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Relative equilibria of the four-body problem

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Abstract. By employing a regularizing transformation, the problem of bifurcation of relative equilibria in the Newtonian 4-body problem is reduced to a study of an algebraic correspondence between real algebraic varieties. The finiteness theorems of algebraic geometry are used to find an upper bound for the number of affine equivalence classes of relative equilibria which holds for all masses in the complement of a proper, algebraic subset of the space of all masses.

1. *Introduction*; *relative equilibria of the N-body problem*

The Newtonian N-body problem concerns the motion of N point masses under the influence of their mutual gravitational attraction. We will consider the case when all the particles move in a fixed plane which we take to be the complex plane, $\mathbb{C}$. The position vector $z \in \mathbb{C}^N$ is the vector $(z_1, \ldots, z_n)$ where $z_j = x_j + iy_j \in \mathbb{C}$ is the position of the jth particle. The mass vector $m \in \mathbb{R}^{n+}$ is $(m_1, \ldots, m_N)$ where m_j is the mass of the jth particle. In complex notation, Newton's laws are:

$$m_j\ddot{z}_j = 2\frac{\partial U}{\partial \bar{z}_j}$$

where

$$U = \sum_{k<l} \frac{m_k m_l}{|z_k - z_l|} \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

We require $z \notin \Delta$, where $\Delta = \{z\colon z_k = z_l \text{ for some } k \neq l\}$.

Let $A_1(\mathbb{C})$ be the affine group, i.e. the group of all affine transformations $w \to aw + b$, where $w, b \in \mathbb{C}$ and $a \in \mathbb{C} \setminus 0$. $A_1(\mathbb{C})$ acts componentwise on $\mathbb{C}^N \setminus \Delta$ and Newton's equations are invariant (up to scale) under the action. We will use the term *configuration* to mean an affine equivalence class of position vectors. Within a configuration the shape of the array of points is constant; only the size and position varies.

An *affine motion* of N points is a curve $z(t)$ in $\mathbb{C}^N \setminus \Delta$ given by

$$z_j(t) = a(t)z_j(0) + b(t); \qquad a(t) \in \mathbb{C} \setminus 0,\ b(t) \in \mathbb{C}.$$

Thus, during an affine motion, the configuration is constant.

Definition. A *relative equilibrium* of the N-body problem with mass vector m is a position vector $z \in \mathbb{C}^N \setminus \Delta$ with the property: there exists an affine motion $z(t)$ with $z(0) = z$ which satisfies Newton's equations.

Note that if z is a relative equilibrium so is every position vector which is affine-equivalent to z. So it is sensible to speak of relative equilibrium configurations.

Following Siegel and Moser we derive the equations for relative equilibria [21].

The potential function U has the property that $\sum_j \partial U/\partial \bar{z}_j = 0$. Then Newton's equations applied to an affine motion give:

$$0 = \sum_j m_j \ddot{z}_j = \ddot{a} \sum_j m_j z_j + \ddot{b} \sum_j m_j.$$

Let $M = \sum_j m_j$ and $c = M^{-1} \sum_j m_j z_j$, the centre of mass of the initial position vector. then the equation becomes:

$$\ddot{b} = -c\ddot{a}. \tag{1.1}$$

Consequently,

$$\ddot{z}_j = \ddot{a}(z_j - c).$$

If we substitute this into Newton's equations we find:

$$\ddot{a}(z_j - c) = -\frac{a}{|a|^3} \sum_{k \neq j} \frac{m_k(z_j - z_k)}{|z_j - z_k|^3}.$$

Here z is the time-independent initial position vector. It follows that $|a|^3 a^{-1} \ddot{a}$ is time-independent. Let Λ be its constant value. Thus

$$\ddot{a} = \frac{\Lambda a}{|a|^3} \tag{1.2}$$

and

$$\Lambda(z_j - c) = -\sum_{k \neq j} \frac{m_k(z_j - z_k)}{|z_j - z_k|^3}. \tag{1.3}$$

Equation (1.3) is a necessary and sufficient condition for a position vector z to be a relative equilibrium. Equations (1.1) and (1.2) determine the corresponding affine motions.

Equation (1.2) can be written

$$\ddot{a} = 2\frac{\partial V}{\partial \bar{a}}$$

with $V = -\Lambda/|a|$. This is just the Kepler problem in complex notation. Therefore the possible motions $a(t)$ are motions on conic sections with one focus at the origin obeying Kepler's laws. The resulting affine motion $z(t)$ can be described as follows. The centre of mass of the N particles moves uniformly along some line in $\mathbb{C}$. The motion of each point relative to the centre of mass is on a conic section and obeys Kepler's laws. All of the N conic sections are similar and the position $z(t)$ of the N particles is always similar to the initial position vector z. As a special case we can take the circular solution of the Kepler problem. So if z is a relative equilibrium, one possible motion is that the array of particles rotates uniformly around their centre of mass. Relative to a rotating coordinate system with origin at the centre of mass, the position z would be an equilibrium.

When $N = 2$, all positions z are equivalent under $A_1(\mathbb{C})$ so every z is a relative equilibrium. Equation (1.3) holds automatically in this case. The discussion above is just the reduction of the 2-body problem to the Kepler problem. When $N = 3$ there are just five affine equivalence classes of relative equilibria [21]. For any choice of m the two equilateral triangular configurations (corresponding to clockwise or counter-clockwise arrangement of the points) are relative equilibria. In addition, for each of the three possible orders of the points along a line there is a unique collinear relative equilibrium. The spacing of the points depends on the choice of m. For $N \geq 4$ there are no configurations which are relative equilibria for all masses [**29**].

The collinear relative equilibria were discovered by Euler in 1767 [**4**]. Lagrange found the equilateral solutions in 1772 [**7**]. It seems fair to say that when $N \geq 4$, equation (1.3) has eluded explicit solution. For example, it is not known how many relative equilibria there are for the 4-body problem with equal masses.

By exploiting symmetries to reduce the number of variables, many special solutions have been found. Moulton has proved the existence of a unique collinear relative equilibrium for each of the $\frac{1}{2}N!$ affine-inequivalent orderings of the N points on the line [**13**]. The existence of at least one equilibrium for each ordering was proved in 1891 [**8**]. A regular polygon is a relative equilibrium in the case of N equal masses and only in this case [**18**], [**29**]. More intricate configurations involving nested polygons have been shown to be relative equilibria when the masses on each polygon are equal [**1**], [**6**], [**10**].

Morse theory has been used to derive lower bounds for the number of relative equilibria valid for arbitrary masses [**15**], [**16**], [**24**]. It is possible to apply Morse theory because equation (1.3) can be interpreted as the equation for critical points of $U(z)$ subject to the constraint:

$$\sum_j m_j |z_j - c|^2 = 1. \tag{1.4}$$

Here Λ plays the role of a Lagrange multiplier. After passing to a quotient manifold of the manifold (1.4) under the action of the affine group one can use Morse theory to prove the existence of non-collinear relative equilibria for all N and m. Note that the manifold of interest here is not compact since we had to delete Δ from $\mathbb{C}^N$ at the outset. Thus it is an important fact (due to Shub [**23**]) that for fixed m one can bound the relative equilibria away from Δ. It follows from this that if the critical points of U on the quotient manifold are isolated, then there are only finitely many of them. Since the problem has no other obvious symmetries it is natural to conjecture that this is in fact the case for all, or at least most, m [**24**]. The isolation condition is satisfied for m if the corresponding U has only non-degenerate critical points, i.e. if U is a Morse function. Palmore has shown that this is not always true [**17**]. The set of masses for which he can give examples of degenerate critical points is of codimension one in the set of all masses. Palmore has also announced that the set of masses which admit degenerate critical points has measure zero in the set of all masses [**17**]; however, no proof has appeared.

Much of the work described above deals with the problem of determining for each fixed m the corresponding relative equilibria. Another approach to the subject is to ask which position vectors z are relative equilibria for some m, and then, given such a relative equilibrium, to find all of the corresponding mass vectors. Such a study was carried out by Dziobek [**3**] and by W. D. MacMillan and W. Bartky [**11**] in the case $N=4$ and by W. L. Williams [**28**] in the case $N=5$. Among the results are beautiful characterizations of the set of all non-collinear relative equilibria by explicit equations and uniqueness theorems for the corresponding mass vectors. Part of this paper is devoted to presenting new proofs of these results for $N=4$ (see theorems 1 and 2). We then apply these results to study the bifurcation theory of relative equilibria in the 4-body problem.

