COMPOSITIO MATHEMATICA

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Compositio Math. **150** (2014), 942–978.

doi:10.1112/S0010437X13007641

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

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Abstract

We introduce a new class of autoequivalences that act on the derived categories of certain vector bundles over Grassmannians. These autoequivalences arise from Grassmannian flops: they generalize Seidel–Thomas spherical twists, which can be seen as arising from standard flops. We first give a simple algebraic construction, which is well suited to explicit computations. We then give a geometric construction using spherical functors which we prove is equivalent.

1. Introduction

Derived equivalences corresponding to flops were first explored by Bondal and Orlov [BO95]. They exhibited an equivalence of bounded derived categories of coherent sheaves corresponding to the standard flop of a projective space $\mathbb{P}^{d-1}$ in a smooth algebraic variety with normal bundle $\mathcal{N} \simeq \mathcal{O}(-1)^{\oplus d}$ (see [BO95, Theorem 3.6]). More generally, it is conjectured [Kaw02, Conjecture 5.1] that for any flop between smooth projective varieties there exists a derived equivalence. This follows for 3-folds by work of Bridgeland [Bri02], but is still an open question in higher dimensions.

Examples of flops, including the standard flop, may be obtained by variation of GIT, and in this case there is a particular approach to constructing derived equivalences. Suppose X_+ and X_- are a pair of varieties related by a flop, and that both are possible GIT quotients of a larger space M by the action of a group G. Then X_+ and X_- are open substacks of the Artin stack $\mathfrak{X} = [M/G]$, and there are restriction functors from $D^b(\mathfrak{X})$ to both $D^b(X_+)$ and $D^b(X_-)$. So one way to construct an equivalence between $D^b(X_+)$ and $D^b(X_-)$ is to find a subcategory inside $D^b(\mathfrak{X})$ which is equivalent to both of them. We call such a subcategory a 'window'.

This technique was inspired by the physical analysis carried out by Herbst, Hori and Page in [HHP08], and was introduced into the mathematics literature by the second author in [Seg11]. Both of these papers were concerned with Landau–Ginzburg models, where the derived category is modified by a superpotential; however, the technique is still interesting when applied to ordinary derived categories.

In this paper we study a particular class of examples, which are local models of 'Grassmannian flops'. For us, X_+ is the total space of the vector bundle

$$\mathrm{Hom}(V, S) \longrightarrow \mathbb{Gr}(r, V)$$

where S is the tautological subspace bundle on the Grassmannian $\mathbb{Gr}(r, V)$ of r-dimensional subspaces of a vector space V, where $0 < r < \dim V$. This can be flopped to a second space X_-,

Received 8 January 2013, accepted in final form 22 August 2013, published online 7 April 2014.
2010 Mathematics Subject Classification 14F05, 18E30 (primary), 14M15 (secondary).
Keywords: derived categories, Grassmannians, derived equivalences, spherical functors.

which is the total space of a vector bundle over the dual Grassmannian $\mathbb{Gr}(V, r)$. (When V is two-dimensional, and $r = 1$, this is the standard Atiyah flop.) This flop arises from a GIT problem, and we show that it is possible to find a window. In fact we find a whole set of windows, indexed by $\mathbb{Z}$, and hence show the following theorem.

Theorem A (Theorem 3.7). *For $k \in \mathbb{Z}$ there exist equivalences*

$$\psi_k : D^b(X_+) \xrightarrow{\sim} D^b(X_-).$$

The fact that there are many different choices of windows is not a surprise, as it was present in the original analysis of Herbst, Hori and Page. It has an important consequence: if we combine equivalences corresponding to different windows, we produce *autoequivalences* of $D^b(X_+)$.

Definition (Definition 3.9). We define *window-shift autoequivalences* $\omega_{k,l}$ by

$$\omega_{k,l} := \psi_k^{-1}\psi_l : D^b(X_+) \xrightarrow{\sim} D^b(X_+).$$

Most of this paper is devoted to studying these autoequivalences, and in particular to proving that they are equivalently described by a geometric construction discovered by the first author in [Don11]. In the case of a standard 3-fold flop this geometric construction is well known: the skyscraper sheaf along the flopping $\mathbb{P}^1$ is a spherical object, and we can get a derived autoequivalence by performing a Seidel–Thomas spherical twist [ST01]. In the Grassmannian examples, the construction gives something more complicated: we will explain that it is a twist autoequivalence [Ann07] associated to a spherical functor which involves a push-down by a resolution of singularities. Generically it acts as a family spherical twist [Hor05], but acts more elaborately on a certain closed locus: in the case $r = 2$, this interesting locus is the zero section $\mathbb{Gr}(2, V)$ of the bundle X_+ (see [Don11] for further discussion of this case). We prove the following theorem relating window-shift autoequivalences to twists.

Theorem B (Theorem 3.12). *There exists a natural isomorphism*

$$\omega_{0,1} \simeq T_F$$

where T_F is a twist of a spherical functor $F : D^b(Y_+) \to D^b(X_+)$, defined in § 3.2.

The space Y_+, which is the source of the spherical functor, is the exact analogue of X_+ but with the subspace dimension r replaced by $r - 1$. As such, we also have a set of window-shift autoequivalences acting on $D^b(Y_+)$. Moreover, the spherical functor F, as well as inducing a twist autoequivalence on the target category $D^b(X_+)$, also induces a *cotwist* autoequivalence on its source category $D^b(Y_+)$. This cotwist may also be related to a suitable window shift, as follows.

Theorem C (Theorem 3.13). *There exists a natural isomorphism*

$$\omega_{-1,0}^{Y_+}[\sigma] \simeq C_F^{-1}$$

where C_F is the cotwist of F, acting on $D^b(Y_+)$, and σ is a suitable integer shift defined in § 3.2.

Hence we have an example of a spherical functor F where both twist and cotwist are non-trivial autoequivalences (in the sense that they are not generated by shifts and twists by line bundles). To our knowledge, this is the first example where such behaviour has been studied: see [Add11] for a review.

Recent work of Halpern-Leistner [Hal12] and Ballard, Favero and Katzarkov [BFK12], also using ideas from [HHP08, Seg11], develops a general theory of derived equivalences corresponding to certain variations of GIT. These equivalences are controlled by the geometry of the Hesselink/Kirwan–Ness stratification on the respective unstable loci of the GIT quotients. Our spherical functor F can readily be defined in terms of this geometry, so it is natural to ask whether window-shift autoequivalences occurring in the framework of [BFK12, Hal12] can also be related to twists of suitable spherical functors. Indeed [HS13] relates window shifts for a variation of GIT quotient involving a single Hesselink stratum to family spherical twists. We hope that more general theorems comparing window shifts and twists will be the subject of future work.

Finally, we say a few words about the possible physical interpretation of our results. The physics in [HHP08] concerns B-branes in gauged linear σ-models. The input data for such a model consists of a vector space M with an action of a group G; then by standard prescriptions one can build a supersymmetric gauge theory in two dimensions. The theory has a complex parameter t, called the Fayet–Iliopoulos parameter, and in certain 'large-radius' limits this gauge theory reduces to a non-linear σ-model with target space given by a GIT quotient $M/\!\!/G$: different quotients appear at different limits. The parameter t becomes identified, in the limit, with the (complexified) Kähler class of the target space, so the space in which t lives is called the stringy Kähler moduli space.

The B-branes in the theory form a category, which in the limit is the derived category of $M/\!\!/G$. Furthermore, when $G \subset \mathrm{SL}(M)$ this category is actually independent of t, so all the GIT quotients are derived equivalent. However, to produce a derived equivalence one must vary t from one large-radius limit point to a different one, and in between the description of the B-branes as the derived category of a space breaks down. Herbst, Hori and Page instead study the B-branes at a different kind of limit, the 'Coloumb phase' of the theory, and in doing so discover 'grade-restriction rules', which we choose to call 'windows'.

The Coulomb phase description arises when t is near certain singularities in the stringy Kähler moduli space. Because of these singularities, when we move from one large-radius limit to another there are many homotopy classes of paths that we can choose to move along, which is why there are many different choices of windows with different corresponding equivalences. In this picture, we see our autoequivalences as coming from monodromy of B-branes as t moves along loops around the singularities.

Herbst, Hori and Page restrict to the case where G is a torus, whereas in our class of examples we consider the non-abelian gauge group $U(r)$. Gauged linear σ-models with non-abelian gauge groups have certainly been studied [HT07], so we hope that our calculations of brane monodromy in these theories will be of interest to some physicists.

The plan of this paper is as follows. Section 2 is intended to give a readable introduction to our methods, without the morass of Schur functors that arises in the general case. We describe the case of the standard flop in some detail, and provide some discussion of the simplest Grassmannian example. In § 3 we give precise descriptions of all the algebraic and geometric constructions, and give the proofs that they are equivalent. In the appendix we prove various technical results that are required. In particular, we make extensive use of some long exact sequences on Grassmannians, and since these are non-standard we give an explicit description of them.

2. Examples and heuristics

Notation. When discussing derived categories, functors are derived unless stated otherwise. Curly braces denote a complex of sheaves understood as an object of a derived category: an underline records the position of the degree 0 term.

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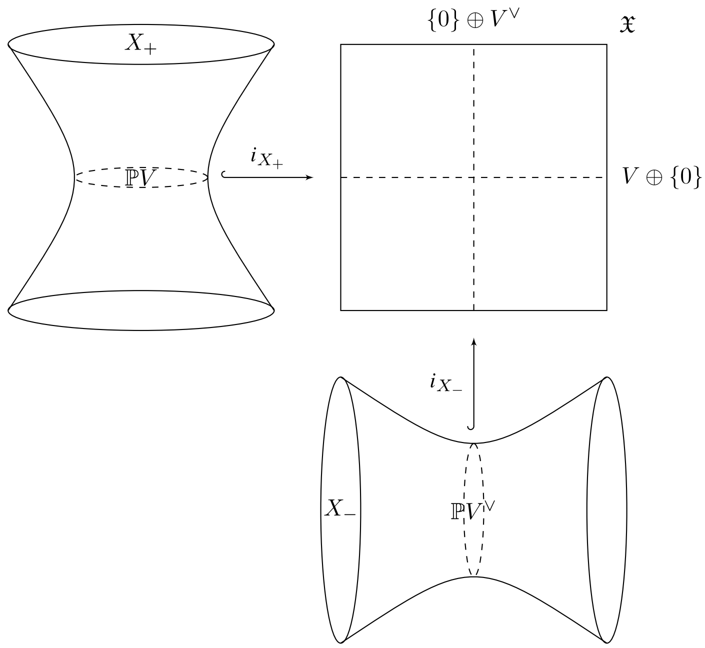

Figure 1. Notation for GIT quotients $X_\pm$, viewed as substacks of the stack $\mathfrak{X}$.

2.1 Windows and window shifts

2.1.1 *The standard 3-fold flop.* We will start by considering the example of the standard 3-fold flop. We let V be a two-dimensional vector space over $\mathbb{C}$, and we let $\mathbb{C}^*$ act on V via the vector space structure. This induces an action

$$\mathbb{C}^* \curvearrowright V \oplus V^\vee.$$

We consider the two possible GIT quotients under this action. For the first one we throw away the subspace $\{0\} \oplus V^\vee$ and get a quotient

$$X_+ = \mathrm{Tot}(\mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}V}).$$

For the second one we throw away $V \oplus \{0\}$ and get

$$X_- = \mathrm{Tot}(\mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}V^\vee}).$$

So both X_+ and X_- are non-compact Calabi–Yau 3-folds, and they are birational (they also happen to be isomorphic). It is well known [BO95, Theorem 3.6] that X_+ and X_- are also derived equivalent.

A particular way of viewing this derived equivalence was introduced by the second author in [Seg11], based on the work of Herbst *et al.* [HHP08]. What we do is view X_+ and X_- as open substacks of the Artin stack

$$\mathfrak{X} = [V \oplus V^\vee / \mathbb{C}^*]$$

and write i_{X_+} and i_{X_-} for the respective inclusions, as illustrated in Figure 1.

On $\mathfrak{X}$ we have a tautological line bundle $\mathcal{O}(1)$. We want to consider the subcategory

$$\mathcal{W}_0 := \langle \mathcal{O}, \mathcal{O}(1) \rangle \subset D^b(\mathfrak{X})$$

which is by definition split-generated by the trivial and tautological line bundles. We call this subcategory a *window*. Its significance is the following proposition.

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Proposition 2.1. *Both functors*

$$i^*_{X_\pm} : \mathcal{W}_0 \longrightarrow D^b(X_\pm)$$

are equivalences.

This proposition is easy to prove: it follows rapidly (see Proposition 3.6) from the statement that the bundle $\mathcal{O} \oplus \mathcal{O}(1)$ is tilting on both X_+ and X_-. This is deduced from Beilinson's theorem [Bei78], which says that $\mathcal{O}$ and $\mathcal{O}(1)$ form a full strong exceptional collection on $\mathbb{P}^1$. Hence we have a derived equivalence

$$\psi_0 : D^b(X_+) \xrightarrow{\sim} D^b(X_-)$$

defined as the composition

$$D^b(X_+) \xrightarrow[(i^*_{X_+})^{-1}]{\sim} \mathcal{W}_0 \xrightarrow[i^*_{X_-}]{\sim} D^b(X_-).$$

We can calculate the effect of this equivalence quite explicitly. Take a sheaf (or complex) $E \in D^b(X_+)$. Resolve E by the bundles $\mathcal{O}$ and $\mathcal{O}(1)$; this determines an extension of E to an object $\mathcal{E} \in \mathcal{W}_0 \subset D^b(\mathfrak{X})$. Now we can restrict $\mathcal{E}$ to get an object in $D^b(X_-)$.

This gets more interesting when we notice that $\mathcal{W}_0$ is not the only window that we could have chosen. Indeed, for any $k \in \mathbb{Z}$ we can define

$$\mathcal{W}_k := \langle \mathcal{O}(k), \mathcal{O}(k+1) \rangle \subset D^b(\mathfrak{X})$$

and Proposition 2.1 will hold for $\mathcal{W}_k$. So we have a whole set of derived equivalences $\{\psi_k\}$, according to which window we choose to pass through, and it turns out they are all distinct. If we combine them, we can produce autoequivalences

$$\omega_{k,l} := \psi_k^{-1}\psi_l : D^b(X_+) \xrightarrow{\sim} D^b(X_+).$$

We call these *window-shift* autoequivalences. Of course they are not independent, rather they obey the following relations:

$$\begin{aligned} &\omega_{m,k} \circ \omega_{k,l} = \omega_{m,l}, \\ &\omega_{k+m,l+m} = (\otimes\, \mathcal{O}(m)) \circ \omega_{k,l} \circ (\otimes\, \mathcal{O}(-m)). \end{aligned} \tag{1}$$

Window shifts can be calculated explicitly, at least in principle. As an example, let us calculate the effect of the window shift $\omega_{-1,0}$ on the two line bundles $\mathcal{O}$ and $\mathcal{O}(1)$. Applying the first functor ψ_0 is easy: these two bundles immediately lift to $\mathcal{W}_0$ so we have

$$\psi_0(\mathcal{O}) = \mathcal{O}, \quad \psi_0(\mathcal{O}(1)) = \mathcal{O}(1),$$

in $D^b(X_-)$. We are adopting a particular sign convention here: since $\mathbb{C}^*$ is acting with weight -1 on $V^\vee$, it seems reasonable to declare that on $\mathbb{P}V^\vee$ it is the $\mathcal{O}(-1)$ line bundle that has global sections, not the $\mathcal{O}(1)$ line bundle. If we were not using this convention then we would have $\psi_0(\mathcal{O}(1)) = \mathcal{O}(-1)$.

