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ON A FUNCTIONAL EQUATION

BY

NADHLA A. AL-SALAM AND WALEED A. AL-SALAM

1. **Introduction.** Let P stand for a polynomial set (p.s.), i.e., a sequence $\{P_0(x), P_1(x), P_2(x), \ldots\}$ such that for each n $P_n(x)$ is a polynomial in x of exact degree n and $P_0(x) \neq 0$. We refer to $P_n(x)$ as the nth component of P.

In this note we consider the set Π of all polynomial sets in which multiplication is defined in the following sense: if $P \in \Pi$, $Q \in \Pi$ such that $P_n(x) = \sum_{k=0}^{n} p(n, k)x^k$ then PQ is defined as the polynomial set whose nth component is given by $P_n(Q) = \sum_{k=0}^{n} p(n, k)Q_k(x)$. An interesting example of such umbral composition of polynomial sets is furnished by the Hermite polynomials defined by

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2}.$$

For this set it is well known [1, p. 246] that

$$H_n(H) = 5^{\frac{1}{2}n}H_n\left(\frac{2}{\sqrt{5}}x\right) \qquad (n = 0, 1, 2, \ldots)$$

In this note we examine the functional equations

(1.1) $$f_n(f(x)) = b^n f_n(cx) \qquad (n = 0, 1, 2, \ldots)$$

where b and c are real constants and find all solutions of (1.1) in the set Π. Clearly the Hermite case arises when $b = \sqrt{5}$ and $c = 2/\sqrt{5}$. We must assume that $b \neq 0$ and $c \neq 0$.

In the following we shall always take $q = bc$.

2. **Solution of (1.1).** Put

(2.1) $$f_n(x) = \sum_{k=0}^{n} p(n, k)x^k \qquad p(n, n) \neq 0.$$

Substituting in (1.1) and equating coefficients of powers of x we get that

(2.2) $$\sum_{k=j}^{n} p(n, k)p(k, j) = b^n c^j p(n, j) \qquad (j = 0, 1, \ldots, n).$$

The case $j = n$ leads to $p(n, n) = (bc)^n = q^n$. Next we move the $k = j$ and $k = n$

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terms to the right hand side to get after some rearrangements

$$(2.3)\quad p(n, n-j)q^{n-j}(b^j-1-q^j)=\sum_{k=0}^{j-2} p(n, n-j+k+1)p(n-j+k+1, n-j).$$

It is easy to see that $b=1$ leads to $f_n(x)=c^n x^n$. Similarly if $c=1$ then $f_n(x)=b^n x^n$. Thus we may assume that $b\neq 1$ and $c\neq 1$.

Next if there is no positive integer s so that $b^s-1-q^s=0$ then putting $j=1$ in (2.3) we get that $p(n, n-1)=0$ and, by induction, that $p(n, r)=0$ $r\neq n$. Hence in this case $f_n(x)=q^n x^n$.

Finally assume that there is an integer $s\geq 1$ so that $b^s-1-q^s=0$. It is easy to see that in this case $b^r-1-q^r\neq 0$ for all non-negative integers $r\neq s$.

Now (2.3) implies that $p(n, n-1)=p(n, n-2)=\cdots=p(n, n-s+1)=0$. If we further assume that $p(n, n-s)=0$ then all $p(n, n-j)=0$ for $j=1, 2, \ldots, n$. Thus we have again $f_n(x)=q^n x^n$.

On the other hand if we put $p(n, n-s)=k_n$ for all n then we make the assertion that for $r=1, 2, 3, \ldots$

$$(2.4)\qquad p(n, n-rs)=\frac{k_n k_{n-s}\cdots k_{n-rs+s}}{r!}q^{\frac{1}{2}r(r-1)s-(r-1)n}$$

The proof of this assertion can be carried out by induction on r.

To summarize we state:

THEOREM. *The solutions of the functional equations*

$$f_n(f(x))=b^n f_n(cx)\qquad (n=0, 1, 2, \ldots)$$

for $f\in\Pi$ and b and c real are given by

(i) *In case either $b=1$ or $c=1$ or in case $b^s-1-(bc)^s\neq 0$ for all integer $s: f_n(x)=(bc)^n x^n$.*

(ii) *In case $b^s-1-(bc)^s=0$, putting $bc=q$, we have*

$$(2.5)\qquad f_n(x)=\sum_{r=0}^{(n/s)}\frac{k_n k_{n-s}\cdots k_{n-rs+s}}{r!}q^{\frac{1}{2}(r-1)(rs-2n)}x^{n-rs}$$

for an arbitrary sequence $\{k_n\}$.

In (2.5) the coefficient of x^n is to be q^n.

In case $q=1$ we put $k_{sn+m}=\alpha_n^{(m)}\neq 0$ $(m=0, 1, \ldots, s-1)$. Then we can write

$$f_{sn+m}(x)=\sum_{r=0}^{n}\frac{\alpha_n^{(m)}\alpha_{n-1}^{(m)}\cdots\alpha_{n-r+1}^{(m)}}{r!}x^{sn+m-rs}$$

which yields the generating function

$$(2.6)\qquad \sum_{n=0}^{\infty}\frac{f_{sn+m}(x)}{\alpha_0^{(m)}\alpha_1^{(m)}\cdots\alpha_n^{(m)}}t^n=x^m e^t\psi_m(x^s t)\qquad (m=0, 1, \ldots, s-1)$$

where $\psi_m(u)$ is a formal power series with non-zero coefficients. Replacing t by t^s and multiplying (2.6) by t^m for $m=0,1,2,\ldots,s-1$ and adding the resulting equations we get

$$\sum_{n=0}^{\infty} \frac{f_n(x)}{c_0c_1\cdots c_n} t^n = e^{t^s}\Phi(xt) \tag{2.7}$$

where $c_n=\alpha_k^{(m)}$ if $n=sk+m$. Note that these polynomials are known as Brenke polynomials.

SPECIAL CASES.

(i) If $b=\sqrt{5}$, $c=2/\sqrt{5}$ so that $q=2$ and $s=2$ and if we put further $k_n=-2^{n-2}n(n-1)$ we get $f_n(x)=H_n(x)$, the Hermite polynomials.

(ii) In case $b-1-bc=0$ so that $s=1$ we have if we also assume that $k_n=c_n=n(1-q^n)$ we get

$$f_n(x)=\sum_{r=0}^{n}\binom{n}{r}\begin{bmatrix} n \\ r \end{bmatrix}(q)_r q^{\frac{1}{2}r(r-1)-(r-1)n}x^{n-r}$$

where $\begin{bmatrix} n \\ r \end{bmatrix}$ is the q-binomial coefficient defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}=\prod_{j=1}^{r}\frac{1-q^{n-j+1}}{(1-q^j)}.$$

REFERENCE

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA, CANADA
T6G 2G1