The first step is an analysis of the equations for the set of all non-collinear relative equilibria. We are able to prove that the equations are independent and that this set is a real analytic manifold (theorem 4). The uniqueness theorem for mass vectors then provides a mapping from the manifold of relative equilibria to the space of (normalized) mass vectors which is shown to be real analytic and proper. The bifurcation theory reduces to a study of this mapping.

In celestial mechanics, transformations discovered by Levi-Civita [**9**] can be used to eliminate the singularities caused by double collisions. We apply similar transformations to the equations for relative equilibria with the result that the mapping described above is replaced by a polynomial mapping of real, projective varieties. In other words, real analytic equations become polynomial equations. This brings the forces of real algebraic geometry into play.

Our main results could be described as finiteness theorems. Theorem 5 states that the set of all non-collinear relative equilibria has a finite number of components. These components correspond to components of the bifurcation diagram of the problem. The difficulty here is that the parameter space, $\mathbb{R}^{4+}$, is not compact. Now we fix a mass vector $m \in \mathbb{R}^{4+}$ and study the corresponding relative equilibria. Theorem 6 gives a bound for the number of components of this set of relative equilibria which is independent of the choice of m. This shows, in particular, that whenever m admits only finitely many relative equilibrium configurations, the number of configurations cannot exceed this fixed bound. The last result, theorem 7, concerns the bifurcation set. We take this to be the complement of the set of masses which admit only non-degenerate relative equilibrium configurations. We show that the bifurcation set is contained in a proper, real algebraic subset of $\mathbb{R}^{4+}$, i.e. the zero set of polynomials in the masses. Such a set is small in any sense of the word. The complement is open and dense and has full measure. On this complementary set of masses, the fixed bound on the number of relative equilibrium configurations holds.

The proofs of theorems 5 and 6 extend easily to the N-body problem. We do not know whether the analogues of theorems 4 and 7 hold for $N \geq 5$. For more information on the significance of relative equilibria in classical mechanics see [**24**], [**25**], [**20**], [**19**]. For a numerical study of the bifurcation problem when $N=4$ see [**22**].

2. *The set of all relative equilibria*

We saw that $z \in \mathbb{C}^N \setminus \Delta$ is a relative equilibrium for some fixed mass m if and only if there is some $\Lambda \in \mathbb{R}$ such that equation (1.3) holds. From now on we will call z a relative equilibrium if (1.3) is satisfied for some Λ and for some m. We want to characterize the set of all relative equilibria. This set is invariant under the affine

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group and it is sometimes more convenient to consider the set of all configurations (affine equivalence classes) whose representatives are relative equilibria. These will be called relative equilibrium configurations. To study this question it is desirable to transform (1.3) to make the mass m appear in the simplest possible way. To this end we define (following [**11**]):

$$\lambda = M^{-1}\Lambda,$$

where $M = \sum_j m_j$. If we recall that $c = M^{-1}\sum_j m_j z_j$ we find that (1.3) becomes a linear, homogeneous function of m. A little algebra yields

$$Am = 0, \tag{2.1}$$

where A is the $N \times N$ complex matrix with entries

$$A_{jk} = S_{jk}(z_j - z_k), \qquad S_{jk} = |z_j - z_k|^{-3} + \lambda. \tag{2.2}$$

The matrix A is anti-symmetric (not anti-Hermitian).

It is useful to recall some of the theory of anti-symmetric matrices. An anti-symmetric matrix defines a complex-valued two-form ω on $\mathbb{C}^N$ by the formula $\omega(v, w) = v^T A w$. The use of the transpose means we are viewing $\mathbb{C}^N$ as a Euclidean space instead of as a Hermitian space.

Definition. A two-form ω is *degenerate* if there is some $v \in \mathbb{C}^N$ for which the interior product $v \lrcorner \omega = 0$, i.e. if $\omega(v, w) = 0$ for all $w \in \mathbb{C}^N$. A two-form is *decomposable* if it is the product of two one-forms: $\omega = \alpha \wedge \beta$.

The next proposition summarizes some of the theory.

PROPOSITION. *A two-form ω on $\mathbb{C}^N$ is decomposable if and only if $\omega \wedge \omega = 0$. If N is odd, every two-form is degenerate. If $N = 2n$ then ω is degenerate if and only if the volume form $\omega^n = \omega \wedge \cdots \wedge \omega = 0$.*

COROLLARY. *A two-form ω in $\mathbb{C}^4$ is degenerate if and only if it is decomposable if and only if $\omega \wedge \omega = 0$.*

From now on suppose $N = 2n$. Let $e_1, \ldots, e_{2n}$ be the standard basis of $\mathbb{C}^{2n}$ and let $e_1^*, \ldots, e_{2n}^*$ be the dual basis of $\mathbb{C}^{2n*}$. Every volume form is a multiple of $e_1^* \wedge \cdots \wedge e_{2n}^*$.

Definition. The *Pfaffian* of a two-form ω is the unique complex number $\mathrm{pf}(\omega)$ such that

$$\omega^n = \mathrm{pf}(\omega) e_1^* \wedge \cdots \wedge e_{2n}^*.$$

COROLLARY. *ω is degenerate if and only if* $\mathrm{pf}(\omega) = 0$.

Returning to the anti-symmetric matrix A we see that ω is degenerate if and only if $\det(A) = 0$. This leads one to suspect a connection between $\mathrm{pf}(\omega)$ and $\det(A)$.

PROPOSITION. $\det(A) = [\mathrm{pf}(\omega)]^2$

Proof. See [**5**].

In the case of a 4×4 anti-symmetric matrix $A = (a_{ij})$ we have by direct computation of $\omega \wedge \omega$:

$$\mathrm{pf}(\omega) = a_{13}a_{24} - a_{12}a_{34} - a_{14}a_{23} \tag{2.3}$$

and $\det(A) = \mathrm{pf}^2$.

We are going to reformulate the equations for relative equilibria as equations on the space of complex anti-symmetric matrices. To any position vector $z \in \mathbb{C}^N \setminus \Delta$ we associate a decomposable two-form

$$\sigma = z^* \wedge 1^* \tag{2.4}$$

where $z^* = \sum z_j e_j^*$ and $1^* = \sum e_j^*$. In terms of matrices z^* and 1^* are represented by the row vectors $(z_1, z_2, \ldots, z_N)$ and $(1, 1, \ldots, 1)$ respectively and σ is represented by the anti-symmetric matrix with entries $z_j - z_k$. It is easy to characterize, in the space of anti-symmetric $N \times N$ complex matrices, those arising from position vectors z via this construction.

PROPOSITION. *A two-form σ represented by an anti-symmetric matrix (σ_{ij}) is of the form $\sigma = z^* \wedge 1^*$ if and only if $\sigma \wedge 1^* = 0$, or equivalently if the 'cocycle equations' $\sigma_{ik} = \sigma_{ij} + \sigma_{jk}$ hold for all i, j, k.*

Proof. The first part is an application of the 'division theorem' for forms: if θ is a 1-form and α a k-form then $\alpha \wedge \theta = 0$ if and only if $\alpha = \eta \wedge \theta$ for some $(k-1)$-form η. The proof of this is elementary. The cocycle equations result from expressing $\sigma \wedge 1^*$ in the standard basis for 3-forms and setting all coefficients to zero.

The vector space of 2-forms on $\mathbb{C}^N$ is isomorphic to $\mathbb{C}^{N(N-1)/2}$. The map $z \to z^* \wedge 1^*$ of $\mathbb{C}^N$ into $\mathbb{C}^{N(N-1)/2}$ has a one-dimensional kernel (spanned by 1) and so the image has dimension $N-1$. The image is the zero set of the cocycle equations. However only $\frac{1}{2}N(N-1)-(N-1)$ of the $\binom{N}{3}$ cocycle equations are independent. For $N=4$ the space of 2-forms is 6-dimensional and 3 of the cocycle equations are independent. □

PROPOSITION. *A 2-form σ on $\mathbb{C}^4$ is of the form $z^* \wedge 1^*$ for some $z \in \mathbb{C}^4$ if and only if*

$$\text{(C)} \qquad \sigma_{13} = \sigma_{12} + \sigma_{23}, \quad \sigma_{14} = \sigma_{12} + \sigma_{24}, \quad \sigma_{14} = \sigma_{13} + \sigma_{34}.$$

Let Δ denote the set of 2-forms on $\mathbb{C}^N$ with $\sigma_{ij} = 0$ for some $i \neq j$. If $\sigma \in \mathbb{C}^{N(N-1)/2} \setminus \Delta$ and $\lambda \in \mathbb{R}$ then we can construct $S_{ij} = |\sigma_{ij}|^{-3} + \lambda$ for $i \neq j$ and another 2-form $\omega(\sigma, \lambda)$ with $\omega_{ij} = S_{ij}\sigma_{ij}$. If σ is $z^* \wedge 1^*$ then, by (2.1), z is a relative equilibrium if and only if for some λ and some positive, real vector m, $m \lrcorner \omega = 0$.