To apply the second functor ψ_{-1}^{-1} we have to resolve $\mathcal{O}(1)$ in terms of $\mathcal{O}(-1)$ and $\mathcal{O}$, so that we can move back through the window $\mathcal{W}_{-1}$. On X_- we have an exact sequence given by

$$0 \longrightarrow \mathcal{O}(1) \otimes \det(V) \longrightarrow \mathcal{O} \otimes V \longrightarrow \mathcal{O}(-1) \longrightarrow 0 \tag{2}$$

which is the pull-up of the Euler sequence on $\mathbb{P}V^\vee$. Consequently, after picking a basis for V we have

$$\omega_{-1,0}(\mathcal{O}) = \mathcal{O}, \quad \omega_{-1,0}(\mathcal{O}(1)) = \{\underline{\mathcal{O}^{\oplus 2}} \longrightarrow \mathcal{O}(-1)\}. \tag{3}$$

It is straightforward, but more fiddly, to compute the effect of $\omega_{-1,0}$ on $\mathcal{O}(k)$ for other k.

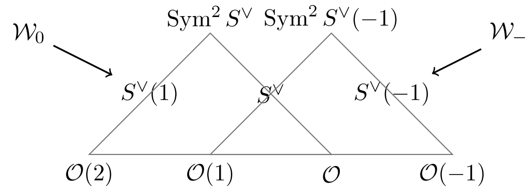

Figure 2. Windows used in calculation of window shift $\omega_{-1,0}$ for Grassmannian example $d = 4$, $r = 2$.

2.1.2 *Grassmannian flops.* The strategy given in § 2.1.1 should lead to derived equivalences, and autoequivalences, in many more examples. In this paper, we will only generalize in the following way. Let V now be a vector space of arbitrary dimension d. Also, let S be another vector space with dimension r, where $r \leqslant d$. We form the Artin stack

$$\mathfrak{X}^{(d,r)} = [\mathrm{Hom}(S, V) \oplus \mathrm{Hom}(V, S)/\mathrm{GL}(S)].$$

We then have two possible GIT quotients given by open substacks $X_{\pm}^{(d,r)}$ of $\mathfrak{X}^{(d,r)}$. It is straightforward to establish (cf. [Tho05, Proposition 4.14]) that one quotient $X_{+}^{(d,r)}$ is the locus where the map from S to V is full rank: it is the total space of a vector bundle over the Grassmannian $\mathbb{Gr}(r, V)$. Similarly $X_{-}^{(d,r)}$ is the total space of a vector bundle over the dual Grassmannian $\mathbb{Gr}(V, r)$. Note that setting $d = 2$ and $r = 1$ recovers the 3-fold flop. As before, $X_{\pm}^{(d,r)}$ are non-compact Calabi–Yau [Don13a, § 3.2].

To apply our strategy we first need to know a (full strong) exceptional collection on $\mathbb{Gr}(r, V)$: such a collection was discovered by Kapranov [Kap88]. It consists of particular Schur powers of the tautological bundle S; for example, in the case $d = 4$, $r = 2$ the exceptional collection is

$$\{\mathcal{O}, S^\vee, \mathrm{Sym}^2\, S^\vee, \mathcal{O}(1), S^\vee(1), \mathcal{O}(2)\}$$

where $\mathcal{O}(1) = \det S^\vee$. Now we can define our windows: this same set of Schur powers determines a set of bundles on the stack $\mathfrak{X}^{(d,r)}$, and we let

$$\mathcal{W}_0 \subset D^b(\mathfrak{X}^{(d,r)})$$

be the subcategory that they split-generate. To get the other windows $\mathcal{W}_k$ we tensor every bundle in the collection by $\mathcal{O}(k)$. The analogue of Proposition 2.1 still holds (see Proposition 3.6), so we get equivalences

$$\psi_k : D^b(X_{+}^{(d,r)}) \xrightarrow{\sim} D^b(X_{-}^{(d,r)})$$

by passing through each window $\mathcal{W}_k$, and combining them we get window-shift autoequivalences

$$\omega_{k,l} := \psi_k^{-1}\psi_l : D^b(X_{+}^{(d,r)}) \xrightarrow{\sim} D^b(X_{+}^{(d,r)}).$$

With very little work, we have produced some novel derived autoequivalences. However, the method is very algebraic, and it would be nice to have some geometric understanding of them. This is a much harder question, which we will turn to in the next section.

Before we do that, let us present one more explicit calculation. Hopefully this will give the reader some feel for the computations that are going to arise later on in the paper. Let us look at the effect of the window shift $\omega_{-1,0}$, as we did before, but this time let us do it in the case $d = 4$, $r = 2$. As before, let us make life easy by only looking at the effect of $\omega_{-1,0}$ on the generating bundles for $\mathcal{W}_0$. Then we have immediately that

$$\psi_0(\mathcal{O}) = \mathcal{O}, \quad \psi_0(S^\vee) = S^\vee, \ldots, \quad \psi_0(\mathcal{O}(2)) = \mathcal{O}(2).$$

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As shown in Figure 2, the bundles $\mathcal{O}$, $S^\vee$ and $\mathcal{O}(1)$ also lie in the generating set for the window $\mathcal{W}_{-1}$, so applying ψ_{-1}^{-1} to them is easy, and we have

$$\omega_{-1,0}(\mathcal{O}) = \mathcal{O}, \quad \omega_{-1,0}(S^\vee) = S^\vee, \quad \omega_{-1,0}(\mathcal{O}(1)) = \mathcal{O}(1).$$

Obviously this is a general phenomenon: $\omega_{k,l}$ fixes any bundles that lie in the generating sets for both $\mathcal{W}_k$ and $\mathcal{W}_l$.

Now let us calculate the effect of $\omega_{-1,0}$ on $\mathrm{Sym}^2 S^\vee$. To apply ψ_{-1}^{-1} we have to resolve $\mathrm{Sym}^2 S^\vee$ in terms of the window $\mathcal{W}_{-1}$. It turns out that there is an exact sequence on $X_-^{(4,2)}$ given by

$$0 \longrightarrow \mathrm{Sym}^2 S^\vee \otimes \wedge^4 V \longrightarrow S^\vee \otimes \wedge^3 V \longrightarrow \mathcal{O} \otimes \wedge^2 V \longrightarrow \mathcal{O}(-1) \longrightarrow 0 \tag{4}$$

which is the pull-up from $\mathbb{G}\mathrm{r}(4,2)$ of (a twist of) an Eagon–Northcott complex [EN62] (see Example A.8). Hence, after picking a basis for V again, we have:

$$\omega_{-1,0}(\mathrm{Sym}^2 S^\vee) = \{\underline{S^{\vee\oplus 4}} \longrightarrow \mathcal{O}^{\oplus 6} \longrightarrow \mathcal{O}(-1)\}.$$

To calculate $\omega_{-1,0}(S^\vee(1))$ we use the exact sequence

$$0 \longrightarrow S^\vee(1) \otimes \wedge^4 V \longrightarrow \mathcal{O}(1) \otimes \wedge^3 V \longrightarrow \mathcal{O} \otimes V \longrightarrow S^\vee(-1) \longrightarrow 0 \tag{5}$$

which is the pull-up from $\mathbb{G}\mathrm{r}(4,2)$ of (a twist of) a Buchsbaum–Rim complex [BR64] (see Example A.9). Then

$$\omega_{-1,0}(S^\vee(1)) = \{\underline{\mathcal{O}(1)^{\oplus 4}} \longrightarrow \mathcal{O}^{\oplus 4} \longrightarrow S^\vee(-1)\}.$$

The calculation for $\omega_{-1,0}(\mathcal{O}(2))$ requires a third sort of 'generalized Koszul complex': it is the complex denoted $\mathcal{C}^2$ in [Eis94, Appendix A.2]. Pulling it up to $X_-^{(4,2)}$ and twisting we get

$$0 \longrightarrow \mathcal{O}(2) \otimes \wedge^4 V \longrightarrow \mathcal{O}(1) \otimes \wedge^2 V \longrightarrow S^\vee \otimes V \longrightarrow \mathrm{Sym}^2 S^\vee(-1) \longrightarrow 0 \tag{6}$$

so

$$\omega_{-1,0}(\mathcal{O}(2)) = \{\underline{\mathcal{O}(1)^{\oplus 6}} \longrightarrow S^{\vee\oplus 4} \longrightarrow \mathrm{Sym}^2 S^\vee(-1)\}.$$

Evidently to do these calculations in general we would need to know a lot of exact sequences on Grassmannians. In fact for $r = 2$ the complexes $\mathcal{C}^i$ in [Eis94] suffice, but for higher r we need generalizations. We will return to this point later.

2.2 Spherical twists

Let us return to the example of the 3-fold flop. We have our 3-fold $X_+ = X_+^{(2,1)}$, and we may consider the window-shift autoequivalence

$$\omega_{0,1}: \quad D^b(X_+) \xrightarrow{\sim} \mathcal{W}_1 \xrightarrow{\sim} D^b(X_-) \xrightarrow{\sim} \mathcal{W}_0 \xrightarrow{\sim} D^b(X_+).$$

Observe that the zero section $\mathbb{P}V$ inside X_+ is precisely the locus that becomes unstable when we pass to the other GIT quotient X_-. Away from $\mathbb{P}V$ the two quotients are isomorphic, and the equivalences ψ_k are just the identity, so the effect of the window shift is concentrated along $\mathbb{P}V$. It was argued (somewhat imprecisely) in [Seg11] that $\omega_{0,1}$ is in fact a Seidel–Thomas spherical twist [ST01] around the spherical object

$$\mathcal{O}_{\mathbb{P}V} \in D^b(X_+).$$

This result was already folklore, at least in the physics literature. To define this spherical twist, we consider $\mathbb{P}V$ as a correspondence:

$$\begin{tikzcd} & \mathbb{P}V \arrow[dl, "\pi"'] \arrow[dr, "j"] & \\ \mathrm{pt} & & X_+ \end{tikzcd} \tag{7}$$

Then we have a functor

$$F = j_*\pi^* : D^b(\mathrm{pt}) \longrightarrow D^b(X_+),$$

and its right adjoint

$$R = \pi_* j^! : D^b(X_+) \longrightarrow D^b(\mathrm{pt}).$$

The adjunction gives a natural transformation

$$j_*\pi^*\pi_*j^! \longrightarrow \mathrm{id},$$

and the *spherical twist*

$$T_F : D^b(X_+) \longrightarrow D^b(X_+)$$

is the cone on this natural transformation. It is immediate that

$$T_F(E) = \mathrm{Cone}(\mathrm{Hom}(\mathcal{O}_{\mathbb{P}V}, E) \otimes \mathcal{O}_{\mathbb{P}V} \longrightarrow E)$$

which is perhaps a more standard definition (but of course we are anticipating a generalization).

PROPOSITION 2.2. *The window shift $\omega_{0,1}$ and the spherical twist T_F coincide.*

This is a special case of our later Theorem 3.12, but we will sketch the proof here. Suppose we wanted to compute the effect of the window shift $\omega_{0,1}$ on some object $E \in D^b(X_+)$. Firstly, we resolve E by the bundles in $\mathcal{W}_1$; then we can apply ψ_1 and get an object $\psi_1 E \in D^b(X_-)$. Secondly, we need to rewrite $\psi_1 E$ in terms of the other window $\mathcal{W}_0$; then we can apply ψ_0^{-1} and bring it back to $D^b(X_+)$. The key idea of the proof is to find an endofunctor

$$T_{\mathcal{F}} : D^b(\mathfrak{X}) \longrightarrow D^b(\mathfrak{X})$$

on the stack $\mathfrak{X}$ that carries out this second step of the window shift, i.e. it rewrites objects from $\mathcal{W}_1$ in terms of $\mathcal{W}_0$. Then we need to know that on X_+ the functor $T_{\mathcal{F}}$ acts as the spherical twist. Specifically, we want a functor that has the following three properties.

(i) The effect of $T_{\mathcal{F}}$ is concentrated along the locus $V \oplus \{0\}$, so it acts as the identity on X_-. More precisely, we want

$$i^*_{X_-} T_{\mathcal{F}} = i^*_{X_-}.$$

In fact it is enough that this equality holds on the subcategory $\mathcal{W}_1$.

(ii) $T_{\mathcal{F}}$ maps the window $\mathcal{W}_1$ to the window $\mathcal{W}_0$.

(iii) When we restrict $T_{\mathcal{F}}$ to X_+ it acts as the spherical twist T_F, i.e. the diagram

$$\begin{tikzcd} D^b(\mathfrak{X}) \arrow[r, "T_{\mathcal{F}}"] \arrow[d, "i^*_{X_+}"'] & D^b(\mathfrak{X}) \arrow[d, "i^*_{X_+}"] \\ D^b(X_+) \arrow[r, "T_F"] & D^b(X_+) \end{tikzcd}$$

commutes. Again, it is actually enough that the diagram commutes when we restrict to the subcategory $\mathcal{W}_1$.

We call a functor with these properties a *transfer functor*, since it transfers between windows. If we have a transfer functor $T_{\mathcal{F}}$, then the proof of Proposition 2.2 is an immediate formality.

Proof of Proposition 2.2. Using property (ii), we have a diagram

$$\begin{tikzcd} \mathcal{W}_1 \arrow[rr, "T_{\mathcal{F}}"] \arrow[dr, "i^*_{X_-}", "\sim"'] \arrow[dd, "i^*_{X_+}"', "\wr"] & & \mathcal{W}_0 \arrow[dl, "i^*_{X_-}"', "\sim"] \arrow[dd, "i^*_{X_+}", "\wr"'] \\ & D^b(X_-) & \\ D^b(X_+) \arrow[rr, "T_F"] \arrow[ur, "\psi_1", "\sim"'] & & D^b(X_+) \arrow[ul, "\psi_0"', "\sim"] \end{tikzcd}$$

The left- and right-hand triangles commute by definition, and the top triangle and the outer square both commute by properties (i) and (iii). Noting that the left-hand side of the outer square is an isomorphism, we then see that the bottom triangle commutes. $\square$

It is not difficult to guess what the transfer functor $T_{\mathcal{F}}$ is: it is the exact analogue of T_F for the stack $\mathfrak{X}$. We consider the correspondence

$$\begin{tikzcd} & {[V/\mathbb{C}^*]} \arrow[dl, "\pi"'] \arrow[dr, "j"] & \\ \mathrm{pt} & & \mathfrak{X} \end{tikzcd} \tag{8}$$

then $T_{\mathcal{F}}$ is the cone

$$\mathrm{Cone}(j_*\pi^*\pi_*j^! \longrightarrow \mathrm{id}) : D^b(\mathfrak{X}) \longrightarrow D^b(\mathfrak{X}).$$

Now it is just a matter of checking properties (i)–(iii), but as this gets rather involved in the general case it is probably worth saying a few words about it here.

(i) This property is obvious from the definition.

(ii) We need to calculate $T_{\mathcal{F}}(\mathcal{O}(1))$ and $T_{\mathcal{F}}(\mathcal{O}(2))$ and check that they both end up in $\mathcal{W}_0$. Firstly, the relative canonical bundle K_j of j is $\mathcal{O}(-2)$, and its relative dimension is -2. So

$$j^!(\mathcal{O}(1)) = K_j \otimes j^*(\mathcal{O}(1))[\dim j] = \mathcal{O}(-1)[-2].$$

Hence $\pi_*j^!(\mathcal{O}(1)) = 0$, but then $T_{\mathcal{F}}(\mathcal{O}(1)) = \mathcal{O}(1)$, and this is indeed in $\mathcal{W}_0$.

The calculation of $T_{\mathcal{F}}(\mathcal{O}(2))$ is a little more complicated. We have

$$j^!(\mathcal{O}(2)) = \mathcal{O}[-2]$$

so $\pi_*j^!(\mathcal{O}(2)) = \mathcal{O}_{\mathrm{pt}}[-2]$, and

$$j_*\pi^*\pi_*j^!(\mathcal{O}(2)) = \mathcal{O}_V[-2] \in D^b(\mathfrak{X}).$$

We know from the adjunction that there is supposed to be a natural map

$$\mathcal{O}_V[-2] \longrightarrow \mathcal{O}(2). \tag{9}$$

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To see it explicitly, we need to use the Koszul resolution of $\mathcal{O}_V$ given by

$$0 \longrightarrow \mathcal{O}(2) \longrightarrow \mathcal{O}(1)^{\oplus 2} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_V \longrightarrow 0. \tag{10}$$

When we take the cone on (9) the two copies of $\mathcal{O}(2)$ cancel out, and the result is quasi-isomorphic to the complex

$$\{\underline{\mathcal{O}(1)^{\oplus 2}} \longrightarrow \mathcal{O}\}.$$

This is in $\mathcal{W}_0$, as required.

