Lemma 6.14), and (54) for $\bar{\imath} = (i'_1, \dots, i'_{q-1}, 1, i'_q - 1)$ if $j_0 = q$ (which holds due to Lemma 6.19) imply that

$$\operatorname{Im}(\tilde{\pi}_{\bar{\imath}}^{\tilde{M}} \circ \tilde{\Delta}_q) \subseteq M_{i'_1} \otimes \cdots \otimes M_{i'_{j_0-1}} \otimes (V \otimes C_{i'_{j_0}-1}) \otimes \tilde{M}_{i'_{j_0+1}} \otimes \cdots \otimes \tilde{M}_{i'_q}. \tag{60}$$

Indeed, (60) holds for $j_0 = q$ due to the mentioned equation (54), and the other cases follow from an inductive argument on j_0 and $|\bar{\imath}|$ using (9), (35) and (36). Since $M_{i'} = (C_{i'-1} \otimes V) \cap (V \otimes C_{i'-1})$ for any $i' \in 2\mathbb{N}_0 + 1$ (by (10)), then (58) for $j = j_0$ holds. The lemma is thus proved. □

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