PROPOSITION. *Let $z \in \mathbb{C}^N \setminus \Delta$ and $\sigma = z^* \wedge 1^* \in \mathbb{C}^{N(N-1)/2} \setminus \Delta$. Then z is a relative equilibrium if and only if for some λ and some real, positive vector m,*

$$\text{(D)} \qquad m \lrcorner \omega(\sigma, \lambda) = 0.$$

The last two propositions can be combined to state the whole problem in $\mathbb{C}^{N(N-1)/2}$.

PROPOSITION. *A 2-form $\sigma \in \mathbb{C}^{N(N-1)/2}$ is $z^* \wedge 1^*$ for some relative equilibrium z if and only if equations* (C) *and* (D) *hold for some real λ and real positive vector m.*

From the general theory we can derive some other equations from (C) and (D) which are especially useful in the case $N=4$.

Since σ satisfies (C) it is decomposable. Of course it is then also degenerate. Since ω satisfies (D) it is also degenerate. So we have $\operatorname{pf}(\sigma) = \operatorname{pf}(\omega) = 0$.

PROPOSITION. *Suppose $N=4$ and* (C) *and* (D) *hold. then*

$$\sigma_{13}\sigma_{24}-\sigma_{12}\sigma_{34}-\sigma_{14}\sigma_{23}=0, \tag{P1}$$

and

$$S_{13}S_{24}\sigma_{13}\sigma_{24}-S_{12}S_{34}\sigma_{12}\sigma_{34}-S_{14}S_{23}\sigma_{14}\sigma_{23}=0. \tag{P2}$$

These equations are the key to the characterization of non-collinear relative equilibria discovered by Dziobek and later by MacMillan and Bartky. It is actually more convenient to consider the set of pairs (z, λ) such that z is a relative equilibrium for λ and for some m or equivalently the pairs (σ, λ) such that (C) and (D) hold for some real, positive m. First we relax the constraints to allow any real vector m in equation (D).

PROPOSITION A. *The set of pairs $(\sigma, \lambda)\in(\mathbb{C}^6\setminus\Delta)\times\mathbb{R}$ such that $\sigma=z^*\wedge 1^*$ for some non-collinear position vector z and* (D) *holds for some non-zero real (but not necessarily positive) vector m is precisely the solution set of the real analytic equations* (C) *and*

$$S_{12}S_{34}=S_{13}S_{24}=S_{14}S_{23}. \tag{MB}$$

The only additional condition needed to guarantee that σ arises from a relative equilibrium is the positivity of the vector m. Hence,

THEOREM 1. *The pairs $(\sigma, \lambda)\in(\mathbb{C}^6\setminus\Delta)\times\mathbb{R}$ arising from non-collinear relative equilibria form an open subset of the solution set of equations* (C) *and* (MB). *The boundary is contained in the set where* (D) *holds for some real m with at least one component vanishing.*

The theorem follows immediately from proposition A. We will prove the proposition by means of two lemmas.

Lemma 1. *If $\sigma=z^*\wedge 1^*$ and z_1, z_2, z_3, z_4 are collinear then* (MB) *does not hold.*

Proof. Assume that the points lie on the real line in the order $z_1<z_2<z_3<z_4$. Set $z_2-z_1=x$, $z_3-z_2=y$ and $z_4-z_3=z$. Equations (MB) become

$$(x^{-3}+\lambda)(z^{-3}+\lambda)=((x+y)^{-3}+\lambda)((y+z)^{-3}+\lambda)=((x+y+z)^{-3}+\lambda)(y^{-3}+\lambda).$$

We will show that these equations have no positive solutions. The first equation gives:

$$\lambda=\frac{(x+y)^{-3}(y+z)^{-3}-x^{-3}z^{-3}}{x^{-3}+z^{-3}-(x+y)^{-3}-(y+z)^{-3}}.$$

The second equation gives:

$$\lambda=\frac{(x+y)^{-3}(y+z)^{-3}-y^{-3}(x+y+z)^{-3}}{y^{-3}+(x+y+z)^{-3}-(x+y)^{-3}-(y+z)^{-3}}.$$

Both expressions are homogeneous so we may assume without loss of generality that $x+y+z=1$. Eliminating y and equating the two expressions gives

$$\frac{x^{-3}z^{-3}-(1-x)^{-3}(1-z)^{-3}}{x^{-3}+z^{-3}-(1-x)^{-3}-(1-z)^{-3}}=\frac{(1-x-z)^{-3}-(1-x)^{-3}(1-z)^{-3}}{1+(1-x-z)^{-3}-(1-x)^{-3}-(1-z)^{-3}}.$$

We will show that the left side is strictly greater than 1 and the right side is strictly less than 1 to complete the proof.

Since $x+y+z=1$ we have $x<1-z$ and $z<1-x$. So the numerator and denominator on the left are positive. Also

$$(x^{-3}-1)(z^{-3}-1)>((1-x)^{-3}-1)((1-z)^{-3}-1)$$

or

$$x^{-3}z^{-3}-(1-x)^{-3}(1-z)^{-3}>x^{-3}+z^{-3}-(1-x)^{-3}-(1-z)^{-3}.$$

This shows that the left side is strictly bigger than 1.

The numerator and denominator on the right are also positive and we can derive

$$((1-x)^{-3}-1)((1-z)^{-3}-1)>0$$

or

$$1-(1-x)^{-3}-(1-z)^{-3}>-(1-x)^{-3}(1-z)^{-3}.$$

Adding $(1-x-z)^{-3}$ to both sides shows that the right side is strictly less than 1.

□

LEMMA 2. *If* (C) *and* (D) *hold for some non-zero, real vector m then either* (MB) *holds or* z_1, z_2, z_3, z_4 *are collinear.*

Proof. If (C) and (D) hold then both of the 2-forms, σ and ω, are degenerate. Hence (P1) and (P2) hold. Suppose first that the complex numbers $\sigma_{13}\sigma_{24}$, $\sigma_{12}\sigma_{34}$, and $\sigma_{14}\sigma_{23}$ are not proportional over $\mathbb{R}$. Then (P1) and (P2) represent non-trivial dependence relations between them (with real coefficients). Therefore the coefficients in (P2) are proportional to those in (P1). This is exactly the condition (MB).

The case when the products of the σ's are proportional over $\mathbb{R}$ uses the fact that m is real. However we can prove immediately that z_1, z_2, z_3, z_4 lie either on a circle or on a line. Namely, we find that the cross ratio $\sigma_{13}\sigma_{24}\sigma_{14}^{-1}\sigma_{23}^{-1}$ is real and use a theorem of projective geometry [2]. Also it is easy to see that if one of the equations (MB) holds so does the other. So we suppose that neither equation holds and prove that z_1, z_2, z_3, z_4 are collinear.

It is impossible to construct a quadrilateral with all six mutual distances equal. Therefore at least one of the quantities $\omega_{ij}=S_{ij}\sigma_{ij}$ is non-zero (see 2.2). Assume without loss of generality that $S_{12}\sigma_{12}\neq 0$. Then also $S_{21}\sigma_{21}=-S_{12}\sigma_{12}\neq 0$ and it follows that the matrix of ω has rank at least 2. On the other hand the two independent vectors $(-\omega_{24}, \omega_{14}, 0, -\omega_{12})$ and $(\omega_{23}, -\omega_{13}, \omega_{12}, 0)$ are in the kernel. Therefore they are a basis over $\mathbb{C}$. Any real vector in the kernel must take the form

$$(-r\bar{\omega}_{12}\omega_{24}+s\bar{\omega}_{12}\omega_{23},\ r\bar{\omega}_{12}\omega_{14}-s\bar{\omega}_{12}\omega_{13},\ s|\omega_{12}|^2,\ -r|\omega_{12}|^2),$$

where r and s are real. For this to be real for some non-trivial r and s it is necessary and sufficient that

$$S_{13}S_{24}\,\mathrm{Im}\,\bar{\sigma}_{12}\sigma_{13}\,\mathrm{Im}\,\bar{\sigma}_{12}\sigma_{24}=S_{14}S_{23}\,\mathrm{Im}\,\bar{\sigma}_{12}\sigma_{14}\,\mathrm{Im}\,\bar{\sigma}_{12}\sigma_{23}. \tag{2.5}$$

Now setting $\sigma_{ij}=z_i-z_j$ one finds that $\mathrm{Im}\,\bar{\sigma}_{12}\sigma_{13}=\mathrm{Im}\,\bar{\sigma}_{12}\sigma_{23}$ and $\mathrm{Im}\,\bar{\sigma}_{12}\sigma_{14}=\mathrm{Im}\,\bar{\sigma}_{12}\sigma_{24}$. Since $S_{13}S_{24}\neq S_{14}S_{23}$ by hypothesis, some of these expressions must vanish. Suppose without loss of generality that $\mathrm{Im}\,\bar{\sigma}_{12}\sigma_{13}=0$. Then z_1, z_2, z_3 are collinear. But since we already know that z_4 lies on the circle determined by z_1, z_2, z_3 we find that all four particles are collinear. □

Proof of proposition A. Suppose (C) and (D) hold for some non-zero real vector m and that z is not collinear. Then by lemma 2, equations (MB) hold.