(iii) This property is not surprising given that the definitions of T_F and $T_{\mathcal{F}}$ are so closely related, but there is something to check. The issue is that calculating π_* from the space $\mathbb{P}V$ can give a different answer than if we calculate it from the stack $[V/\mathbb{C}^*]$, because on $\mathbb{P}V$ sheaves can have higher cohomology. However, the two bundles $\mathcal{O}(-1)$ and $\mathcal{O}$ have no higher cohomology, which means that the functors $T_F i^*_{X_+}$ and $i^*_{X_+} T_{\mathcal{F}}$ give the same results when we restrict to the window $\mathcal{W}_1$. They are not, however, the same on the whole of $D^b(\mathfrak{X})$.

Remark 2.3. Unlike the twist T_F on $D^b(X_+)$, the transfer functor $T_{\mathcal{F}}$ is not in general an autoequivalence on the derived category $D^b(\mathfrak{X})$ of the stack. This can be seen explicitly in the case above: noting that $\pi_* j^!(\mathcal{O}_V) = \mathbb{C}$ using the Koszul resolution (10) of $\mathcal{O}_V$, it follows that $T_{\mathcal{F}}(\mathcal{O}_V) = 0$, so that $T_{\mathcal{F}}$ cannot be an autoequivalence.

2.3 Spherical cotwists

We now look for a geometric interpretation for the window-shift autoequivalence

$$\omega_{-1,0} : D^b(X_+) \xrightarrow{\sim} D^b(X_+)$$

of the derived category of our 3-fold. We can then compare this with an algebraic interpretation which we already have: by the relations (1), the autoequivalence $\omega_{-1,0}$ is inverse to $\omega_{0,1}$ up to tensoring with $\mathcal{O}(1)$. The relevant geometrical functor is an example of an (inverse) *spherical cotwist*, around a functor with source $D^b(X_+)$. Note that, by contrast, the twist in § 2.2 was around a functor with target $D^b(X_+)$.

We need to use the natural map

$$X_+ \longrightarrow \mathrm{Hom}(V, V) \simeq \mathbb{C}^4$$

which contracts the zero section and has a 3-fold ordinary double point $\mathrm{Im}(X_+)$ as its image. To maintain symmetry with § 2.2 (and the general case which we will meet later), we will write this as a correspondence as follows:

$$\begin{tikzcd} & X_+ \arrow[dl, "\mathrm{id}"'] \arrow[dr, "j"] & \\ X_+ & & \mathrm{Hom}(V,V) \end{tikzcd} \tag{11}$$

Then analogously we have a functor

$$F = j_* : D^b(X_+) \longrightarrow D^b(\mathrm{Hom}(V, V)).$$

This time we will use its *left* adjoint, which is

$$L = j^* : D^b(\mathrm{Hom}(V, V)) \longrightarrow D^b(X_+),$$

and form the cone

$$\mathrm{Cone}(j^* j_* \longrightarrow \mathrm{id}) : D^b(X_+) \longrightarrow D^b(X_+).$$

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Proposition 2.4. *The window shift $\omega_{-1,0}$ is equal to the shifted cone*

$$\mathrm{Cone}(j^* j_* \longrightarrow \mathrm{id})\,[-2].$$

The proof of this proposition follows exactly the same structure as the proof of Proposition 2.2, i.e. we find a transfer functor on $D^b(\mathfrak{X})$ which restricts to the given functor on $D^b(X_+)$. However, describing this transfer functor would require us to go into more detail than we wish to at this point, so for the moment we will just do a heuristic calculation (for the full proof, see Theorem 3.13). What we will do is show that these two functors give the same answer on the bundles $\mathcal{O}$ and $\mathcal{O}(1)$. These bundles generate all of $D^b(X_+)$, so this is some evidence that the functors are the same. We will also see some of the kinds of computation that will be needed in the proof of the general case.

We have already calculated the effect of $\omega_{-1,0}$ on $\mathcal{O}$ and $\mathcal{O}(1)$: the answer is given in (3). So we now calculate the functor $[j^* j_* \to \mathrm{id}]$ on these two bundles and compare. We have

$$j_*(\mathcal{O}) = \mathcal{O}_{\mathrm{Im}(X_+)}$$

on $\mathrm{Hom}(V, V)$. To apply j^*, we need to know that this has a free resolution

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{\mathrm{Im}(X_+)} \longrightarrow 0. \tag{12}$$

Taking the cone to the identity kills the copy of $\mathcal{O}$ corresponding to the middle term in (12), so

$$\mathrm{Cone}(j^* j_* \longrightarrow \mathrm{id}) : \mathcal{O} \longmapsto \mathcal{O}[2],$$

and hence the shifted cone agrees with the window shift on the bundle $\mathcal{O}$. To do the calculation for $\mathcal{O}(1)$, we observe that $j_*(\mathcal{O}(1))$ is a sheaf supported on $\mathrm{Im}(X_+)$, and it has a free resolution

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow j_*(\mathcal{O}(1)) \longrightarrow 0 \tag{13}$$

so $[j^* j_* \to \mathrm{id}]$ maps $\mathcal{O}(1)$ to

$$\{\mathcal{O}^{\oplus 2} \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow \underline{\mathcal{O}(1)}\}$$

which is quasi-isomorphic to

$$\{\mathcal{O}^{\oplus 2} \longrightarrow \underline{\mathcal{O}(-1)}\}[1],$$

using the analogue of the exact sequence (2) on X_+. After shifting by $[-2]$, this agrees with $\omega_{-1,0}$.

Remark 2.5. The relations (1) imply that the cotwist of Proposition 2.4 is inverse to the twist of § 2.2 (up to a shift, and tensoring by a line bundle). This relation remains somewhat surprising to us geometrically. In particular, it is immediate from the definition of the twist that it acts as the identity on sheaves supported on the complement of the zero section $\mathbb{P}V$ of X_+. The relation then gives that the cotwist acts on such sheaves simply by a shift: this may also be verified from the definition of the cotwist, but it is significantly less obvious. In the following section we will remark on how this relation generalizes.

2.4 Grassmannian twists

2.4.1 *Statement of results.* Now let us turn to the general case (at least for us!), namely the Grassmannian flop

$$X_+^{(d,r)} \dashleftarrow\dashrightarrow X_-^{(d,r)}.$$

We seek a geometric interpretation for our window-shift autoequivalences

$$\omega_{k,l} : D^b(X_+^{(d,r)}) \xrightarrow{\sim} D^b(X_+^{(d,r)}).$$

As before, the effect of the window shift is concentrated on the flopping locus where the birational map fails to be an isomorphism, i.e. the locus in $X_+^{(d,r)}$ that becomes unstable when we pass to the other GIT quotient. In the 3-fold flop case $X_+^{(2,1)}$ this locus was just the zero section $\mathbb{P}V$, but in a general $X_+^{(d,r)}$ it is much more complicated, and in particular is usually non-compact. Nevertheless, the geometric constructions of §§ 2.2 and 2.3 can be generalized.

Consider the correspondences (7) and (11) that we used for $X_+^{(2,1)}$. The key point to notice is that $\mathrm{Hom}(V, V)$ is actually $X_+^{(2,2)}$, and that $X_+^{(2,0)}$ is a point! So in general we should be looking for correspondences as follows:

$$\begin{tikzcd}[column sep=small] \cdots \arrow[dr, "j"] & & Z^{(d,r-1,r)} \arrow[dl, "\pi"'] \arrow[dr, "j"] & & Z^{(d,r,r+1)} \arrow[dl, "\pi"'] \arrow[dr, "j"] & & \cdots \arrow[dl, "\pi"'] \\ & X_+^{(d,r-1)} & & X_+^{(d,r)} & & X_+^{(d,r+1)} & \end{tikzcd} \tag{14}$$

The relevant correspondences for the $r = 2$ case were described by the first author in [Don13a]: we describe the general case in § 3.1. Then we have functors

$$F = j_*\pi^* : D^b(X_+^{(d,r)}) \longrightarrow D^b(X_+^{(d,r+1)})$$

which have right and left adjoints R and L, and we form the *twist* functors

$$T_F := \mathrm{Cone}(FR \longrightarrow \mathrm{id}) : D^b(X_+^{(d,r)}) \longrightarrow D^b(X_+^{(d,r)})$$

and *inverse cotwist* functors

$$C_F^{-1} := \mathrm{Cone}(LF \longrightarrow \mathrm{id})[-1] : D^b(X_+^{(d,r)}) \longrightarrow D^b(X_+^{(d,r)}).$$

We then prove (Theorems 3.12 and 3.13) that the twist functor T_F is equal to the window shift $\omega_{0,1}$, and the inverse cotwist functor C_F^{-1} is equal to the window shift $\omega_{-1,0}$ (up to a shift in homological degree).

In particular, the relations (1) then imply that the twist T_F and the cotwist C_F are inverse, up to a shift and tensoring by a line bundle. See Corollary 3.14 for details, and also [Don13b, Figure 1] for further discussion. Although we prove this result in an algebraic manner using window shifts, the statement is geometric, and so it would be interesting to seek some purely geometric explanation for it. We hope this will be the subject of future work.

2.4.2 *Remarks on the proofs.* The structure of our proofs remains the same as in the 3-fold flop example: we find transfer functors on the stack $\mathfrak{X}^{(d,r)}$ that transfer between the relevant pairs of windows, and restrict to T_F and C_F^{-1} on $X_+^{(d,r)}$. To find these transfer functors, we embed the correspondences $Z^{(d,r,r+1)}$ into correspondences of Artin stacks

$$\begin{tikzcd} & \mathcal{Z}^{(d,r,r+1)} \arrow[dl, "\pi"'] \arrow[dr, "j"] & \\ \mathfrak{X}^{(d,r)} & & \mathfrak{X}^{(d,r+1)} \end{tikzcd}$$

in the same way that the correspondence (7) sits inside the correspondence (8).

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When we discussed the 3-fold flop case in § 2.2, one of the ingredients that we needed in the proof was the locally-free resolution (10) of the skyscraper sheaf $\mathcal{O}_V$ on $\mathfrak{X}$. We hit a similar step in our calculations in § 2.3: we needed the locally-free resolutions (12) and (13) of two sheaves that lived on $\operatorname{Im}(j) \subset \operatorname{Hom}(V, V)$. In our proof of the general case we will need to generalize these examples, i.e. we will need to produce explicit locally-free resolutions of various sheaves that live on the unstable loci in $\mathfrak{X}^{(d,r)}$.

These locally-free resolutions are very closely related to the exact sequences of bundles on Grassmannians that we mentioned at the end of § 2.1.2. For example, we already noted that when we restrict the resolution (10) to X_- we get the pull-up of the Euler sequence on $\mathbb{P}V^\vee$. For another example, consider the exact sequence (4) on $X_-^{(4,2)}$. If we consider this as a complex on $\mathfrak{X}^{(4,2)}$ then it is no longer exact, but we claim that it only fails to be exact at the last term, so it gives a resolution of a sheaf. The last two terms are the twist by $\mathcal{O}(-1)$ of the map

$$\wedge^2 V(1) \longrightarrow \mathcal{O}$$

and the cokernel of this map is the skyscraper sheaf along the unstable locus $\mathcal{U} = \mathfrak{X}^{(4,2)} \backslash X_-^{(4,2)}$. So (4) arises from the locally-free resolution of $\mathcal{O}_{\mathcal{U}}(-1)$. The other two sequences (5) and (6) on $\mathfrak{X}^{(4,2)}$ arise from locally-free resolutions of more complicated sheaves supported on $\mathcal{U}$.

These locally-free resolutions/exact sequences on Grassmannians do not appear to be very well known. They are present implicitly in the book of Weyman [Wey03], and most of the exact sequences were described explicitly in [Fon13]. We describe them in excruciating detail in the Appendix A.2, as applications of Theorem A.7.

Remark 2.6. To obtain the window equivalences corresponding to the Grassmannian flop, we make crucial use of the fact that the generators of our window $\mathcal{W}_0$ restrict to $X_+^{(d,r)}$ to give a tilting bundle, in Proposition 3.6. It would be desirable to extend our approach to settings where a tilting bundle is not so easily available. For instance, replacing $X_+^{(d,r)}$ with the cotangent bundle $T^\vee \mathbb{Gr}(r, V)$, the analogue of this tilting result no longer holds, as explained in [Don13a, Remark C.3]. Derived equivalences corresponding to the flop of $T^\vee \mathbb{Gr}$ have, however, been constructed by other methods [CKL13]. It would be interesting to find a way to apply our approach to this case, and compare the resulting functors with these established constructions.

3. Proofs

Notation. For a Young diagram δ we write $\delta = (\delta^1, \dots, \delta^h)$ where the $\delta^\bullet$ are the (non-increasing sequence of) row lengths of δ. Trailing zeros may be omitted. Given V a vector space of dimension h, we write $\mathbb{S}^\delta V$ for the associated Schur power [Wey03].

Let γ be a Young diagram of width less than or equal to w and height less than or equal to h, so that we can draw γ inside a $w \times h$ rectangle. Take the complement of γ inside this rectangle and rotate it by 180°: this produces a new Young diagram which we denote by $\operatorname{Comp}_w^h(\gamma)$. Figure 3 gives an example.

Remark 3.1. We will frequently use the following result from [Wey03, Exercise 2.18(a)]:

$$\mathbb{S}^\gamma V^\vee \otimes \det V^{\otimes w} = \mathbb{S}^{\operatorname{Comp}_w^h(\gamma)} V.$$

3.1 Windows on Grassmannian flops

Let V be a vector space of dimension d, and S be another vector space of dimension r, where $0 < r \leqslant d$. For simplicity, we will fix a trivialization of $\det V$ throughout.

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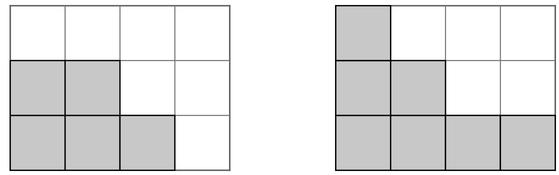

$\gamma = (3,2)$ $\qquad$ $\mathrm{Comp}^h_w(\gamma) = (4,2,1)$

FIGURE 3. Young diagram $\gamma = (3,2)$ and its complement for $h = 3$, $w = 4$.

Our first space is the affine Artin stack

$$\mathfrak{X} = [\mathrm{Hom}(S, V) \oplus \mathrm{Hom}(V, S)/\mathrm{GL}(S)].$$

In § 2.1.2 we denoted this by $\mathfrak{X}^{(d,r)}$, but from now on we will drop the (d, r) from our notation. There are two possible GIT quotients of this stack, which correspond to open substacks denoted by

$$X_\pm \xhookrightarrow{i_{X_\pm}} \mathfrak{X}.$$

Remark 3.2. One quotient X_+ is the locus where the map from S to V is full rank: it is the total space of the vector bundle $\mathrm{Hom}(V, S)$ over $\mathbb{Gr}(r, V)$, where we reuse the notation S to denote the tautological subspace bundle on the Grassmannian $\mathbb{Gr}(r, V)$.

Dually, X_- is the locus where the map from V to S is of full rank: it is the total space of the vector bundle $\mathrm{Hom}(S, V)$ over the dual Grassmannian $\mathbb{Gr}(V, r)$, where now S denotes the tautological quotient bundle.

As anticipated in § 2.1, we will define some derived equivalences between X_+ and X_- using 'windows' in $D^b(\mathfrak{X})$. To define these windows we need to recall Kapranov's exceptional collection for a Grassmannian [Kap88].

Let δ be a partition of some integer, which as usual we can draw as a Young diagram. Then associated to δ we have a Schur power $\mathbb{S}^\delta S^\vee$ of $S^\vee$. This is a representation of $\mathrm{GL}(S)$ and so induces a vector bundle on $\mathbb{Gr}(r, V)$. Now we put the following definition.

DEFINITION 3.3.

$$\Gamma_{d,r} := \{\text{Young diagrams } \gamma \text{ with height} \leqslant r \text{ and width} \leqslant d - r\}.$$

Kapranov's exceptional collection for $\mathbb{Gr}(r, V)$ (see [Kap88]) is the set

$$\{\mathbb{S}^\delta S^\vee \bigm| \delta \in \Gamma_{d,r}\}.$$

We can also consider this as a set of vector bundles on $\mathbb{Gr}(V, r)$, on $X_\pm$, or on $\mathfrak{X}$. These bundles give us our zeroth window, i.e. we define $\mathcal{W}_0 \subset D^b(\mathfrak{X})$ as the full subcategory split-generated by this set of vector bundles. The other windows $\mathcal{W}_k$ are obtained by tensoring $\mathcal{W}_0$ by powers of the tautological line bundle

$$\mathcal{O}(1) := \det S^\vee.$$

DEFINITION 3.4. $\mathcal{W}_k$ is the full subcategory of $D^b(\mathfrak{X})$ split-generated by the set

$$\{\mathbb{S}^\delta S^\vee(k) \bigm| \delta \in \Gamma_{d,r}\}.$$

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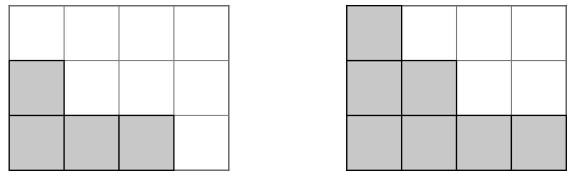

$\delta = (3,1)$ $\tilde{\delta} = (4,2,1)$

Figure 4. Young diagram $\delta \in \Gamma^{(1)}_{d,r}$ and its twist $\tilde{\delta}$.

Now observe that if the width of δ is strictly less than $d-r$ then we can create a new diagram $\tilde{\delta} \in \Gamma_{d,r}$ by adding on a new column of height r to δ (see Figure 4), and

$$\mathbb{S}^{\delta} S^{\vee}(k) = \mathbb{S}^{\tilde{\delta}} S^{\vee}(k-1).$$

These are the bundles that lie in the generating set for both $\mathcal{W}_k$ and the neighbouring window $\mathcal{W}_{k-1}$. Since this observation will be useful for us later, we make the following definition.

Definition 3.5.

$$\Gamma^{(1)}_{d,r} = \{\delta \in \Gamma_{d,r} \,|\, \mathrm{width}(\delta) < d-r\},$$
$$\Gamma^{(2)}_{d,r} = \{\delta \in \Gamma_{d,r} \,|\, \mathrm{width}(\delta) = d-r\}.$$

We will also frequently switch between Schur powers of $S^{\vee}$ and of S. In terms of the latter, $\mathcal{W}_k$ is generated by the set

$$\{\mathbb{S}^{\gamma} S(d-r+k) \;\big|\; \gamma \in \Gamma_{d,r}\}.$$

The following proposition is the crucial ingredient in constructing our window equivalences.

Proposition 3.6. *For any k and $0 < r < d$, both functors*

$$i^*_{X_{\pm}} : \mathcal{W}_k \longrightarrow D^b(X_{\pm})$$

are equivalences.

Proof. By symmetry we only need the argument for $i^*_{X_+}$. We observe that $i^*_{X_+} S^{\vee} = \pi^* S^{\vee}$ where π is the projection $X_+ \to \mathbb{Gr}(r,V)$. It immediately follows, because Schur powers commute with pullbacks, that $i^*_{X_+} \mathcal{W}_k = \pi^* \mathcal{T}$, where $\mathcal{T}$ is the Kapranov tilting bundle for $\mathbb{Gr}(r,V)$ given by

$$\mathcal{T} := \bigoplus_{\delta \in \Gamma_{d,r}} \mathbb{S}^{\delta} S^{\vee}.$$

An extended exercise in Schur functors [Don13a, Appendix C] gives that $\pi^* \mathcal{T}$ is tilting on X_+.

It then suffices to show that the natural restriction map of derived functors

$$\mathbb{R}\mathrm{Hom}_{\mathfrak{X}}(\mathcal{W}_k, \mathcal{W}_k) \longrightarrow \mathbb{R}\mathrm{Hom}_{X_+}(i^*_{X_+}\mathcal{W}_k, i^*_{X_+}\mathcal{W}_k)$$

induces isomorphisms on cohomology. There is no higher cohomology on the left-hand side because $\mathcal{W}_k$ is locally free and the stack $\mathfrak{X}$ is affine, and none on the right-hand side by the above tilting property. It therefore remains to show that the restriction map of *ordinary* Hom functors

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{W}_k, \mathcal{W}_k) \longrightarrow \mathrm{Hom}_{X_+}(i^*_{X_+}\mathcal{W}_k, i^*_{X_+}\mathcal{W}_k)$$

is an isomorphism. Consider, then, the complement of X_+ in $\mathfrak{X}$. This is the pull-up via the projection $\pi : \mathfrak{X} \to \mathrm{Hom}(S,V)$ of the locus in $\mathrm{Hom}(S,V)$ consisting of maps of rank strictly less than $r = \dim S$. We then see from [BV88, Proposition 1.1(b)] that its codimension is $d-r+1 \geqslant 2$, and hence the required isomorphism follows by normality. □

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The next theorem immediately follows.

Theorem 3.7. *For $0 < r < d$ there exists a* window equivalence ψ_k *defined by the composition*

$$\psi_k : \quad D^b(X_+) \xrightarrow[(i^*_{X_+})^{-1}]{\sim} \mathcal{W}_k \xrightarrow[i^*_{X_-}]{\sim} D^b(X_-).$$

Remark 3.8. Theorem 3.7 is obtained in [BLvdB11, § 5] using a different method which works in arbitrary characteristic.

Consequently, we have the following definition.

Definition 3.9. We define *window-shift autoequivalences* $\omega_{k,l}$ by

$$\omega_{k,l} := \psi_k^{-1}\psi_l : D^b(X_+) \xrightarrow{\sim} D^b(X_+).$$

3.2 The geometric construction

In [Don13a], the first author constructed an endofunctor of $D^b(X_+)$ using more geometric techniques, and proved that it was an autoequivalence when $r \leqslant 2$. In this section we will show that this endofunctor agrees with the window shift $\omega_{0,1}$, and hence that it is in fact an autoequivalence for all r.

In addition to the vector spaces V and S, let H be a third vector space, of dimension $r-1$. Consider the affine Artin stack

$$\mathcal{Y} = [\mathrm{Hom}(H,V) \oplus \mathrm{Hom}(V,H)/\mathrm{GL}(H)]$$

which is of course the same thing as $\mathfrak{X}^{(d,r-1)}$. It contains an open substack Y_+ $(= X_+^{(d,r-1)})$ consisting of the locus where the map from H to V has full rank: this is the total space of a vector bundle over $\mathbb{Gr}(r-1,V)$.

Now let

$$\bar{\mathcal{Z}} = [\mathrm{Hom}(S,V) \oplus \mathrm{Hom}(V,H) \oplus \mathrm{Hom}(H,S)/\mathrm{GL}(H) \times \mathrm{GL}(S)].$$

There are obvious maps

$$\mathcal{Y} \xleftarrow{\pi} \bar{\mathcal{Z}} \xrightarrow{j} \mathfrak{X}$$

given by composing the relevant two linear maps in $\bar{\mathcal{Z}}$, and then forgetting the redundant group action. This defines a correspondence between $\mathcal{Y}$ and $\mathfrak{X}$, however it is not exactly the correspondence that we want; rather we will define

$$\mathcal{Z} \subset \bar{\mathcal{Z}}$$

to be the substack where the map from H to S is an injection. This is the correspondence that we want to consider.

There is an open substack $Z \subset \mathcal{Z}$ where the map from S to V is also required to be an injection. We have a commutative diagram as follows:

$$\begin{tikzcd}
\mathcal{Y} & \mathcal{Z} \arrow[l, "\pi"'] \arrow[r, "j"] & \mathfrak{X} \\
Y_+ \arrow[u, "i_{Y_+}"] & Z \arrow[l, "\pi"'] \arrow[r, "j"] \arrow[u, "i_Z"] & X_+ \arrow[u, "i_{X_+}"]
\end{tikzcd} \tag{15}$$

Remark 3.10. The lower line of this diagram gives a correspondence between Y_+ and X_+. This was introduced in [Don13a] (in that paper X_0 denotes the space that we are calling Y_+, and $\hat{B}$ denotes the correspondence that we are calling Z), where it was used to construct endofunctors of $D^b(X_+)$ and $D^b(Y_+)$ as we shall now explain. Note also that the correspondence is analogous to the Hecke correspondences in [CKL13].

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Consider the functor

$$F := j_* \pi^* : D^b(Y_+) \longrightarrow D^b(X_+).$$

It has a right adjoint

$$R := \pi_* j^! : D^b(X_+) \longrightarrow D^b(Y_+)$$

where

$$j^!(-) = j^*(-) \otimes K_j[\dim j].$$

It also has a left adjoint

$$L := \pi_*(j^*(-) \otimes K_\pi)[\dim \pi].$$

Both X_+ and Y_+ are Calabi–Yau [Don13a, § 3.2], so $K_\pi = K_j$, so we deduce that

$$R = L[-\sigma]$$

where

$$\sigma = \dim \pi - \dim j = 2(d-r) + 1.$$

This means that F and R are *biadjoint* functors up to a shift. By applying standard Fourier–Mukai techniques [Don13a, Appendix A] we may take cones on units and counits to give four endofunctors.

Definition 3.11. (i) The **twist** functor $T_F : D^b(X_+) \longrightarrow D^b(X_+)$ is the cone

$$T_F := \mathrm{Cone}(FR \longrightarrow \mathrm{id}).$$

It has a right adjoint

$$T_F^\dagger := \mathrm{Cone}(\mathrm{id} \longrightarrow FR[\sigma]).$$

(ii) The **cotwist** functor $C_F : D^b(Y_+) \longrightarrow D^b(Y_+)$ is the cone

$$C_F := \mathrm{Cone}(\mathrm{id} \longrightarrow RF).$$

It has a right adjoint

$$C_F^\dagger := \mathrm{Cone}(RF[\sigma] \longrightarrow \mathrm{id})[-1].$$

General theory [AL10] says that T_F is an equivalence if and only if C_F is an equivalence, given the fact that X_+ and Y_+ are Calabi–Yau.

For $r \leqslant 2$ these functors were proven to be equivalences in [Don13a]: it is an immediate corollary of the following theorem that in fact T_F is an equivalence for all $r < d$.

Theorem 3.12. *For $r < d$, the twist functor T_F is naturally isomorphic to the window shift $\omega_{0,1}$.*

Proof. This proceeds formally from Lemmas 3.15, 3.16 and 3.17, as in the proof of Proposition 2.2. □

Theorem 3.12 implies that C_F is also an equivalence, and so $C_F^\dagger = C_F^{-1}$, at least for $r < d$. However, we can say more: since $Y_+ = X_+^{(d,r-1)}$, we also have window-shift autoequivalences

$$\omega_{k,l}^Y : D^b(Y_+) \longrightarrow D^b(Y_+)$$

and we prove the following theorem.

Theorem 3.13. *The shifted inverse cotwist functor $C_F^{-1}[-\sigma]$ is naturally isomorphic to the window shift $\omega_{-1,0}^Y$.*

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Proof. This follows from Lemmas 3.18, 3.19 and 3.20, once again using the method of proof in Proposition 2.2. □

Theorem 3.13 was proved in [Don13a] for the $r \leqslant 2$ case.

We now temporarily reinstate the ds and rs into our notation, and state these theorems in a slightly different way. We have a whole chain of correspondences, going between $X_+^{(d,r)}$ and $X_+^{(d,r+1)}$ for every r (see diagram (14)). So for every d and r, we have both twist and cotwist endofunctors

$$T_F^{(d,r)} \text{ and } C_F^{(d,r)} : D^b(X_+^{(d,r)}) \xrightarrow{\sim} D^b(X_+^{(d,r)}).$$

We also have our window-shift autoequivalences $\omega_{k,l}^{(d,r)}$ of $D^b(X_+^{(d,r)})$, and our result is that

$$T_F^{(d,r)} = \omega_{0,1}^{(d,r)},$$
$$C_F^{(d,r)}[-\sigma_{d,r}] = \omega_{0,-1}^{(d,r)},$$

where $\sigma_{d,r} = 2(d-r)-1$. This means these two functors are almost inverse to each other. More precisely, the relations (1) between window shifts imply the following corollary.

Corollary 3.14.

$$(T_F^{(d,r)})^{-1} = (\otimes\mathcal{O}(1)) \circ C_F^{(d,r)}[-\sigma_{d,r}] \circ (\otimes\mathcal{O}(-1))$$

Notation. Finally, we introduce a little more notation. On $\mathcal{Z}$ there are two tautological line bundles given by the determinants of S and H. To distinguish between them we put

$$\mathcal{O}(1) := (\det S)^\vee,$$
$$\mathcal{O}\langle 1\rangle := (\det H)^\vee.$$

3.2.1 *Analysis of correspondences.* Before we begin the proofs of our two theorems, let us make an observation about the diagram (15). The left-hand square is trivial in the $\mathrm{Hom}(V,H)$ directions, and at various points in the following proofs we will be considering sheaves and maps that are constant over these trivial directions. Therefore it is helpful to introduce the notation

$$\begin{array}{ccc} \mathcal{Q} & \xleftarrow{\;\pi\;} & \mathcal{P} \\ {\scriptstyle i_Q}\big\uparrow & & \big\uparrow{\scriptstyle i_P} \\ Q & \xleftarrow{\;\pi\;} & P \end{array} \tag{16}$$

for the square that we obtain by deleting the $\mathrm{Hom}(V,H)$ directions, so for example $\mathcal{Q}$ is the stack $[\mathrm{Hom}(H,V)/\mathrm{GL}(H)]$.

Now look at the right-hand square in (15). This square is a fibre product, with the map $j : Z \to X_+$ being just the restriction of $j : \mathcal{Z} \to \mathfrak{X}$ to the open substack $X_+ \subset \mathfrak{X}$. Furthermore, the map $j : \mathcal{Z} \to \mathfrak{X}$ is trivial in the $\mathrm{Hom}(S,V)$ directions, and removing them gives a map which we denote

$$j : \mathcal{S} \longrightarrow \mathcal{T}.$$

We do some more analysis of these two squares in the appendix.

3.2.2 *Twist.* We now turn to the proof of Theorem 3.12. The structure of the proof is as outlined in § 2.2: the result follows formally from the existence of a transfer functor, as in the

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proof of Proposition 2.2. The functor we need to consider is

$$T_{\mathcal{F}} := \mathrm{Cone}(\mathcal{F}\mathcal{R} \longrightarrow \mathrm{id}) : D^b(\mathfrak{X}) \longrightarrow D^b(\mathfrak{X})$$

where

$$\mathcal{F} := j_* \pi^* : D^b(\mathcal{Y}) \longrightarrow D^b(\mathfrak{X})$$

and

$$\mathcal{R} := \pi_* j^! : D^b(\mathfrak{X}) \longrightarrow D^b(\mathcal{Y}).$$

Then we just need to establish the following three properties.

Lemma 3.15. $i^*_{X_-} T_{\mathcal{F}} = i^*_{X_-}$

Lemma 3.16. *$T_{\mathcal{F}}$ maps the window $\mathcal{W}_1$ to the window $\mathcal{W}_0$.*

Lemma 3.17. *The following diagram commutes.*

$$\begin{tikzcd} \mathcal{W}_1 \arrow[r, "T_{\mathcal{F}}"] \arrow[d, "i^*_{X_+}"'] & D^b(\mathfrak{X}) \arrow[d, "i^*_{X_+}"] \\ D^b(X) \arrow[r, "T_F"] & D^b(X) \end{tikzcd}$$

Lemma 3.15 is obvious, since the image of j is exactly the unstable locus that gets deleted to form X_-. The remaining two lemmas are rather more involved.