Conversely, suppose equations (C) and (MB) hold. From equations (C) we have $\sigma = z^* \wedge 1^*$ for some z and equation (P1) (which says σ is decomposable). By lemma 1, the vector z cannot be collinear. (P1) together with (MB) immediately yield (P2) so ω is also decomposable and degenerate. This means that there are some non-zero vectors m (real or complex) in the kernel. As in the proof of lemma 2 we may suppose that $\omega_{12} = -\omega_{21} \neq 0$ and construct a basis for the kernel. The condition for a real vector in the kernel is exactly (2.5) but this time we have $S_{13}S_{24} = S_{14}S_{23}$. As before the imaginary parts in the equation are equal in pairs so (2.5) does in fact hold. This completes the proof. □

We introduce the symbol $\mathcal{R}$ for the set of all pairs (σ, λ) corresponding to non-collinear relative equilibria. Theorem 1 asserts that $\mathcal{R}$ is an open subset of a certain real analytic variety in $(\mathbb{C}^6 \setminus \Delta) \times \mathbb{R}$, the solution set of (C) and (MB). Before studying the structure of $\mathcal{R}$ we turn to the other main result of MacMillan and Bartky concerning the uniqueness of the mass vector.

THEOREM 2. *Let $(\sigma, \lambda) \in \mathcal{R}$. Then the real mass vector m is unique up to a constant multiple.*

Proof. First we allow any (σ, λ) in the solution set of (C) and (MB). We will make use of the description of the kernel of ω obtained during the proof of lemma 2. The real vectors in the kernel form a real subspace and we must show that its dimension is 1. Refering to the expression for an arbitrary real vector in the kernel in the proof of lemma 2 we see that to get a two-dimensional set of real vectors we would have to have

$$\operatorname{Im} \bar{\omega}_{12}\omega_{24} = \operatorname{Im} \bar{\omega}_{12}\omega_{23} = \operatorname{Im} \bar{\omega}_{12}\omega_{14} = \operatorname{Im} \bar{\omega}_{12}\omega_{13} = 0.$$

Remember that we are assuming $\omega_{12} \neq 0$. From the first equation we have either $S_{24} = 0$ or $\bar{\sigma}_{12}\sigma_{24} \in \mathbb{R}$. Similar dichotomies arise from the other equations.

First suppose $S_{24} \neq 0$. Then $\bar{\sigma}_{12}\sigma_{24} \in \mathbb{R}$ so z_1, z_2, z_4 are collinear. Then the distances from z_3 to the others are not all equal so at least one of the quantities S_{i3}, $i = 1, 2, 4$, is non-zero. Equations (MB) show that S_{34} cannot be the only one which is non-zero. From the dichotomies we conclude that at least one of $\bar{\sigma}_{12}\sigma_{13}$ or $\bar{\sigma}_{12}\sigma_{23}$ is real. Then z_3 is on the same line as the other particles which contradicts lemma 1. The same conclusion can be drawn if we assume some other S_{ij} besides S_{24} is non-zero. The only other possibility is that $S_{13} = S_{14} = S_{23} = S_{24} = S_{34} = 0$. Then one finds that the kernel of ω is $\{m_1 = m_2 = 0\}$ contradicting the existence of a positive mass vector in the kernel. □

The results to follow are easier to state if we eliminate the constant multiple from the mass vector. Let

$$\mathcal{M} = \{m \in \mathbb{R}^{4+} : m_1 + m_2 + m_3 + m_4 = 1\}.$$

By theorem 2 there is a well-defined mapping $\varphi : \mathcal{R} \to \mathcal{M}$ taking a pair (σ, λ) arising from a non-collinear relative equilibrium to the corresponding normalized mass vector.

PROPOSITION. *$\varphi:\mathcal{R}\to\mathcal{M}$ is continuous.*

Proof. Let $(\sigma, \lambda)\in\mathcal{R}$ and $m=\varphi(\sigma, \lambda)$. Since m is the unique element of $\mathcal{M}$ in the kernel of $\omega(\sigma, \lambda)$ we have for each ε, a positive δ such that $|m'\lrcorner\,\omega(\sigma,\lambda)|>\delta$ for $|m'-m|>\varepsilon$. For $(\tilde\sigma, \tilde\lambda)$ sufficiently close to (σ, λ), $|m'\lrcorner\,\omega(\tilde\sigma,\tilde\lambda)|>\delta/2$ holds for the same m'. But for $\tilde m=\varphi(\tilde\sigma,\tilde\lambda)$ we have $\tilde m\lrcorner\,\omega(\tilde\sigma,\tilde\lambda)=0$, hence $|\tilde m-m|<\varepsilon$. □

Stronger statements can be made after we eliminate another constant from the description. The group $\mathbb{C}\setminus 0$ acts on $(\mathbb{C}^6\setminus\Delta)\times\mathbb{R}:(\sigma,\lambda)\to(c\sigma,|c|^{-3}\lambda)$; $c\in\mathbb{C}\setminus 0$. This multiplies $\omega(\sigma,\lambda)$ by $c|c|^{-3}$ and so has no effect on the kernel. The action has a global cross section $\{(\sigma, \lambda)\colon \sigma_{12}=1\}$. We identify pairs which are equivalent by this action. The quotient space is of the form $(\mathbb{C}P(5)\setminus\Delta)\times\mathbb{R}$.

$\mathcal{R}$ is invariant under the action. Let $\tilde{\mathcal{R}}$ denote the quotient space of $\mathcal{R}$. The map φ descends to a continuous map $\tilde\varphi:\tilde{\mathcal{R}}\to\mathcal{M}$. The bifurcation theory of relative equilibria of the 4-body problem reduces to a study of $\tilde\varphi$.

THEOREM 3: *$\tilde\varphi:\tilde{\mathcal{R}}\to\mathcal{M}$ is a continuous, proper mapping.*

Proof. To show $\tilde\varphi$ is proper, consider a compact set $K\subset\mathcal{M}$. We must show $\tilde\varphi^{-1}(K)$ is compact. First we show that $\tilde\varphi^{-1}(K)$ is closed in $(\mathbb{C}P(5)\setminus\Delta)\times\mathbb{R}$. For this we can revert to φ and consider $\varphi^{-1}(K)$ in $(\mathbb{C}^6\setminus\Delta)\times\mathbb{R}$. By continuity $\varphi^{-1}(K)$ is closed in $\mathcal{R}$. The second sentence of theorem 1 shows $\varphi^{-1}(K)$ is closed in the solution set of equations (C) and (MB). Since this set is closed in $(\mathbb{C}^6\setminus\Delta)\times\mathbb{R}$ so is $\varphi^{-1}(K)$.

Next we must show that $\tilde\varphi^{-1}(K)$ is contained in some compact subset of $(\mathbb{C}P(5)\setminus\Delta)\times\mathbb{R}$. For this we identify $(\mathbb{C}P(5)\setminus\Delta)\times\mathbb{R}$ with $\{(\sigma,\lambda)\colon \sigma_{12}=1\}$ as above. First we must see that σ is bounded away from Δ. A result of Shub for the N-body problem shows that if m is fixed and the size of the position vector z is normalized the relative equilibria can be bounded away from Δ **[23]**. This extends immediately to a compact set K of mass vectors and when translated to the σ notation is precisely what we require. Finally we obtain a bound for λ by means of an explicit formula. If equation 1.3 is multiplied by $m_j\overline{(z_j-c)}$ and summed for $j=1,\ldots,N$ we find

$$\Lambda\sum_j m_j|z_j-c|^2=-U(z),$$

where we have used the homogeneity of $U(z)$. If we recall that $\Lambda=M^{-1}\lambda$ and $M=1$ we have

$$\lambda=-U(z)\left(\sum_j m_j|z_j-c|^2\right)^{-1}.$$

For σ with $\sigma_{12}=1$, Shub's result gives positive lower bounds for the denominators of the potential function. The particles cannot then all be close to c so we have a lower bound for $\sum_j m_j|z_j-c|^2$ as n varies over K. Hence λ is bounded on $\varphi^{-1}(K)$. Since $\varphi^{-1}(K)$ is closed and is contained in a compact set, it is compact. Note for later use that $\lambda<0$ on $\mathcal{R}$. □

In **[11]**, MacMillan and Bartky also show that for $(\sigma,\lambda)\in\mathcal{R}$, λ is uniquely determined by σ except in the case that σ represents the following configuration: three particles form an equilateral triangle with the fourth at the centre. So except in this case, the configuration determines the mass ratios uniquely.