Proof of Lemma 3.16. We need to calculate the effect of $T_{\mathcal{F}} = [\mathcal{F}\mathcal{R} \to \mathrm{id}]$ on the vector bundles

$$\{\mathbb{S}^\delta S^\vee(1) \mid \delta \in \Gamma_{d,r}\}$$

and verify that each one ends up in the window $\mathcal{W}_0$. Recall that

$$\mathcal{F}\mathcal{R} = j_* \pi^* \pi_* j^!.$$

We will send our vector bundles through each of these functors in turn. The first one is

$$j^!(-) = (K_j \otimes j^*(-))[\dim j] = j^*(-)(r-d-1)\langle d-r\rangle[r-d-1]. \tag{17}$$

Note that the calculation of the canonical bundle here is straightforward as the spaces involved are open substacks of quotients of vector spaces: recall also that $\det V$ is trivialized. Let $\gamma = \mathrm{Comp}^r_{d-r}(\delta)$, so

$$\mathbb{S}^\delta S^\vee(1) = \mathbb{S}^\gamma S(d-r+1)$$

and then we have

$$j^!(\mathbb{S}^\delta S^\vee(1)) = \mathbb{S}^\gamma S\langle d-r\rangle[r-d-1]. \tag{18}$$

Next we apply π_* to this object, and by the projection formula it suffices to know what $\pi_*\mathbb{S}^\gamma S$ is. Everything here is constant along the $\mathrm{Hom}(V,H)$ directions in $\mathcal{Z}$ and $\mathcal{Y}$, so Proposition A.1(i) tells us that

$$\pi_* \mathbb{S}^\gamma S = \mathbb{S}^\gamma H$$

and so

$$\pi_* j^!(\mathbb{S}^\delta S^\vee(1)) = \mathbb{S}^\gamma H\langle d-r\rangle[r-d-1]. \tag{19}$$

This expression will be zero if and only if the height of γ is r, or equivalently if and only if the width of δ is less than $d-r$. So if $\delta \in \Gamma^{(1)}_{d,r}$ then

$$T_{\mathcal{F}}(\mathbb{S}^\delta S^\vee(1)) = \mathbb{S}^\delta S^\vee(1).$$

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Since these bundles are the ones that also lie in the target window $\mathcal{W}_0$, we have verified the lemma on this subset of our generating set.

For the remaining bundles, we continue with the calculation of $T_{\mathcal{F}}$. Let $\delta \in \Gamma^{(2)}_{d,r}$, so the height of γ is less than r and we can define $\hat{\delta} = \mathrm{Comp}^{r-1}_{d-r}(\gamma)$, which is δ with its first row deleted. Then

$$\mathbb{S}^{\gamma} H\langle d-r\rangle = \mathbb{S}^{\hat{\hat{\delta}}} H^{\vee}$$

and

$$\mathcal{FR}(\mathbb{S}^{\delta} S^{\vee}(1)) = j_*(\mathbb{S}^{\hat{\hat{\delta}}} H^{\vee})[r-d-1].$$

This is a (shift of a) torsion sheaf on $\mathfrak{X}$. Since everything here is constant in the $\mathrm{Hom}(S,V)$ directions, the sheaf is evidently the pull-up of the sheaf $j_*(\mathbb{S}^{\hat{\hat{\delta}}} H^{\vee})$ from the stack $\mathcal{T}$. In the appendix (Theorem A.7) we construct a free resolution of $j_*(\mathbb{S}^{\hat{\hat{\delta}}} H^{\vee})$, of length $d-r+1$. If we pull this resolution up to $\mathfrak{X}$ and shift it by $[r-d-1]$ then we get a complex of vector bundles situated in non-negative degrees

$$\{\underline{\mathbb{S}^{\hat{\hat{\delta}}_K} S^{\vee} \otimes \wedge^{s_K} V} \longrightarrow \cdots \longrightarrow \mathbb{S}^{\hat{\hat{\delta}}_1} S^{\vee} \otimes \wedge^{s_1} V \longrightarrow \mathbb{S}^{\hat{\hat{\delta}}_0} S^{\vee}\} \tag{20}$$

which is quasi-isomorphic to $\mathcal{FR}(\mathbb{S}^{\delta} S^{\vee}(1))$. Remarks A.10 following Theorem A.7 tell us more about the terms in this complex. Firstly, by Remark A.10(v), $\hat{\hat{\delta}}_K$ is the Young diagram of height r and width $d-r+1$ such that if we delete the first row and the first column we get back $\hat{\hat{\delta}}$, so we find that

$$\mathbb{S}^{\hat{\hat{\delta}}_K} S^{\vee} = \mathbb{S}^{\delta} S^{\vee}(1).$$

Also $s_K = d$ by Remark A.10(iii), and $\det(V)$ is trivialized, so the term in degree zero is $\mathbb{S}^{\delta} S^{\vee}(1)$. Secondly, for $k < K$ we have $\hat{\hat{\delta}}_k \in \Gamma_{d,r}$ by Remark A.10(iv), so every term in positive degree lies in the target window $\mathcal{W}_0$.

From this description, we see that the natural map $\mathcal{FR}(\mathbb{S}^{\delta} S^{\vee}(1)) \to \mathbb{S}^{\delta} S^{\vee}(1)$ is given by some non-zero map of bundles

$$\iota : \mathbb{S}^{\delta} S^{\vee}(1) \longrightarrow \mathbb{S}^{\delta} S^{\vee}(1)$$

since there are no higher Ext groups between vector bundles on the stack $\mathfrak{X}$. This map arises, via adjunction, from the natural map

$$\pi^*\pi_* j^!(\mathbb{S}^{\delta} S^{\vee}(1)) \longrightarrow j^!(\mathbb{S}^{\delta} S^{\vee}(1))$$

which is (by (18) and (19)) a shift and twist of a map

$$\mathbb{S}^{\gamma} H \longrightarrow \mathbb{S}^{\gamma} S.$$

It follows from the proof of Proposition A.1 that this is actually the tautological map, so in particular it is constant over the $\mathrm{Hom}(S,V)$ directions. Therefore the map ι must also be constant in those directions, and so is the pull-up of a map that lives on the smaller stack $\mathcal{S}$. By Lemma A.13, it must be an isomorphism. Consequently, the cone $T_{\mathcal{F}}(\mathbb{S}^{\delta} S^{\vee}(1))$ is quasi-isomorphic to the positive-degree part of the complex (20), so it lives in the target window $\mathcal{W}_0$. $\square$

Proof of Lemma 3.17. Recall the diagram (15). $T_{\mathcal{F}}$ is the cone $[j_*\pi^*\pi_* j^! \to \mathrm{id}]$ of endofunctors of $D^b(\mathfrak{X})$, and T_F is the same cone of endofunctors of $D^b(X_+)$. We wish to compare $i^*_{X_+} T_{\mathcal{F}}$ with $T_F i^*_{X_+}$.

The right-hand square in (15) is a fibre square and the open inclusion i_{X^+} is flat, so

$$i^*_{X_+} j_* = j_* i^*_Z$$

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and hence

$$i_{X_+}^* j_* \pi^* \pi_* j^! = j_* i_Z^* \pi^* \pi_* j^! = j_* \pi^* i_{Y_+}^* \pi_* j^!.$$

The left-hand square is not a fibre square, so we have only a natural transformation

$$\tau : i_{Y_+}^* \pi_* \longrightarrow \pi_* i_Z^*.$$

Then for any $\mathcal{E} \in D^b(\mathfrak{X})$ we get a morphism

$$j_* \pi^* i_{Y_+}^* \pi_* j^! \mathcal{E} \xrightarrow{j_* \pi^*(\tau_{j^! \mathcal{E}})} j_* \pi^* \pi_* i_Z^* j^! \mathcal{E} = j_* \pi^* \pi_* j^! i_{X_+}^* \mathcal{E} \tag{21}$$

where the final equality holds because i_{X_+} and i_Z are open inclusions. Thus we have a square

$$\begin{tikzcd} i_{X_+}^* j_* \pi^* \pi_* j^! \mathcal{E} \arrow[r] \arrow[d, "j_* \pi^*(\tau_{j^! \mathcal{E}})"'] & i_{X_+}^* \mathcal{E} \arrow[d, equal] \\ j_* \pi^* \pi_* j^! i_{X_+}^* \mathcal{E} \arrow[r] & i_{X_+}^* \mathcal{E} \end{tikzcd}$$

which commutes by naturality of adjunctions. This means we have a natural transformation from $i_{X_+}^* T_{\mathcal{F}}$ to $T_F i_{X_+}^*$. We claim that this becomes a natural isomorphism when we restrict it to the window $\mathcal{W}_1$. It is sufficient to check this on the generating vector bundles, i.e. we just need to check that (21) is an isomorphism when $\mathcal{E}$ is a vector bundle $\mathbb{S}^\gamma S(d-r+1)$ for some $\gamma \in \Gamma_{d,r}$. By (17) in the proof of Lemma 3.16 we know that $j^! \mathcal{E}$ is a shift of the bundle

$$\mathbb{S}^\gamma S \langle d - r \rangle$$

so it is sufficient to prove that

$$i_{Y_+}^* \pi_* \mathbb{S}^\gamma S \xrightarrow{\tau_{\mathbb{S}^\gamma S}} \pi_* i_Z^* \mathbb{S}^\gamma S$$

is an isomorphism. Everything here is constant in the $\mathrm{Hom}(V, H)$ directions, so we can actually work on the smaller square (16), and Proposition A.1(ii) is the required statement. □

3.2.3 *Cotwist.* Now we prove Theorem 3.13. The structure of the proof is exactly the same, although curiously enough in this case the transfer functor is

$$\mathcal{RF} : D^b(\mathcal{Y}) \longrightarrow D^b(\mathcal{Y}),$$

i.e. we do not take the cone from the identity.

We denote the windows on $\mathcal{Y}$ by $\mathcal{V}_k$, where each $\mathcal{V}_k$ is split-generated by the set

$$\{\mathbb{S}^\delta H^\vee \langle k \rangle \mid \delta \in \Gamma_{d,r-1}\}.$$

The three lemmas that we need are as follows.

Lemma 3.18. *$i_{Y_-}^* \mathcal{RF} = i_{Y_-}^*$ when restricted to the window $\mathcal{V}_0$.*

Lemma 3.19. *$\mathcal{RF}$ maps the window $\mathcal{V}_0$ to the window $\mathcal{V}_{-1}$.*

Lemma 3.20. *The following diagram commutes.*

$$\begin{tikzcd} \mathcal{V}_0 \arrow[r, "\mathcal{RF}"] \arrow[d, "i_{Y_+}^*"'] & D^b(\mathcal{Y}) \arrow[d, "i_{Y_+}^*"] \\ D^b(Y_+) \arrow[r, "C_F^\dagger[-\sigma]"] & D^b(Y_+) \end{tikzcd}$$

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Notice that unlike the twist case the first lemma is not obvious, and only holds on the window $\mathcal{V}_0$ and not on the whole of $D^b(\mathcal{Y})$. We will prove the first two lemmas in reverse order.

Proof of Lemma 3.19. We compute the effect of $\mathcal{RF}$ on the generating vector bundles

$$\{\mathbb{S}^\delta H^\vee \;\big|\; \delta \in \Gamma_{d,r-1}\}$$

of the window $\mathcal{V}_0$. By Corollary A.11 we have that $j^! j_* \pi^* \mathbb{S}^\delta H^\vee$ is the complex

$$\{\underline{\mathbb{S}^{\epsilon_K} S\langle d-r\rangle \otimes \wedge^{s_K} V} \longrightarrow \cdots \longrightarrow \mathbb{S}^{\epsilon_1} S\langle d-r\rangle \otimes \wedge^{s_1} V \longrightarrow \mathbb{S}^{\epsilon_0} S\langle d-r\rangle\} \tag{22}$$

where $K = d-r+1$, and the partitions ϵ_k and the numbers s_k are defined by Algorithm A.4 and (A4).

Suppose that $\delta \in \Gamma^{(1)}_{d,r-1}$, i.e. width$(\delta) < d-r+1$. Then Remark A.12(iv*) tells us that all the ϵ_k except for ϵ_K have height r, and Remark A.12(iii*) that $s_K = d$ and hence $\wedge^{s_K} V$ is trivial. So by Proposition A.1 applying π_* to (22) kills all the terms in positive degree, and leaves only

$$\pi_*(\mathbb{S}^{\epsilon_K} S)\langle d-r\rangle = \mathbb{S}^{\epsilon_K} H\langle d-r\rangle = \mathbb{S}^\delta H^\vee$$

where the final equality is because of Remark A.12(v*). So if $\delta \in \Gamma^{(1)}_{d,r-1}$ then

$$\mathcal{RF}(\mathbb{S}^\delta H^\vee) = \mathbb{S}^\delta H^\vee.$$

These bundles already lie in the target window $\mathcal{V}_{-1}$, so this verifies the lemma on this subset.

Now take $\delta \in \Gamma^{(2)}_{d,r-1}$. By Proposition A.1 again we have that for any k,

$$\pi_*(\mathbb{S}^{\epsilon_k} S)\langle d-r\rangle = \mathbb{S}^{\epsilon_k} H\langle d-r\rangle = \mathbb{S}^{\hat{\delta}_k} H^\vee\langle -1\rangle$$

where $\hat{\delta}_k = \mathrm{Comp}^{r-1}_{d-r+1}(\epsilon_k)$, which is well defined by Remark A.12(vi*). So we have represented $\mathcal{RF}(\mathbb{S}^\delta H^\vee)$ by a complex of bundles, with each term lying in the target window $\mathcal{V}_{-1}$. $\square$

Proof of Lemma 3.18. Composing $i^*_{Y_-}$ with the unit of the adjunction gives a natural transformation

$$i^*_{Y_-} \longrightarrow i^*_{Y_-}\mathcal{RF}. \tag{23}$$

It is sufficient to show that the components of this natural transformation are isomorphisms on the generating set of vector bundles for $\mathcal{V}_0$.

Pick $\delta \in \Gamma_{d,r-1}$. We know that $j^! j_* \pi^* \mathbb{S}^\delta H^\vee$ is given by the complex (22), so the unit of the j_*-$j^!$ adjunction is given by some map of bundles on $\mathcal{Z}$,

$$\eta : \mathbb{S}^\delta H^\vee \longrightarrow \mathbb{S}^{\epsilon_K} S\langle d-r\rangle \otimes \wedge^{s_K} V.$$

Furthermore, this map is constant over the $\mathrm{Hom}(S, V)$ directions, i.e. it is pulled up from the stack $\mathcal{S}$.

If δ lies in $\Gamma^{(1)}_{d,r-1}$ then we have $s_K = d$ and $\mathbb{S}^\delta H^\vee = \mathbb{S}^{\epsilon_K} H\langle d-r\rangle$, so by Lemma A.13(ii) the map η must be a twist of the tautological map, up to a scalar. Therefore the adjoint map under the π^*–π_* adjunction given by

$$\mathbb{S}^\delta H^\vee \longrightarrow \mathcal{RF}(\mathbb{S}^\delta H^\vee) = \mathbb{S}^\delta H^\vee$$

is a scalar multiple of the identity. So on this subset of the generating bundles we have shown that (23) is an isomorphism even before restricting to Y_-.

Now let $\delta \in \Gamma^{(2)}_{d,r-1}$. As we argued in the proof of Lemma 3.19, applying π_* to the complex (22) shows that $\mathcal{RF}(\mathbb{S}^\delta H^\vee)$ is given by a complex

$$\{\mathbb{S}^{\hat{\delta}_K} H^\vee \otimes \wedge^{s_K} V\langle -1\rangle \longrightarrow \cdots \longrightarrow \mathbb{S}^{\hat{\delta}_1} S \otimes \wedge^{s_1} V\langle -1\rangle \longrightarrow \underline{\mathbb{S}^{\hat{\delta}_0} H^\vee\langle -1\rangle}\}.$$

The diagrams $\hat{\delta}_k$ arose in the following way. Starting from $\delta_0 = \delta$, we applied Algorithm A.4 to get a sequence of diagrams δ_k. These all have width $d-r+1$, and so, using (A4), we have

$$\hat{\delta}_k = \mathrm{Comp}^{r-1}_{d-r+1}(\mathrm{Comp}^{r}_{d-r+1}(\delta_k)).$$

Hence $\hat{\delta}_k$ is the diagram obtained from δ_k by deleting the first row. This means that if we apply Algorithm A.4 to the starting diagram $\hat{\delta}_0$, and with the parameter r replaced by $r-1$, then it produces the sequence of diagrams $\hat{\delta}_k$.

Now recall that $\mathcal{Y}$ is the analogue of $\mathfrak{X}$ but with r replaced by $r-1$. Therefore by Theorem A.7 there is a complex of bundles on $\mathcal{Y}$

$$\{\mathbb{S}^{\hat{\delta}_{K+1}} H^\vee \otimes \wedge^{s_{K+1}} V \xrightarrow{\tilde{\eta}} \mathbb{S}^{\hat{\delta}_K} H^\vee \otimes \wedge^{s_K} V \longrightarrow \cdots \longrightarrow \mathbb{S}^{\hat{\delta}_1} H^\vee \otimes \wedge^{s_1} V \longrightarrow \underline{\mathbb{S}^{\hat{\delta}_0} H^\vee}\}$$

which is a free resolution of a sheaf supported on the unstable locus that we remove when we form Y_-. Furthermore, since $\mathrm{width}(\hat{\delta}_0) < d-r+2$, Remark A.10(iii) tells us that $s_{K+1} = d$, and

$$\mathbb{S}^{\hat{\delta}_{K+1}} H^\vee\langle -1\rangle = \mathbb{S}^\delta H^\vee$$

because removing the first column of $\hat{\delta}_{K+1}$ gives δ by Remark A.10(v) and $\hat{\delta}_{K+1}$ has height $r-1$ by Remark A.10(ii). So we have found a map on $\mathcal{Y}$,

$$\tilde{\eta} : \mathbb{S}^\delta H^\vee \longrightarrow \mathbb{S}^{\hat{\delta}_K} H^\vee \otimes \wedge^{s_K} V\langle -1\rangle,$$

which induces a quasi-isomorphism

$$i^*_{Y_-}\mathbb{S}^\delta H^\vee \longrightarrow i^*_{Y_-}\mathcal{FR}(\mathbb{S}^\delta H^\vee).$$

We claim that $\tilde{\eta}$ is the adjoint to η under the π^*–π_* adjunction, at least up to a scalar factor. If we can show this claim then the proof of the lemma is complete, because then applying (23) to $\mathbb{S}^\delta H^\vee$ gives the above quasi-isomorphism.

To show the claim, observe that the adjoint of $\tilde{\eta}$ is given by the composition

$$\begin{aligned}\mathbb{S}^\delta H^\vee &\xrightarrow{\tilde{\eta}} \mathbb{S}^{\hat{\delta}_K} H^\vee \otimes \wedge^{s_K} V\langle -1\rangle\\ &=\!= \mathbb{S}^{\epsilon_K} H\langle d-r\rangle \otimes \wedge^{s_K} V\\ &\xrightarrow{\tau} \mathbb{S}^{\epsilon_K} S\langle d-r\rangle \otimes \wedge^{s_K} V\end{aligned}$$

on the stack $\mathcal{Z}$, where τ is the tautological map. By construction, $\tau\tilde{\eta}$ is independent of the $\mathrm{Hom}(S,V)$ directions, so it is pulled up from $\mathcal{S}$. Also, both η and $\tau\tilde{\eta}$ must be $\mathrm{SL}(V)$-equivariant, because our entire construction is, so by Lemma A.15 they agree up to a scalar factor. □

Proof of Lemma 3.20. The functor $C^\dagger_F[-\sigma]$ is a (shifted) cone on the natural transformation

$$RF \longrightarrow \mathrm{id}[-\sigma].$$

The arguments in the proof of Lemma 3.17 show that there is a natural transformation

$$i^*_{Y_+}\mathcal{RF} \longrightarrow RF i^*_{Y_+}.$$

We claim this induces a natural isomorphism

$$i^*_{Y_+}\mathcal{RF} \longrightarrow C^\dagger_F[-\sigma] i^*_{Y_+}$$

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of functors from $\mathcal{V}_0$ to $D^b(Y_+)$, i.e. for every object $\mathcal{E} \in \mathcal{V}_0$ the two natural morphisms

$$i^*_{Y_+}\mathcal{RF}(\mathcal{E}) \longrightarrow RFi^*_{Y_+}(\mathcal{E}) \longrightarrow i^*_{Y_+}\mathcal{E}[-\sigma] \tag{24}$$

form (two-thirds of) an exact triangle. It is sufficient to prove this claim on the generating set of vector bundles.

Fix $\mathbb{S}^\delta H^\vee \in \mathcal{V}_0$. Arguing again as in Lemma 3.17, there is a natural isomorphism

$$i^*_Z j^! j_* \pi^* \mathbb{S}^\delta H^\vee \simeq j^! j_* \pi^* i^*_{Y_+} \mathbb{S}^\delta H^\vee.$$

Combining this with a component of the natural transformation from $i^*_{Y_+}\pi_*$ to $\pi_* i^*_Z$ gives us the natural morphism

$$i^*_{Y_+}\mathcal{RF}(\mathbb{S}^\delta H^\vee) \longrightarrow RFi^*_{Y_+}(\mathbb{S}^\delta H^\vee). \tag{25}$$

By Corollary A.11, $j^! j_* \pi^* \mathbb{S}^\delta H^\vee$ is a complex

$$\{\underline{\mathbb{S}^{\epsilon_K} S\langle d-r\rangle \otimes \wedge^{s_K} V} \longrightarrow \cdots \longrightarrow \mathbb{S}^{\epsilon_1} S\langle d-r\rangle \otimes \wedge^{s_1} V \longrightarrow \mathbb{S}^{\epsilon_0} S\langle d-r\rangle\}.$$

We can understand the morphism (25) term by term in this complex, i.e. it is the aggregate of the natural maps

$$i^*_{Y_+}\pi_* \mathbb{S}^{\epsilon_k} S \longrightarrow \pi_* i^*_Z \mathbb{S}^{\epsilon_k} S$$

twisted by powers of $\mathcal{O}\langle 1\rangle$ and exterior powers of V. These maps are constant in the $\mathrm{Hom}(V, H)$ directions in Y_+.

Now consider the natural transformation from RF to $\mathrm{id}[-\sigma]$. It arises in the following way. For any object $\mathcal{E} \in D^b(Y_+)$, the natural morphism

$$j^* j_* \pi^* \mathcal{E} \longrightarrow \pi^* \mathcal{E}$$

induces a morphism

$$j^! j_* \pi^* \mathcal{E} \longrightarrow \pi^! \mathcal{E}[-\sigma]$$

because $\sigma = \dim \pi - \dim j$ by definition and the relative canonical bundles K_π and K_j are equal, as $\mathfrak{X}$ and $\mathcal{Y}$ are Calabi–Yau. Then the π_*–$\pi^!$ adjunction gives the morphism

$$RF(\mathcal{E}) = \pi_* j^! j_* \pi^* \mathcal{E} \longrightarrow \mathcal{E}[-\sigma].$$

We apply this to the case $\mathcal{E} = i^*_{Y_+}\mathbb{S}^\delta H^\vee$. We know that $j^* j_* \pi^* \mathbb{S}^\delta H^\vee$ is a complex

$$\{\mathbb{S}^{\delta_K} S^\vee \otimes \wedge^{s_K} V \longrightarrow \cdots \longrightarrow \mathbb{S}^{\delta_1} S^\vee \otimes \wedge^{s_1} V \longrightarrow \underline{\mathbb{S}^\delta S^\vee}\}.$$

Hence the natural map from $j^* j_* \pi^* \mathbb{S}^\delta H^\vee$ to $\pi^* \mathbb{S}^\delta H^\vee$ must be given by some non-zero map of bundles

$$\mathbb{S}^\delta S^\vee \longrightarrow \pi^* \mathbb{S}^\delta H^\vee.$$

This map must be independent of the $\mathrm{Hom}(S, V)$ directions in Z because both j and $\pi^* \mathbb{S}^\delta H^\vee$ are, i.e. it is the pull-up of a map from the stack $\mathcal{S}$. So by Lemma A.13(ii) it must be the tautological map (up to a scalar multiple), and in particular it is also constant over the $\mathrm{Hom}(V, H)$ directions in Z. Consequently, the natural map

$$j^! j_* \pi^* \mathbb{S}^\delta H^\vee \longrightarrow \pi^! \mathbb{S}^\delta H^\vee[-\sigma]$$

is given by a map of bundles

$$\mathbb{S}^{\epsilon_0} S\langle d-r\rangle \longrightarrow \pi^! \mathbb{S}^\delta H^\vee[-\sigma]$$

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where $\epsilon_0 = \mathrm{Comp}^r_{d-r+1}(\delta)$ as before, and the natural map from $RFi^*_{Y_+}(\mathbb{S}^\delta H^\vee)$ to $i^*_{Y_+}\mathbb{S}^\delta H^\vee[-\sigma]$ is obtained by restricting this map to Z and taking its adjoint. Note that

$$\mathbb{S}^\delta H^\vee = \mathbb{S}^{\tilde{\epsilon}_0} H\langle d-r+1\rangle$$

where $\tilde{\epsilon}_0 = \mathrm{Comp}^{r-1}_{d-r+1}(\delta)$ and so $\tilde{\epsilon}_0$ is ϵ_0 with its first row removed.

Now we evaluate (24) on the object $\mathbb{S}^\delta H^\vee$ and verify that we obtain an exact triangle as required. Combining the above arguments, the result can be written as the twist by $\mathcal{O}\langle d-r\rangle$ of a diagram as follows:

$$\begin{tikzcd}
i^*_{Y_+}\pi_*\mathbb{S}^{\epsilon_K}S\otimes\wedge^{s_K}V \arrow[r] \arrow[d] & \cdots \arrow[r] & i^*_{Y_+}\pi_*\mathbb{S}^{\epsilon_1}S\otimes\wedge^{s_1}V \arrow[r] \arrow[d] & i^*_{Y_+}\pi_*\mathbb{S}^{\epsilon_0}S \arrow[d] \\
\pi_*i^*_Z\mathbb{S}^{\epsilon_K}S\otimes\wedge^{s_K}V \arrow[r] & \cdots \arrow[r] & \pi_*i^*_Z\mathbb{S}^{\epsilon_1}S\otimes\wedge^{s_1}V \arrow[r] & \pi_*i^*_Z\mathbb{S}^{\epsilon_0}S \arrow[d] \\
 & & & i^*_{Y_+}\mathbb{S}^{\tilde{\epsilon}_0}H\langle 1\rangle[-\sigma]
\end{tikzcd}$$

All the vertical arrows are constant in the $\mathrm{Hom}(V,H)$ directions, so we can analyse them on the smaller space Q. By Remark A.12(ii*), the width of ϵ_0 is $d-r+1$, and the width of ϵ_k is at most $d-r$ for $k>0$. Then by Proposition A.1(ii) and (iii) (and the discussion following), the first K columns of the above diagram are isomorphisms, and the final column gives an exact triangle. So (24) yields an exact triangle on each object $\mathbb{S}^\delta H^\vee \in \mathcal{V}_0$ as claimed. □

Acknowledgements

We would like to thank Nick Addington, Timothy Logvinenko and Richard Thomas for helpful conversations and suggestions, and also Debbie Levene for a useful observation on the combinatorics. W.D. would like to express his gratitude for the support of Iain Gordon and EPSRC grant EP/G007632. E.S. is grateful for the support of an Imperial College Junior Research Fellowship.

Appendix A. Geometric correspondences and locally-free resolutions

In this appendix we study the behaviour of tautological vector bundles as we push them around the two squares in the diagram (15) as discussed in § 3.2.1.

Notation. As in the rest of the paper, we let V, S and H be vector spaces of dimensions d, r and $r-1$ respectively, under the assumption that $d \geqslant r > 0$. We also fix a trivialization of $\det V$.

A.1 The left-hand square

Let $\mathcal{Q}$ be the affine stack

$$\mathcal{Q} = [\mathrm{Hom}(H,V)/\mathrm{GL}(H)]$$

and define a second stack

$$\mathcal{P} \subset [\mathrm{Hom}(H,S)\oplus\mathrm{Hom}(S,V)/\mathrm{GL}(H)\times\mathrm{GL}(S)]$$

to be the locus where the map from H to S is an injection. Then we have a composition map $\pi : \mathcal{P}\to\mathcal{Q}$.

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Let $P \subset \mathcal{P}$ be the locus where both maps are injections, so P is a partial flag variety. The image of P under π is the Grassmannian $Q \cong \mathbb{Gr}(r-1, V)$ which forms an open substack of $\mathcal{Q}$. So we have a commutative diagram as follows:

$$\begin{tikzcd} \mathcal{Q} & \mathcal{P} \arrow[l, "\pi"'] \\ Q \arrow[u, hook, "i_Q"] & P \arrow[l, "\pi"'] \arrow[u, hook, "i_P"'] \end{tikzcd}$$

If we choose a point $q \in Q$ then the fibre of P over q is the projective space $\mathbb{P}(V/H)$. The fibre of $\mathcal{P}$ over q is slightly larger: it is given by the affine stack

$$[\mathrm{Hom}(L, V/H)/\mathrm{GL}(L)] \tag{A1}$$

where L is the one-dimensional space S/H. In particular, the above diagram is not a fibre square. The fibres of $\mathcal{P}$ over points $q \in \mathcal{Q}\backslash Q$ have the same description, although the dimension of V/H will jump.

Observe also that the relative canonical bundle of π is

$$K_\pi = (\det S)^{d-r+1} \otimes (\det H)^{r-d} = L^{\otimes d-r+1} \otimes \det H$$

and the dimension of π is $d-r$.

Proposition A.1. *Let γ be a Young diagram, and let $\mathbb{S}^\gamma S$ be the associated vector bundle on $\mathcal{P}$.*

(i) *We have an isomorphism of bundles on $\mathcal{Q}$,*

$$\mathbb{S}^\gamma H \simeq \pi_* \mathbb{S}^\gamma S.$$

(ii) *If the width of γ is at most $d-r$ then we have an isomorphism of bundles on Q,*

$$\mathbb{S}^\gamma H = i_Q^* \pi_* \mathbb{S}^\gamma S \simeq \pi_* i_P^* \mathbb{S}^\gamma S,$$

i.e. we have base change for $\mathbb{S}^\gamma S$.

(iii) *Let γ have width $d-r+1$, and let $\tilde{\gamma}$ be the Young diagram obtained by deleting the first row of γ. Then there is an exact triangle*

$$\mathbb{S}^\gamma H \longrightarrow \pi_* i_P^* \mathbb{S}^\gamma S \longrightarrow \mathbb{S}^{\tilde{\gamma}} H \otimes \det H^\vee [r-d] \dashrightarrow$$

in $D^b(Q)$.

Remark A.2. If $d > r$ then (iii) is the statement that the non-zero higher push-down sheaves of $i_P^* \mathbb{S}^\gamma S$ are

$$\mathbb{R}^k \pi_* i_P^* \mathbb{S}^\gamma S = \begin{cases} \mathbb{S}^\gamma H & k = 0, \\ \mathbb{S}^{\tilde{\gamma}} H \otimes \det H^\vee & k = d-r. \end{cases}$$

However if $d = r$ then we may get a non-split extension of bundles on Q.

Proof. On $\mathcal{P}$ we have a short exact sequence of bundles

$$0 \longrightarrow H \longrightarrow S \longrightarrow L \longrightarrow 0.$$

Thus $\mathbb{S}^\gamma S$ has a filtration whose associated graded pieces are

$$\bigoplus_\alpha (\mathbb{S}^\alpha H \otimes L^{\otimes t})^{\oplus c^\gamma_{\alpha,\tau}} \tag{A2}$$

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where τ is a partition of width t and height 1, and $c^{\gamma}_{\alpha,\tau}$ are the Littlewood–Richardson coefficients [FH96]. This means we can compute $\pi_*\mathbb{S}^\gamma S$ and $\pi_* i_P^*\mathbb{S}^\gamma S$ by spectral sequences that start with the push-downs of this graded bundle. Thus what we need to calculate is $\pi_*(L^{\otimes t})$ and $\pi_* i_P^*(L^{\otimes t})$ for $t \geqslant 0$.