To obtain further results, some information on the dimension or measure of $\mathcal{R}$ and its subsets is needed. $\mathcal{R}$ is an open subset of the solution set of equations (C) and (MB). These represent 8 real analytic constraints in a 13-dimensional space. Remarkably, these equations are independent at every point of $\mathcal{R}$. The equations are independent at every point (σ, λ) where the following 8×13 matrix has maximal rank:

$$A = \left[\begin{array}{cccccc:c} I & -I & 0 & I & 0 & 0 & 0 \\ I & 0 & -I & 0 & I & 0 & 0 \\ 0 & I & -I & 0 & 0 & I & 0 \\ A_{12} & A_{13} & A_{14} & A_{23} & A_{24} & A_{34} & A_\lambda \end{array}\right].$$

Here all entries except the last column represent 2×2 real matrices. I is the 2×2 identity matrix. A_{ij} is the matrix of partial derivatives

$$A_{ij} = \begin{pmatrix} \dfrac{\partial F}{\partial x_{ij}} & \dfrac{\partial F}{\partial y_{ij}} \\ \dfrac{\partial G}{\partial x_{ij}} & \dfrac{\partial G}{\partial y_{ij}} \end{pmatrix},$$

where $F = S_{13}S_{24} - S_{12}S_{34}$ and $G = S_{13}S_{24} - S_{14}S_{23}$ and $\sigma_{ij} = x_{ij} + iy_{ij}$. Finally A_λ is the column vector

$$A_\lambda = \begin{pmatrix} \dfrac{\partial F}{\partial \lambda} \\ \dfrac{\partial G}{\partial \lambda} \end{pmatrix}.$$

LEMMA 3. *Let* (σ, λ) *be a point of the solution set of equations* (C) *and* (MB) *where* A *fails to have maximal rank. Then*

$$S_{12}S_{34} = S_{13}S_{24} = S_{14}S_{23} = 0.$$

Before proving this key lemma we apply it to the description of the set of all relative equilibria of the 4-body problem.

THEOREM 4. $\mathcal{R}$ *is a* 5*-dimensional real analytic submanifold of* $(\mathbb{C}^6 \setminus \Delta) \times \mathbb{R}$. $\tilde{\mathcal{R}}$ *is a* 3*-dimensional real analytic submanifold of* $(\mathbb{C}P(5) \setminus \Delta) \times \mathbb{R}$.

Proof. The second statement follows from the first by taking quotient manifolds under the $\mathbb{C} \setminus 0$ action. By lemma 3 it will be enough to show that if $(\sigma, \lambda) \in \mathcal{R}$ then the products $S_{12}S_{34}$, $S_{13}S_{24}$ and $S_{14}S_{23}$ are non-zero. Suppose to the contrary that, for example, $S_{12} = S_{13} = S_{14} = 0$. Then the kernel of ω is spanned over $\mathbb{C}$ by $(1, 0, 0, 0)$ and $(0, \omega_{34}, -\omega_{24}, \omega_{23})$. By theorem 2 the only real vectors are of the form $(r, 0, 0, 0)$ and therefore (σ, λ) cannot be in $\mathcal{R}$. Now suppose $S_{23} = S_{24} = S_{34} = 0$. Then any vector in the kernel satisfies $m_1 = 0$ and again $(\sigma, \lambda) \notin \mathcal{R}$. Finally any other choice for which the S_{ij} vanish reduces to one of these two by permuting subscripts. □

Proof of lemma 3. By elementary operations we can reduce A to the matrix

$$\begin{bmatrix} I & -I & 0 & I & 0 & 0 & \vdots & 0 \\ I & 0 & -I & 0 & I & 0 & \vdots & 0 \\ 0 & I & -I & 0 & 0 & I & \vdots & 0 \\ A_{12}-A_{23}-A_{24} & A_{13}+A_{23}-A_{34} & A_{14}+A_{24}+A_{34} & 0 & 0 & 0 & \vdots & A_\lambda \end{bmatrix}.$$

This will have rank 8 if and only if the 2×13 bottom 'row' has rank 2. For brevity of notation we will replace A_{ij} by the complex vector

$$2\begin{bmatrix} \dfrac{\partial F}{\partial \bar\sigma_{ij}} \\ \dfrac{\partial G}{\partial \bar\sigma_{ij}} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial F}{\partial x_{ij}} + i\dfrac{\partial F}{\partial y_{ij}} \\ \dfrac{\partial G}{\partial x_{ij}} + i\dfrac{\partial G}{\partial y_{ij}} \end{bmatrix}.$$

If A fails to have rank 8 we must have (from the first column)

$$\frac{\partial G}{\partial \lambda}\left(\frac{\partial F}{\partial \bar\sigma_{12}} - \frac{\partial F}{\partial \bar\sigma_{23}} - \frac{\partial F}{\partial \bar\sigma_{24}}\right) = \frac{\partial F}{\partial \lambda}\left(\frac{\partial G}{\partial \bar\sigma_{12}} - \frac{\partial G}{\partial \bar\sigma_{23}} - \frac{\partial G}{\partial \bar\sigma_{24}}\right) \tag{2.6}$$

and two similar equations (from the other columns). Now

$$\frac{\partial S_{ij}}{\partial \bar\sigma_{ij}} = \frac{\partial}{\partial \bar\sigma_{ij}}\left(\sigma_{ij}^{-\frac{3}{2}}\bar\sigma_{ij}^{-\frac{3}{2}} + \lambda\right) = -\frac{3}{2}\frac{\sigma_{ij}}{|\sigma_{ij}|^5} \quad \text{and} \quad \frac{\partial S_{ij}}{\partial \lambda} = 1.$$

From these we compute the derivatives of F and G. One finds

$$\frac{\partial F}{\partial \lambda} = |\sigma_{13}|^{-3} + |\sigma_{24}|^{-3} - |\sigma_{12}|^{-3} - |\sigma_{34}|^{-3}$$

or, using the fact that $F=0$,

$$\begin{aligned} -\lambda\frac{\partial F}{\partial \lambda} &= |\sigma_{13}\sigma_{24}|^{-3} - |\sigma_{12}\sigma_{34}|^{-3}, \\ -\lambda\frac{\partial G}{\partial \lambda} &= |\sigma_{13}\sigma_{24}|^{-3} - |\sigma_{14}\sigma_{23}|^{-3}. \end{aligned} \tag{2.7}$$

After some computation (2.6) becomes

$$\frac{\partial G}{\partial \lambda}(S_{34}|\sigma_{12}|^{-5}\sigma_{12} + S_{13}|\sigma_{24}|^{-5}\sigma_{24}) = \frac{\partial F}{\partial \lambda}(-S_{14}|\sigma_{23}|^{-5}\sigma_{23} + S_{13}|\sigma_{24}|^{-5}\sigma_{24}).$$

The other equations analogous to (2.6) give

$$\frac{\partial G}{\partial \lambda}(-S_{24}|\sigma_{13}|^{-5}\sigma_{13} - S_{12}|\sigma_{34}|^{-5}\sigma_{34}) = \frac{\partial F}{\partial \lambda}(-S_{24}|\sigma_{13}|^{-5}\sigma_{13} + S_{14}|\sigma_{23}|^{-5}\sigma_{23})$$

and

$$\frac{\partial G}{\partial \lambda}(-S_{13}|\sigma_{24}|^{-5}\sigma_{24} + S_{12}|\sigma_{34}|^{-5}\sigma_{34}) = \frac{\partial F}{\partial \lambda}(+S_{23}|\sigma_{14}|^{-5}\sigma_{14} - S_{13}|\sigma_{24}|^{-5}\sigma_{24}).$$

Using (C) we can eliminate σ_{12}, σ_{13} and σ_{23} in terms of σ_{14}, σ_{24} and σ_{34}. Then the three equations just derived become dependence relations among these three complex numbers with real coefficients. Since the three numbers are not proportional over $\mathbb{R}$ (lemma 1 again) the coefficients in these three dependence relations are proportional. After some calculation we obtain the following simple conditions which must hold whenever A fails to have rank 8:

$$\left(\frac{\partial F}{\partial \lambda}\right)^2 \frac{S_{14}S_{23}}{|\sigma_{14}\sigma_{23}|^5} = \left(\frac{\partial G}{\partial \lambda}\right)^2 \frac{S_{12}S_{34}}{|\sigma_{12}\sigma_{34}|^5} = \left(\frac{\partial F}{\partial \lambda} - \frac{\partial G}{\partial \lambda}\right)^2 \frac{S_{13}S_{24}}{|\sigma_{13}\sigma_{24}|^5}.$$

To complete the proof we will assume that the common value of $S_{12}S_{34}$, $S_{13}S_{24}$ and $S_{14}S_{23}$ is non-zero and then study the solutions of these equations. Once we cancel the S's these equations involve only the quantities $r = |\sigma_{12}\sigma_{34}|^{-1}$, $s = |\sigma_{13}\sigma_{24}|^{-1}$ and $t = |\sigma_{14}\sigma_{23}|^{-1}$. Using (2.7) we get

$$(s^3 - r^3)t^5 = (t^3 - s^3)r^5 = (t^3 - r^3)s^5.$$

An obvious positive solution is $r = s = t$. In fact this is the only positive solution. To see this, assume without loss of generality that $r \le s \le t$. Then $t^3 - s^3 \le t^3 - r^3$ and $r^5 \le s^5$. Therefore the second equation holds if and only if equality holds in both inequalities, i.e. $r = s$. But if $r = s$ then the first equation gives $r = s = t$. This then is a necessary condition for a point where A fails to have rank 8. We now show that A does in fact have rank 8 when $r = s = t$.

Suppose $|\sigma_{12}\sigma_{34}| = |\sigma_{13}\sigma_{24}| = |\sigma_{14}\sigma_{23}|$. From equations (MB) we find

$$|\sigma_{12}|^{-3} + |\sigma_{34}|^{-3} = |\sigma_{13}|^{-3} + |\sigma_{24}|^{-3} = |\sigma_{14}|^{-3} + |\sigma_{23}|^{-3}.$$

Together these equations imply that the lengths $|\sigma_{12}|$, $|\sigma_{34}|$ equal the lengths $|\sigma_{13}|$, $|\sigma_{24}|$ in some order and also equal $|\sigma_{14}|$, $|\sigma_{23}|$ in some order. Therefore three of the particles are arranged in an equilateral triangle and the fourth is equidistant from the other three, i.e. at the centre of the triangle. All such position vectors are affine equivalent so we may take (assuming z_4 at the centre) $z_4 = 0$, $z_1 = 1$, $z_2 = -\frac{1}{2} + \sqrt{\frac{3}{2}}i$, $z_3 = -\frac{1}{2} - \sqrt{\frac{3}{2}}i$. A direct computation gives:

$$A_{14} + A_{24} + A_{34} = (3^{-\frac{3}{2}} + \lambda)\begin{bmatrix} 0 & -\sqrt{3} \\ \frac{3}{2} & \sqrt{\frac{3}{2}} \end{bmatrix}$$

and $S_{12} = S_{13} = S_{23} = 3^{-\frac{3}{2}} + \lambda$. Since we have assumed that $S_{12}S_{34} = S_{13}S_{24} = S_{14}S_{23} \neq 0$, the matrix has rank 2 and so A has rank 8. This completes the proof. □

With the structure of $\mathscr{R}$ and $\tilde{\mathscr{R}}$ in mind it is natural to wonder about the smoothness of φ and $\tilde{\varphi}$. Note that the range $\mathscr{M}$ is also a real analytic manifold.

PROPOSITION. *φ and $\tilde{\varphi}$ are real analytic mappings.*

Proof. It is enough to consider φ. Recall that $\varphi(\sigma, \lambda)$ is the normalized real vector in the kernel of $\omega(\sigma, \lambda)$. The entries of ω are real analytic functions of σ and λ. The kernel of ω is the span of $(-\omega_{24}, \omega_{14}, 0, -\omega_{12})$ and $(\omega_{23}, -\omega_{13}, \omega_{12}, 0)$, at least if $\omega_{12} \neq 0$ (compare the proof of lemma 2). These vectors are also real analytic functions of (σ, λ). The real vector in the kernel is the linear combination of these with coefficients $r\bar{\omega}_{12}$, $s\bar{\omega}_{12}$ where the real vector (r, s) satisfies:

$$\begin{bmatrix} -\mathrm{Im}\ \bar{\omega}_{12}\omega_{24} & \mathrm{Im}\ \bar{\omega}_{12}\omega_{23} \\ \mathrm{Im}\ \bar{\omega}_{12}\omega_{14} & -\mathrm{Im}\ \bar{\omega}_{12}\omega_{13} \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = 0.$$

If $(\sigma, \lambda) \in \mathscr{R}$ the rows are automatically proportional and we may choose $r = \mathrm{Im}\ \bar{\omega}_{12}\omega_{23}$ and $s = \mathrm{Im}\ \bar{\omega}_{12}\omega_{24}$. Thus the coefficients in the expansion of a real vector in the kernel are real analytic functions of (σ, λ) as (σ, λ) varies in $\mathcal{R}$. Finally we can normalize this real vector by dividing by the sum of its components. As this sum is non-zero in $\mathcal{R}$, φ is real analytic. □

3. *Regularization of singularities and finiteness theorems*

We have seen that $\mathcal{R}$, the set of all pairs (σ, λ) which arise from non-collinear relative equilibria, is an open subset of a real-analytic variety in $(\mathbb{C}^6\setminus\Delta)\times\mathbb{R}$. This variety, which we denote by V, is just the solution set of equations (C) and (MB). To obtain the deeper results we are after we need more information on the structure of V.

The most important step is to compactify V. Using a transformation reminiscent of the Levi-Civita regularization of double collisions [**9**] we can replace V by an algebraic (and later even a projective) variety. Then the techniques of algebraic geometry can be used.

From the definition of S_{ij}, equations (MB) are:

$$\text{(MB)}\qquad \begin{gathered}(|\sigma_{12}|^{-3}+\lambda)(|\sigma_{34}|^{-3}+\lambda)=(|\sigma_{13}|^{-3}+\lambda)(|\sigma_{24}|^{-3}+\lambda),\\ \text{and}\\ (|\sigma_{12}|^{-3}+\lambda)(|\sigma_{34}|^{-3}+\lambda)=(|\sigma_{14}|^{-3}+\lambda)(|\sigma_{23}|^{-3}+\lambda).\end{gathered}$$

Multiplying the first equation by $|\sigma_{12}\sigma_{34}\sigma_{13}\sigma_{24}|^3$ and the second by $|\sigma_{12}\sigma_{34}\sigma_{14}\sigma_{23}|^3$ gives:

$$\text{(NMB)}\qquad \begin{gathered}|\sigma_{13}\sigma_{24}|^3(1+\lambda|\sigma_{12}|^3)(1+\lambda|\sigma_{34}|^3)=|\sigma_{12}\sigma_{34}|^3(1+\lambda|\sigma_{13}|^3)(1+\lambda|\sigma_{24}|^3)\\ \text{and}\\ |\sigma_{14}\sigma_{23}|^3(1+\lambda|\sigma_{12}|^3)(1+\lambda|\sigma_{34}|^3)=|\sigma_{12}\sigma_{34}|^3(1+\lambda|\sigma_{14}|^3)(1+\lambda|\sigma_{23}|^3).\end{gathered}$$

Let $W=\{(\sigma,\lambda)\in\mathbb{C}^6\times\mathbb{R}\text{: (C) and (NMB) hold}\}$. Clearly $V=W\cap(\mathbb{C}^6\setminus\Delta)\times\mathbb{R}$. W is defined by equations which fail to be analytic on Δ. However, we now introduce the squaring map $S:\mathbb{C}^6\to\mathbb{C}^6$, defined in coordinates τ on the domain and σ on the range by $\sigma_{ij}=\tau_{ij}^2$; $1\le i<j\le 4$. Note that $S(\Delta)=\Delta$, $S^{-1}(\Delta)=\Delta$ and that $S|_{\mathbb{C}^6\setminus\Delta}$ is a 64-fold covering map. Define a map $T:\mathbb{C}^6\times(\mathbb{R}\setminus 0)\to\mathbb{C}^6\times\mathbb{R}^-$ by $T(\tau,\alpha)=(S(\tau),-\alpha^{-6})$. Substitution into (C) and (NMB) and multiplication of the latter by α^6 gives:

$$\text{(C}'\text{)}\qquad \tau_{13}^2=\tau_{12}^2+\tau_{23}^2,\quad \tau_{14}^2=\tau_{12}^2+\tau_{24}^2,\quad \tau_{14}^2=\tau_{13}^2+\tau_{34}^2,$$

and

$$\text{(NMB}'\text{)}\qquad \begin{gathered}|\tau_{13}\tau_{24}|^6(\alpha^6-|\tau_{12}|^6)(\alpha^6-|\tau_{34}|^6)=|\tau_{12}\tau_{34}|^6(\alpha^6-|\tau_{13}|^6)(\alpha^6-|\tau_{24}|^6)\\ |\tau_{14}\tau_{23}|^6(\alpha^6-|\tau_{12}|^6)(\alpha^6-|\tau_{34}|^6)=|\tau_{12}\tau_{34}|^6(\alpha^6-|\tau_{14}|^6(\alpha^6-|\tau_{23}|^6),\end{gathered}$$

The point of this is that $|\tau_{ij}|^6$ is a polynomial function of the real and imaginary parts of τ_{ij}. Furthermore, both sets of equations are homogeneous, of degrees 2 and 18, respectively.