Fix a point $q \in \mathcal{Q}$; the fibre $\mathcal{P}_q$ over this point is the affine stack (A1). The restriction of L to $\mathcal{P}_q$ is the negative tautological line bundle, so any positive powers of it have no (derived) global sections. Since this is true at all fibres it implies that $\pi_*(L^{\otimes t}) = 0$ for $t > 0$. Similarly, $\pi_*(\mathcal{O}) = \mathcal{O}$. The fibres of P, on the other hand, are projective spaces, so we still have $\pi_* i_P^*\mathcal{O} = \mathcal{O}$ and $\pi_* i_P^*(L^{\otimes t}) = 0$ for $0 < t < d-r+1$, but when $t \geqslant d-r+1$ we have a single higher push-down sheaf. In particular, we have

$$\pi_* i_P^*(L^{\otimes d-r+1}) = \det H^\vee[r-d].$$

(i) Apply π_* to (A2). Only the degree-zero piece $\mathbb{S}^\gamma H$ survives, and the spectral sequence collapses.

(ii) By the width restriction on γ, no powers of L above $L^{\otimes d-r}$ occur in (A2), so when we apply $\pi_* i_P^*$ again only the piece $\mathbb{S}^\gamma H$ survives.

(iii) By the Littlewood–Richardson rule (or the simpler Pieri rule [FH96]), the degree $d-r+1$ piece of (A2) is $\mathbb{S}^{\tilde\gamma} H \otimes L^{\otimes d-r+1}$, and there are no pieces of higher degree. So when we apply $\pi_* i_P^*$ we get two surviving terms, the spectral sequence collapses, and $\pi_* i_P^*\mathbb{S}^\gamma S$ is an extension as claimed. $\square$

We now say a little more about the second map that occurs in the exact triangle in (iii). This is used in the proof of Lemma 3.20. Let γ have width $d-r+1$. We observed that the filtration (A2) concludes with a natural map

$$q : \mathbb{S}^\gamma S \longrightarrow \mathbb{S}^{\tilde\gamma} H \otimes L^{\otimes d-r+1}$$

where, as before, $\tilde\gamma$ is γ with its first row removed. The nature of this map is clearer if we switch to Schur powers of the dual bundles. Let

$$\begin{aligned}\delta &= \mathrm{Comp}^r_{d-r+1}(\gamma)\\ &= \mathrm{Comp}^{r-1}_{d-r+1}(\tilde\gamma).\end{aligned}$$

Then

$$\mathbb{S}^\gamma S = \mathbb{S}^\delta S^\vee \otimes (\det S)^{r-d-1}$$

and

$$\begin{aligned}\mathbb{S}^{\tilde\gamma} H \otimes L^{\otimes d-r+1} &= \mathbb{S}^\delta H^\vee \otimes (\det H)^{r-d-1} \otimes L^{\otimes d-r+1}\\ &= \mathbb{S}^\delta H^\vee \otimes (\det S)^{r-d-1}.\end{aligned}$$

The map q is the tautological map from $\mathbb{S}^\delta S^\vee$ to $\mathbb{S}^\delta H^\vee$, twisted by a line bundle. We also have that

$$\mathbb{S}^{\tilde\gamma} H \otimes L^{\otimes d-r+1} = \pi^!(\mathbb{S}^{\tilde\gamma} H \otimes \det H^\vee[r-d]).$$

Now restrict to the space P, and use the π_*–$\pi^!$ adjunction. The map q gets sent to the map

$$\pi_* i_P^*\mathbb{S}^\gamma S \longrightarrow \mathbb{S}^{\tilde\gamma} H \otimes \det H^\vee[r-d]$$

that occurs in the statement of (iii).

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A.2 The right-hand square

We consider the affine stack

$$\mathcal{T} = [\mathrm{Hom}(V, S)/\mathrm{GL}(S)].$$

We also consider a second stack

$$\bar{\mathcal{S}} = [\mathrm{Hom}(V, H) \oplus \mathrm{Hom}(H, S)/\mathrm{GL}(H) \times \mathrm{GL}(S)],$$

and let $\mathcal{S} \subset \bar{\mathcal{S}}$ be the open substack where the map from H to S is an injection. We let j be the map

$$j : \mathcal{S} \longrightarrow \mathcal{T}$$

given by composing the two factors and forgetting the $\mathrm{GL}(H)$ action.

As in the body of the paper, we write $\mathcal{O}(1) := \det S^\vee$, and $\mathcal{O}\langle 1\rangle := \det H^\vee$. Then

$$j^!(-) = j^*(-) \otimes K_j[\dim j] = j^*(-)(r-d-1)\langle d-r\rangle[r-d-1] \tag{A3}$$

(recalling that $\det V$ is trivialized), which of course agrees with (17).

The image of j is the degenerate locus in $\mathcal{T}$ where the rank of the linear map has dropped. More specifically, if we fix a point $t \in \mathcal{T}$ then we have a vector space C defined as the cokernel

$$V \longrightarrow S \longrightarrow C \longrightarrow 0.$$

Generically this will be zero-dimensional, and it will jump up in dimension for non-generic t. The fibre of $\mathcal{S}$ over t is the projective space $\mathcal{S}_t = \mathbb{P}^\vee C$ of hyperplanes in C.

Lemma A.3. *Let δ be any Young diagram. Then $j_*\mathbb{S}^\delta H^\vee$ is just a sheaf, i.e. there are no higher push-down sheaves.*

Proof. Pick $t \in \mathrm{Hom}(V, S)$. The restriction of H to the fibre $\mathcal{S}_t$ is isomorphic to

$$\tilde{H} \oplus \mathcal{O}^{\oplus rk(t)}$$

where $\tilde{H}$ is the tautological subbundle on $\mathbb{P}^\vee C$. Thus the restriction of $\mathbb{S}^\delta H^\vee$ to $\mathcal{S}_t$ is a non-negative bundle, and has no higher cohomology. Since this is true at all fibres, the higher push-down sheaves vanish. □

We will construct a locally-free resolution of the torsion sheaf $j_*\mathbb{S}^\delta H^\vee$, for certain δ. In order to describe this resolution, we first need to introduce some combinatorics with Young diagrams.

Algorithm A.4. *Let δ be a Young diagram of height less than r. We define a sequence of Young diagrams $\delta_1, \delta_2, \dots$ starting from $\delta_0 := \delta$, by the following procedure:*

- *δ_1 is obtained from δ_0 by adding boxes to the first column until it reaches height r;*
- *δ_k is obtained from δ_{k-1} by adding boxes to the kth column, until its height is one more than the height of the $(k-1)$th column of δ_0.*

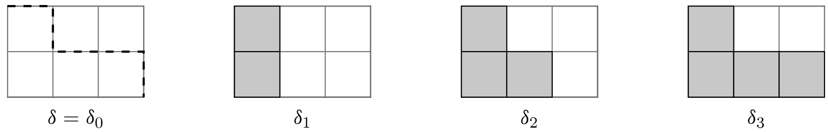
We let s_k denote the total number of boxes added at stage k, i.e. s_k is the difference in size between δ_k and δ_0.

Remark A.5. In Algorithm A.4, the last box added at stage k is immediately to the right of the first box added at stage $k-1$.

Lemma A.6. *Writing h_k for the height of the kth column of a Young diagram δ of height less than r given by $\delta = (\delta^1, \dots, \delta^{r-1})$, we have*

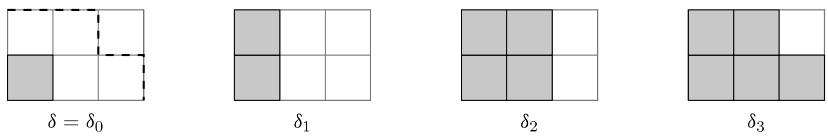
$$\delta_k = (\delta^1, \dots, \delta^{h_k}, k, \delta^{h_k+1} + 1, \dots, \delta^{r-1} + 1).$$

Proof. Induction. □



$$0 \longrightarrow \mathrm{Sym}^2 S^\vee(1) \otimes \wedge^4 V \longrightarrow S^\vee(1) \otimes \wedge^3 V \longrightarrow \mathcal{O}(1) \otimes \wedge^2 V \longrightarrow \mathcal{O} \longrightarrow j_* \mathcal{O} \longrightarrow 0$$

Figure A1. Exact sequence obtained from Theorem A.7 by setting $r = 2$, $d = 4$ and δ to be the empty partition.



$$0 \longrightarrow S^\vee(2) \otimes \wedge^4 V \longrightarrow \mathcal{O}(2) \otimes \wedge^3 V \longrightarrow \mathcal{O}(1) \otimes V \longrightarrow S^\vee \longrightarrow j_* H^\vee \longrightarrow 0$$

Figure A2. Exact sequence from Theorem A.7 for $r = 2$, $d = 4$ and $\delta = (1, 0)$.

Now we can state the following theorem, whose proof is given in § A.3.

Theorem A.7. *Let δ be a Young diagram of height less than r and width less than or equal to $d - r + 1$. We have an exact sequence of sheaves on $\mathcal{T}$,*

$$0 \longrightarrow \mathbb{S}^{\delta_K} S^\vee \otimes \wedge^{s_K} V \longrightarrow \cdots \longrightarrow \mathbb{S}^{\delta_1} S^\vee \otimes \wedge^{s_1} V \longrightarrow \mathbb{S}^{\delta_0} S^\vee \longrightarrow j_* \mathbb{S}^\delta H^\vee \longrightarrow 0,$$

where $K = d - r + 1$, and the δ_k and s_k are defined in Algorithm A.4.

We give some simple examples showing how Theorem A.7 reproduces certain exact sequences used in § 2.1.2.

Example A.8. Set $r = 2$, $d = 4$, and let δ be the empty partition. The partitions δ_k and associated resolution are shown in Figure A1. The dashed line indicates the positions of the boxes added to δ to produce the various δ_k. Restricting the resolution to the full rank locus of $\mathcal{T}$, we obtain a long exact sequence which is the Eagon–Northcott complex used in § 2.1.2.

Example A.9. Set $r = 2$, $d = 4$ again, and let $\delta = (1, 0)$. The partitions and resolution are shown in Figure A2. Restricting to the full rank locus of $\mathcal{T}$, we obtain the Buchsbaum–Rim complex used in § 2.1.2.

We make a few elementary observations on the combinatorics: these all follow from the size restrictions on δ.

Remark A.10.

(i) We have $s_K \leqslant d$ but $s_{K+1} > d$, which explains (at least formally) why the resolution terminates at K terms.

(ii) The height of δ_k is r for $k > 0$, and less than r for $k = 0$. The width of δ_k is less than or equal to $d - r + 1$ for all $k \leqslant K$.

Additionally, if the width of δ is less than $d - r + 1$ then:

(iii) $s_K = d$;

(iv) the width of δ_k is less than $d - r + 1$ for $k < K$, and the width of δ_K is $d - r + 1$;

(v) if we delete the first row and the first column from δ_K then we recover the diagram δ.

On the other hand, if the width of δ is equal to $d-r+1$ then:

(vi) the width of δ_k is $d-r+1$ for all $k \leqslant K$.

Remark (ii) implies that we can define

$$\epsilon_k = \mathrm{Comp}^r_{d-r+1}(\delta_k) \tag{A4}$$

for $0 \leqslant k \leqslant K$. Then the following corollary is immediate using (A3).

Corollary A.11. *For δ as before in Theorem A.7, $j^! j_* \mathbb{S}^\delta H^\vee$ is quasi-isomorphic to the complex*

$$\{\, \underline{\mathbb{S}^{\epsilon_K} S\langle d-r\rangle \otimes \wedge^{s_K} V} \longrightarrow \ \cdots \ \longrightarrow \mathbb{S}^{\epsilon_1} S\langle d-r\rangle \otimes \wedge^{s_1} V \longrightarrow \mathbb{S}^{\epsilon_0} S\langle d-r\rangle \}.$$

Taking complements in Remark A.10 prompts the following remark.

Remark A.12. (ii*) The width of ϵ_k is $d-r+1$ for $k=0$, and less than $d-r+1$ for $k>0$. The height of ϵ_k is less than or equal to r for all k.

Also if the width of δ is less than $d-r+1$ then:

(iii*) $s_K = d$;

(iv*) the height of ϵ_k is r for $k<K$, and the height of ϵ_K is less than r;

(v*) $\epsilon_K = \mathrm{Comp}^{r-1}_{d-r}(\delta)$.

If the width of δ is equal to $d-r+1$ then:

(vi*) the height of ϵ_k is less than or equal to $r-1$ for all k.

We end this section with some observations on the spaces of maps between various bundles on $\mathcal{S}$ and $\mathcal{T}$.

Lemma A.13. *For any partition δ, we have:*

(i)

$$\mathrm{Ext}^0_{\mathcal{S}}(\mathbb{S}^\delta S, \mathbb{S}^\delta S) = \mathbb{C}$$

i.e. any map from this bundle to itself is a scalar multiple of the identity;

(ii)

$$\mathrm{Ext}^0_{\mathcal{S}}(\mathbb{S}^\delta H, \mathbb{S}^\delta S) = \mathbb{C}$$

i.e. any map between these two bundles is a scalar multiple of the tautological map.

Proof. We can work on $\bar{\mathcal{S}}$ instead, since the complement of $\mathcal{S}$ has codimension at least 2. Then for (i) we just have to compute the $\mathrm{GL}(H) \times \mathrm{GL}(S)$ invariants in

$$\mathbb{S}^\delta S^\vee \otimes \mathbb{S}^\delta S \otimes \mathrm{Sym}(V \otimes H^\vee \oplus H \otimes S^\vee).$$

This is an easy calculation using the Littlewood–Richardson rule and the facts that for any vector spaces A and B we have

$$\mathrm{Sym}(A \otimes B) = \bigoplus_{\lambda,\mu} \mathbb{S}^\lambda A \otimes \mathbb{S}^\mu B$$

[Wey03, Theorem 2.3.2] and

$$(\mathbb{S}^\lambda A^\vee \otimes \mathbb{S}^\mu A)^{\mathrm{GL}(A)} = \begin{cases} \mathbb{C} & \lambda = \mu, \\ 0 & \lambda \neq \mu. \end{cases}$$

Part (ii) is identical. $\square$

Lemma A.14. *Let $\delta_0, \delta_1, \ldots$ be a sequence of Young diagrams constructed by Algorithm A.4 above. Then for any k,*

$$\mathrm{Hom}_{\mathcal{T}}(\mathbb{S}^{\delta_{k+1}} S^\vee \otimes \wedge^{s_{k+1}} V, \mathbb{S}^{\delta_k} S^\vee \otimes \wedge^{s_k} V)^{\mathrm{SL}(V)} = \mathbb{C},$$

i.e. the maps in the sequence in Theorem A.7 are determined (up to scalar multiples) by the requirement of $\mathrm{SL}(V)$-equivariance.

Proof. The calculation is very similar to those in Lemma A.13. Note that the Littlewood–Richardson coefficient $c^{\delta_{k+1}}_{\lambda, \delta_k}$ is zero unless λ is the 'column' $(1, 1, \ldots, 1)$ of height $s_{k+1} - s_k$, in which case it equals 1. □

We do not actually use Lemma A.14, but it is interesting to note. It can also be considered a warm-up for the next lemma, which is more technical and is used in the proof of Lemma 3.18.

Lemma A.15. *Let δ have height less than r and width equal to $d-r+1$, and let $\delta_0, \delta_1, \ldots$ be the corresponding sequence of Young diagrams. As above, let $K = d-r+1$, and $\epsilon_K = \mathrm{Comp}^r_{d-r+1}(\delta_K)$. Then*

$$\mathrm{Ext}^0_{\mathcal{S}}(\mathbb{S}^\delta H^\vee, \mathbb{S}^{\epsilon_K} S\langle d-r\rangle \otimes \wedge^{s_K} V)^{\mathrm{SL}(V)} = \mathbb{C}.$$

Proof. As in Lemma A.13 we can work on $\bar{\mathcal{S}}$, and this is a computation of invariants. After taking $\mathrm{GL}(S)$-invariants, we are left with

$$\bigoplus_\lambda \mathbb{S}^\delta H \otimes \mathbb{S}^{\epsilon_K} H \otimes \mathbb{S}^\lambda H^\vee \langle d-r\rangle \otimes \mathbb{S}^\lambda V \otimes \wedge^{s_K} V. \tag{A5}$$

Now consider the expression

$$\begin{aligned} \mathbb{S}^\delta H \otimes \mathbb{S}^{\epsilon_K} H \otimes \mathbb{S}^\lambda H^\vee \langle d-r\rangle &= \mathbb{S}^\delta H\langle d-r+1\rangle \otimes \mathbb{S}^{\epsilon_K} H\langle -1\rangle \otimes \mathbb{S}^\lambda H^\vee \\ &= \mathbb{S}^{\tilde{\epsilon}_0} H^\vee \otimes \mathbb{S}^{\tilde{\epsilon}_K} H \otimes \mathbb{S}^\lambda H^\vee \end{aligned}$$

where $\tilde{\epsilon}_0 = \mathrm{Comp}^{r-1}_{d-r+1}(\delta)$, and $\tilde{\epsilon}_K$ is the diagram obtained from ϵ_K by adding on an extra column of height $r-1$, which is well defined because the height of ϵ_K is less than or equal to $r-1$ (Remark A.12(vi*)). Then the $\mathrm{GL}(H)$-invariants in (A5) are

$$\bigoplus_\lambda \mathbb{S}^\lambda V \otimes \wedge^{s_K} V^{\oplus c^{\tilde{\epsilon}_K}_{\tilde{\epsilon}_0, \lambda}}. \tag{A6}$$

Now let h_K be the height of the Kth column of δ, and use Lemma A.6 to deduce that

$$\tilde{\epsilon}_K = (K - \delta^{r-1}, \ldots, K - \delta^{h_K+1}, 1, \ldots, 1)$$

where the number of rows is $r-1$. But by definition

$$\tilde{\epsilon}_0 = (K - \delta^{r-1}, \ldots, K - \delta^{h_K+1})$$

and so the Littlewood–Richardson coefficient $c^{\tilde{\epsilon}_K}_{\tilde{\epsilon}_0, \lambda}$ is equal to 1 if λ is the column $(1, \ldots, 1)$ of height h_k, and equal to 0 otherwise. Hence (A6) is just

$$\wedge^{h_K} V \otimes \wedge^{s_K} V$$

and this contains a one-dimensional space of $\mathrm{SL}(V)$-invariants, since $h_K + s_K = d$. □

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A.3 Locally-free resolutions

We now prove Theorem A.7, which turns out to be an extreme case of the twisted Lascoux resolution [Wey03, § 6.1]. Weyman gives this resolution implicitly: we present a Borel–Weil–Bott calculation that makes it explicit, as required for our purposes. We could not find this given in the literature, but note that [Fon13] uses the same combinatorics to produce exact sequences on Grassmannians. We briefly review Weyman's construction, slightly modifying his language with the aim of providing a bridge between his account and our application.

Let G be a linearly reductive group, P a parabolic subgroup of G, and T a vector space with a G-action. We also choose a subspace U of T with a compatible P-action. Consider a diagram

$$\begin{tikzcd} U \times_P G \arrow[d] \arrow[dr, "j"] & \\ \operatorname{Im} j \arrow[r, hook] & T \end{tikzcd}$$

where j takes $(u, g) \longmapsto (gu)$. Say we are interested in obtaining resolutions of torsion sheaves on T, supported on $\operatorname{Im} j$, which are obtained as direct images under the map j: [Wey03] uses this setup to calculate syzygies on determinantal varieties, and it turns out to be what we require also. Following [Wey03, §§ 5.1 and 5.4], we form a diagram as follows, factoring j into an embedding i and a flat projection q:

$$\begin{tikzcd} U \times_P G \arrow[r, hook, "i"] \arrow[dr, "j"'] & T \times G/P \arrow[r, "p"] \arrow[d, "q"] & G/P \\ & T & \end{tikzcd} \tag{A7}$$

Here i takes $(u, g) \longmapsto (gu, gP)$.

The relevant result from Weyman is given in the following theorem.

Theorem A.16 [Wey03, Theorem 5.4.1]. *Take a vector bundle $\mathcal{E}$ on G/P induced from a representation of P, and assume that $j_* i^* p^* \mathcal{E}$ is a sheaf on T (i.e. it has no higher push-downs). Then this sheaf has a G-equivariant resolution given by $\mathcal{F}_\bullet$, where*

$$\mathcal{F}_k := \bigoplus_{j \geqslant 0} \mathbb{R}^j q_* (\wedge^{k+j} ((T/U)^\vee) \otimes p^* \mathcal{E}).$$

Remark A.17. The proof uses the commutativity of (A7) and the projection formula to rewrite the sheaf in question as

$$j_* i^* p^* \mathcal{E} = q_* i_* i^* p^* \mathcal{E} = q_* (\mathcal{O}_{\operatorname{Im} i} \otimes p^* \mathcal{E})$$

and then evaluates this using the Koszul resolution of $\mathcal{O}_{\operatorname{Im} i}$. The relevant spectral sequence simplifies because of equivariance considerations [Wey03, § 5.2].

Remark A.18. Note that the bundle $\mathcal{E}$ on G/P is the 'twist' in the twisted Lascoux resolution noted above.

Now we are ready for the following proof.

Proof of Theorem A.7. We take $G = \mathrm{GL}(S)$, choose an inclusion $H \hookrightarrow S$, and define P as the parabolic preserving H. Then we take T to be underlying vector space of our stack $\mathcal{T}$, i.e. $T := \operatorname{Hom}(V, S)$ with its G-action. T acquires a P-action, compatible with the natural P-action on $U := \operatorname{Hom}(V, H)$.

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We apply the construction explained above to obtain a diagram

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "j"'] & \mathrm{Hom}(V,S)\times G/P \arrow[r, "p"] \arrow[d, "q"] & G/P \\ & \mathrm{Hom}(V,S) & \end{tikzcd}$$

where Z is $\mathrm{Hom}(V,H)\times_P G$.

We now just need to reinterpret this in our stacky language: in particular, we relate Z and our stack $\mathcal{S}$ from § A.2. Firstly, we have an equivalence of categories of sheaves on the variety G/P and on the open substack of

$$[\mathrm{Hom}(H,S)/\mathrm{GL}(H)]$$

where the map from H to S is an injection (these are two alternative descriptions of the projective space $\mathbb{P}^\vee S$). It follows that we have a similar equivalence relating Z and the open substack of

$$[\mathrm{Hom}(V,H)\oplus \mathrm{Hom}(H,S)/\mathrm{GL}(H)]$$

where the map from H to S is an injection. Hence we see that working G-equivariantly with the morphism j in (A7) is just the same as working with the morphism $j:\mathcal{S}\to\mathcal{T}$ as defined in § A.2.

Now we consider the representation $\mathbb{S}^\delta H^\vee$ of P. This induces a sheaf on G/P and also on $\mathcal{S}$, as appears in the statement of Theorem A.7. We want then to resolve $j_*i^*p^*\mathbb{S}^\delta H^\vee$. Lemma A.3 gives that there are no higher push-down sheaves, so Theorem A.16 immediately gives us a G-equivariant resolution $\mathcal{F}_\bullet$ on $T=\mathrm{Hom}(V,S)$ where

$$\mathcal{F}_k := \bigoplus_{j\geqslant 0}\mathbb{R}^j q_*(\wedge^{k+j}(V\otimes (S/H)^\vee)\otimes \mathbb{S}^\delta H^\vee).$$

This yields the required resolution on $\mathcal{T}$. Lemma A.19 below shows that these push-forwards evaluate to the terms given in the statement of Theorem A.7 in § A.2. □

Lemma A.19. *For* $0\leqslant k\leqslant K := d-r+1$ *we have*

$$\mathcal{F}_k = \mathbb{S}^{\delta_k}S^\vee\otimes \wedge^{s_k}V$$

where δ_k *and* s_k *are defined in Algorithm A.4, and* $\mathcal{F}_k=0$ *otherwise.*

Proof. Rearranging to give

$$\mathcal{F}_k := \bigoplus_{i\geqslant k}\mathbb{R}^{i-k} q_*((S/H)^{\vee i}\otimes \mathbb{S}^\delta H^\vee)\otimes \wedge^i V,$$

it suffices to work fibrewise and evaluate

$$H^\bullet(\mathrm{GL}(S)/P,(S/H)^{\vee i}\otimes \mathbb{S}^\delta H^\vee).$$

We explain how to calculate this cohomology group explicitly using the Borel–Weil–Bott theorem [Wey03, § 4.1]. According to the standard prescription (see, for example, [Wey03, Corollary 4.1.9]), the bundle in question corresponds to a $\mathrm{GL}(S)$-weight

$$\alpha(i) := (\delta^1,\ldots,\delta^{r-1},i).$$

First we note that if $\alpha=\alpha(i)$ is *dominant* (i.e. given by a sequence of non-increasing integers) then we have $H^0=\mathbb{S}^\alpha S^\vee$ and $H^{>0}=0$. More generally, the theorem determines the cohomology,

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which occurs in at most one degree, according to the behaviour of the weight α under the twisted action of the Weyl group $W = S_r$. For $w \in S_r$ this action is given by

$$w \bullet \alpha := w(\alpha + \rho) - \rho$$

where $\rho := (r, \dots, 2, 1)$. We say that α is *regular* if there exists a unique $w \in S_r$ such that $w \bullet \alpha$ is dominant. We then have three mutually exclusive cases, with the theorem giving the cohomology in each:

(1) α dominant	$\implies$	$H^0 = \mathbb{S}^{\alpha} S^{\vee}$
(2) α regular, non-dominant	$\implies$	$H^l = \mathbb{S}^{w\bullet\alpha} S^{\vee},\ l = \text{length}(w)$
(3) α non-regular	$\implies$	$H^{\bullet} = 0.$

As i varies we classify the weight $\alpha(i)$ as follows.

– *Case* 1: $\alpha(i)$ *dominant* if $0 \leqslant i \leqslant \delta^{r-1}$.
  This is immediate: dominant $\mathrm{GL}(S)$-weights correspond precisely to non-increasing integer sequences.
– *Case* 2: $\alpha(i)$ *regular, non-dominant* if there exists a natural number $l \leqslant r-1$ such that

$$\delta^{r-l} < i - l \leqslant \delta^{r-l-1}. \tag{A8}$$

  (Here for convenience we set $\delta^0 = \infty$ so that when $l = r-1$ the second inequality is redundant.)
  In this case the cycle $w = (r-l\ \dots\ r)$ gives

$$w \bullet \alpha(i) = (\delta^1, \dots, \delta^{r-l-1}, i-l, \delta^{r-l}+1, \dots, \delta^{r-1}+1)$$

  which is dominant by condition (A8).
  Now the crucial point is to observe that in fact $w \bullet \alpha(i) = \delta_{i-l}$, one of the Young diagrams obtained by applying Algorithm A.4. This follows from the description of the δ_k in Lemma A.6 after noting that the height of the $(i-l)$th column of $w \bullet \alpha(i)$ is given by $r-l-1$ by condition (A8).
  We also observe that $s_{i-l} = i$, because $w \bullet \alpha(i) = \delta_{i-l}$ has the same number of boxes as $\alpha(i)$, which is i more than the number of boxes in δ.
– *Case* 3: $\alpha(i)$ *non-regular* if there exists a natural number $l < r-1$ such that

$$i - l = \delta^{r-l}. \tag{A9}$$

  In this case the transposition exchanging $r-l$ and r stabilizes $\alpha(i)$ under the twisted Weyl group action, because

$$(\alpha(i)+\rho)^{r-l} := \delta^{r-l} + l + 1 = i + 1 =: (\alpha(i)+\rho)^r$$

  by condition (A9).

In summary, we have that if $\alpha(i)$ is regular (the first two cases), then there exists a (possibly trivial) Weyl group element $w(i)$ with length $l(i)$ such that:

(i) $w(i) \bullet \alpha(i) = \delta_{i-l(i)}$;
(ii) $s_{i-l(i)} = i$.

We see then that $\alpha(i)$ contributes to $\mathcal{F}_{\bullet}$ via homology in degree $l(i)$ and thence to $\mathcal{F}_k$ when $i - k = l(i)$. This occurs precisely when $k = i - l(i)$, and for each k this equation has a unique solution for i, as non-uniqueness would contradict $s_{i-l(i)} = i$. Hence we deduce that

$$\mathcal{F}_k = \mathbb{R}^{l(i)} q_*((S/H)^{\vee i} \otimes \mathbb{S}^{\delta} H^{\vee}) \otimes \wedge^i V.$$

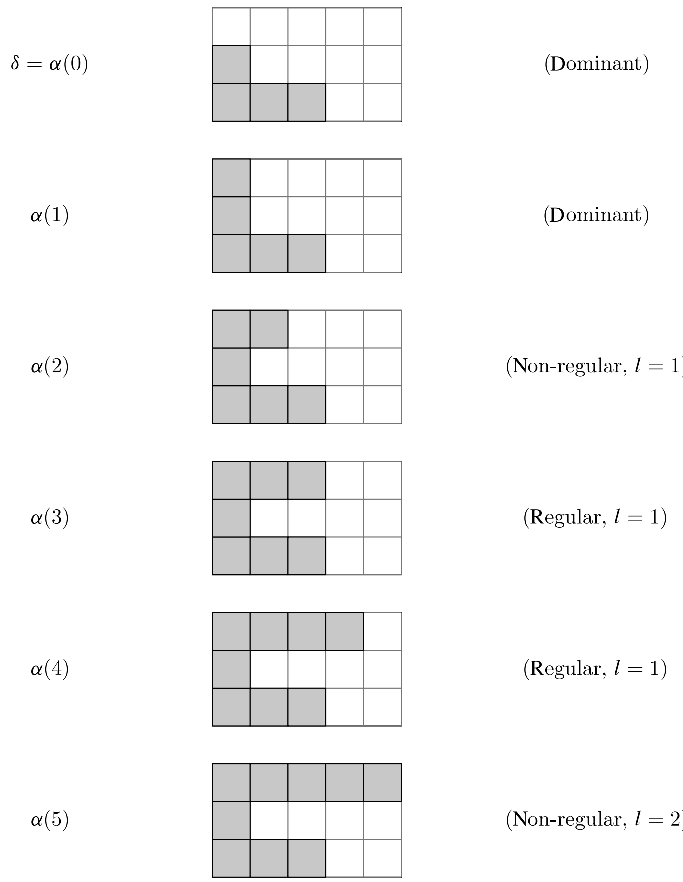

Figure A3. Cases arising for $\alpha(i)$ in the proof of Lemma A.19.

The required push-down then comes from

$$\begin{aligned} H^{l(i)}(\mathrm{GL}(S)/P,\ (S/H)^{\vee i} \otimes \mathbb{S}^{\delta} H^{\vee}) &= \mathbb{S}^{w(i)\bullet\alpha(i)} S^{\vee} \\ &= \mathbb{S}^{\delta_{i-l(i)}} S^{\vee} \\ &= \mathbb{S}^{\delta_k} S^{\vee} \end{aligned}$$

and noting that $i = s_{i-l(i)} = s_k$ gives the result. □

Example A.20. We illustrate in Figure A3 how the three cases in the proof of Lemma A.19 occur in the example $\delta = (3, 1)$ with $r = 3$. We give diagrams corresponding to the $\mathrm{GL}(S)$-weights $\alpha(0), \ldots, \alpha(5)$, with the row lengths of the diagrams corresponding to components of the respective weights. Note that in this example $\alpha(6)$, $\alpha(7)$, $\ldots$ are regular with $l = 2$.

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