Let $W'=\{(\tau,\alpha)\text{: (C}'\text{) and (NMB}'\text{) hold}\}$. Clearly

$$W'\cap\{\alpha\ne 0\}=T^{-1}(W\cap\{\lambda<0\}).$$

Since $\lambda < 0$ on $\mathscr{R}$, $T^{-1}(\mathscr{R})$ is an open subset of W'. We will now show that the map $\varphi \circ T: T^{-1}(\mathscr{R}) \to \mathscr{M}$ is part of an algebraic correspondence. To do this we introduce the analogue of equation (D) in the (τ, α)-space. To each $(\tau, \alpha) \in \mathbb{C}^6 \times \mathbb{R}$ we associate a matrix ω' via:

$$\omega'_{ij} = \chi(\alpha^6 |\tau_{ij}|^{-6} - 1)\tau_{ij}^2,$$

when $\chi = |\tau_{12}\tau_{13}\tau_{14}\tau_{23}\tau_{24}\tau_{34}|^6$. Note that ω'_{ij} is a homogeneous polynomial in the thirteen (τ, α) variables of degree 38. ω' can be viewed as a symmetric 2-tensor.

PROPOSITION. *Let* $(\tau, \alpha) \in \mathbb{C}^6 \times \mathbb{R}$ *satisfy* $\alpha\chi = 0$ *and let* $(\sigma, \lambda) = T(\tau, \alpha)$. *Then* (D) *holds if and only if* (D′) *holds where* (D′) *is*:

$$(D') \qquad m \lrcorner \, \omega'(\tau, \alpha) = 0.$$

Proof. (D′) is obtained from (D) by substitution of $T(\tau, \alpha)$ for (σ, λ) and multiplication by $\alpha^6\chi$. Since $\alpha^6\chi \neq 0$, the equations are equivalent. □

Recall that an algebraic correspondence between $\mathbb{R}P(m)$ and $\mathbb{R}P(n)$ is an algebraic subset $Z \subset \mathbb{R}P(m) \times \mathbb{R}P(n)$. Thus Z is given in homogeneous coordinates $(x_1, \ldots, x_{m+1};\ y_1, \ldots, y_{n+1})$ by polynomial equations which are separately homogeneous in x and in y. This concept generalizes the graph of a polynomial mapping from $X = \Pi_1(Z)$ to $Y = \Pi_2(Z)$ where Π_1 and Π_2 are the projections of $\mathbb{R}P(m) \times \mathbb{R}P(n)$ to its factors. In general Z is not the graph of any mapping.

PROPOSITION. $Z = \{(\tau, \lambda, m)$: (C′), (D′) *and* (NMB′) *hold*$\}$ *is an algebraic correspondence in* $\mathbb{R}P(12) \times \mathbb{R}P(3)$.

Proof. ω' is homogeneous of degree 38 in the 13 variables (τ, λ) and equation (D′) is also homogeneous of degree 1 with respect to $m = (m_1, \ldots, m_4)$. (C′) and (NMB′) are also homogeneous in (τ, λ). □

Suppose $(\tau, \alpha) \in T^{-1}(\mathscr{R})$. Then $(\tau, \alpha, m) \in Z$ if and only if $m = \varphi \circ T(\tau, \alpha)$, by the previous proposition. Thus if Γ is the graph of $\varphi \circ T$ we have $\Gamma = Z \cap \Pi_1^{-1}(T^{-1}(\mathscr{R}))$ where $\Pi_1: \mathbb{R}P(12) \times \mathbb{R}P(3) \to \mathbb{R}P(12)$ is projection. In the following theorem, $Z_0 = \{(\tau, \alpha, m)$: $m_i = 0$ for some i or $\alpha = 0$ or $\tau_{ij} = 0$ for some $i, j\}$. Thus Z_0 is an algebraic set which contains Δ.

PROPOSITION. Γ *is an open and closed subset of* $Z \setminus Z_0$, *the difference of two algebraic varieties.*

Proof. On Γ we have $m_i > 0$, $\alpha \neq 0$ and $\tau_{ij} \neq 0$ so $\Gamma \subset Z \setminus Z_0$. Since $\Pi_1(Z) \subset W'$ and since $T^{-1}(\mathscr{R})$ is open in W', $\Gamma = \Pi_1^{-1}(T^{-1}(\mathscr{R})) \cap Z$ is open in Z and so also in $Z \setminus Z_0$. To show that Γ is closed let $(\bar\tau, \bar\alpha, \bar m)$ represent a point of $\bar\Gamma \cap (Z \setminus Z_0)$. We must show it lies in Γ. By definition of Z_0, $\bar\alpha \neq 0$ and $\overline{\tau_{ij}} \neq 0$. Since $m_i > 0$ on Γ and $\bar m_i \neq 0$, we also have $\overline{m_i} > 0$. We have already seen that under these conditions, equation (D′) implies equation (D) where $(\bar\sigma, \bar\lambda) = T(\bar\tau, \bar\alpha)$. Also, since (C) and (NMB′) hold for $(\bar\tau, \bar\alpha)$ we have (C) and (MB) for $(\bar\sigma, \bar\lambda)$. By definition, $\mathscr{R}$ is the set of pairs (σ, λ) such that (C), (MB) and (D) hold for some positive m. Hence $(\bar\sigma, \bar\lambda) \in \mathscr{R}$ and $(\bar\tau, \bar\alpha, \bar m) \in \Pi_1^{-1}(T^{-1}(\mathscr{R})) \cap Z = \Gamma$. □

As a first application of algebraic geometry we will combine theorem 4 with the following result of Whitney [27]: a set which is the difference of two algebraic varieties has finitely many topological components.

THEOREM 5. *$\mathscr{R}$ has finitely many components.*

Proof. Let $\Pi_1:\mathbb{R}P(12)\times\mathbb{R}P(3)\to\mathbb{R}P(12)$ be projection. Then $T^{-1}(\mathscr{R})=\Pi_1(\Gamma)$ so by the last proposition and Whitney's result, $T^{-1}(\mathscr{R})$ has finitely many components. As the continuous image of a connected set is connected, the same is true for $\mathscr{R}$. □

Now another result of Whitney states that any real algebraic variety is a finite stratified union of submanifolds. Such a set is locally path connected so components and path components agree. Theorem 5 can be rephrased as follows: say that two relative equilibria are equivalent if by changing the masses we can find a continuous path of relative equilibria connecting them; then there are finitely many equivalence classes.

We now turn to the bifurcation theory of relative equilibria. This amounts to the study of the restriction of the projection $\Pi_2:\mathbb{R}P(12)\times\mathbb{R}P(3)\to\mathbb{R}P(3)$ to Γ, the graph of $\varphi\circ T$. We are interested in how the fibres $\Pi_2^{-1}(m)$ change as we vary the mass vector.

LEMMA 4: *Let X be any algebraic set in $\mathbb{R}P(12)\times\mathbb{R}P(3)$. Then $\Gamma\cap X$ is open and closed in $Z\cap X\setminus Z_0\cap X$, and so has finitely many components.*

Proof. $Z\cap X$ and $Z_0\cap X$ are also algebraic. The lemma then follows from the last proposition and Whitney's theorem. □

THEOREM 6. *Fix $m\in\mathcal{M}$. The set of relative equilibria of the 4-body problem with mass vector m has finitely many components. The number of components is majorized by a bound which is independent of m.*

Proof. Let $X\subset\mathbb{R}P(12)\times\mathbb{R}P(3)$ be the algebraic set defined by fixing the mass vector in $\mathbb{R}P(3)$ to be the given m. Then $Z\cap X$ is algebraic and so has finitely many components. In fact a result of Milnor and Thom implies that the number of components is majorized by a bound which depends only on the degrees of the defining equations and the dimension of the ambient space [**12**], [**26**]. In particular this bound is independent of m. A component of relative equilibria for m appears in the (σ,λ)-space as a component of $\varphi^{-1}(m)$ in $\mathscr{R}$. In $T^{-1}(\mathscr{R})$ it has several (≤ 128) pre-images which are components of $T^{-1}(\mathscr{R})\cap T^{-1}(\varphi^{-1}(m))$. Finally in the graph Γ each component of relative equilibria is represented by several components of $\Gamma\cap X$. We will complete the proof by showing that each component of $\Gamma\cap X$ is a component of $Z\cap X$. Now by lemma 4 we need only show $\overline{\Gamma\cap X}\cap Z_0=\emptyset$. On $\Gamma\cap X$ we have definite lower bounds for $|\tau_{ij}|$ and $|\alpha|$ and of course m is constant. Therefore on $\overline{\Gamma\cap X}$, $\tau_{ij}\neq 0$, $\alpha\neq 0$ and $m_i>0$ as required. □

The main bifurcation theorems we will prove depend on the strengthened versions of Sard's theorem which hold for algebraic correspondences. Unfortunately the literature of algebraic geometry deals almost exclusively with the case of complex varieties. To understand how these results can be used for real varieties we must discuss the complexification of real varieties. We refer to [27] for proofs of the basic facts.

Let $X \subset \mathbb{R}^n$ be a real algebraic variety. The ideal of X, $I(X)$, consists of all real polynomials which vanish on X. The rank of a point $x \in X$ is the maximum number of polynomials $f_i \in I(X)$ with $df_i(x)$ independent. The rank of X is the maximum rank of any point. The set of points of maximum rank is an open subset of X and is a submanifold of $\tilde{\mathbb{R}}$ of dimension $n-(\text{rank }(X))$. These are called smooth points. Other points are called singular points. A singular point is smooth in some subvariety of X. Varieties in $\mathbb{R}\mathrm{P}(n)$ and $\mathbb{R}\mathrm{P}(m)\times\mathbb{R}\mathrm{P}(n)$ are handled by choosing affine local coordinates on these spaces.

If $Y \subset \mathbb{C}^n$ is a complex variety we proceed in exactly the same way, except linear independence of differentials means independence over $\mathbb{C}$. Given a complex variety Y we can form $Y \cap \mathbb{R}^n$. This is a real variety, the zero set of all real parts of polynomials in $I(Y)$. If $X \subset \mathbb{R}^n$ is a real variety we can construct a complex variety $X^* \subset \mathbb{C}^n$ merely by viewing all variables as complex. Then one can show that for any complex variety $Y \supset X$, we also have $Y \supset X^*$. In particular, $(Y \cap \mathbb{R}^n)^* \subset Y$ for every complex variety Y.

Consider a polynomial map $\pi : X \to X'$. We call $x \in X$ a *critical point of* π if x is smooth in X, $\pi(x)$ is smooth in X', but $d\pi(x)$ fails to have maximal rank. Note that a singular point is not a critical point. When x and $\pi(x)$ are smooth points this definition agrees with the usual one for maps of manifolds. Critical values are the images of critical points. When X and X' are complex we have a strong version of Sard's theorem: there is a proper subvariety of X' containing all the critical values [**14**]. We will use this to get a similar result for real varieties.

LEMMA 5. *Let* $\Pi : X \to X'$ *be a map of real, algebraic varieties. Then there is a proper subvariety of* X' *which contains all the critical values of* Π.

Proof. Suppose $X \subset \mathbb{R}^m$ and $X' \subset \mathbb{R}^n$. Complexify to obtain $X^* \subset \mathbb{C}^m$ and $X'^* \subset \mathbb{C}^n$. The map Π is defined by restricting a polynomial map $\mathbb{R}^m \to \mathbb{R}^n$ (by definition) and this can be complexified too. Apply the complex Sard's theorem to $\Pi^* : X^* \to X'^*$ to find a proper, complex subvariety $Y \subset X'^*$ which contains all of the critical values of Π^*. Now $Y \cap \mathbb{R}^n$ is a proper subvariety of X'; otherwise $X' = Y \cap \mathbb{R}^n$ and $X'^* = (Y \cap \mathbb{R}^n)^* \subset Y$. But Y is a proper subvariety of X'^*. To complete the proof we will show that if $x \in X$ is a critical point for Π^*, it is also a critical point for Π. Now it is easy to see that a point x of X is smooth if and only if it is also smooth when viewed as a point of X^*. Similarly for $x' \in X'$. Also, the rank of $d\pi(x)$ is the same as the rank over $\mathbb{C}$ of $d\pi^*(x)$ since both of these linear maps are obtained from the same real $m \times n$ matrix by restricting to a subspace, $T_xX \subset \mathbb{R}^m$ or $T_xX^* \subset \mathbb{C}^m$, and these subspaces are given by the same set of $\text{rank}_x(X) = \text{rank}_x(X^*)$ real linear equations. Therefore the critical points of π and π^* in X coincide. □

PROPOSITION. *The set of critical values of* $\varphi : \mathcal{R} \to \mathcal{M}$ *is contained in a proper, algebraic subset of* $\mathcal{M}$. *In particular, the set of regular values is open and dense and has full measure in* $\mathcal{M}$.

Proof. The critical values of φ and $\varphi \circ T$ are the same because $T: T^{-1}(\mathscr{R}) \to \mathscr{R}$ is a covering map and so a local diffeomorphism. The graph Γ of $\varphi \circ T$ is contained in the projective correspondence Z. Since Γ is a manifold its points all have the same rank. It follows that by replacing Z by some subvariety we may assume that Γ consists of smooth points. Now $\pi_2: Z \to \mathbb{R}P(3)$ is a map of real, projective varieties so by lemma 5, the set of its critical values is contained in some proper algebraic subset of $\mathbb{R}P(3)$. Since Γ and $\mathbb{R}P(3)$ consist of smooth points the critical values of $\pi_2: \Gamma \to \mathbb{R}P(3)$ viewed as a map of manifolds lie in this set. Since the critical values of a mapping are the same as the critical values of the projection of its graph, the proof is complete. □

Since φ factors through $\tilde{\varphi}: \tilde{\mathscr{R}} \to \mathscr{M}$, theorem 6 also applies to $\tilde{\varphi}$. Since $\tilde{\mathscr{R}}$ and $\mathscr{M}$ are both 3-dimensional manifolds, a regular point is a point where $\tilde{\varphi}$ is a local diffeomorphism. Since φ is a proper mapping, the pre-image of a regular value is a finite set. Moreover, some neighbourhood of each pre-image maps diffeomorphically into $\mathscr{M}$.

We now define the bifurcation set, B, of the relative equilibrium problem for 4 bodies. B is the complement in $\mathscr{M}$ of the set of masses, m, with the following property: in some neighbourhood $\mathscr{U}$ of m, each mass admits the same finite number, k, of relative equilibrium configurations; the equilibria corresponding to masses in $\mathscr{U}$ fall into k components in the space of configurations, each of which is continuously parametrized by $\mathscr{U}$. Recall that each point of $\tilde{\mathscr{R}}$ is associated to a unique relative equilibrium configuration. Namely, each point of $\tilde{\mathscr{R}}$ is represented by pairs (σ, λ) in $\mathscr{R}$ which differ only by the $\mathbb{C} \setminus 0$ action. The affine equivalence class of position vectors z with $z^* \wedge 1^* = \sigma$ is then uniquely (and analytically) determined by the original point.

THEOREM 7. *The bifurcation set, B, of the relative equilibrium problem for* 4 *bodies is contained in a proper algebraic subset of the normalized mass space, $\mathscr{M}$. There is a positive integer K such that each mass in the complement of B admits no more than K affine equivalence classes of relative equilibria.*

For the sake of concreteness we note that if the degrees of the various polynomials are used in the theorem of Thom and Milnor mentioned above, the inequality $K \leq 39 \cdot 77^{14}$ can be proved.

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REFERENCES

[1] M. H. Andoyer. Sur l'equilibre relatif de n corps. *Bull. Astron.* **25** (1906), 50–59.
[2] R. Artzy. *Linear Geometry.* Addison-Wesley, 1978, p. 42.
[3] O. Dziobek. Über einen merkwürdigen Fall der Vielkörperproblems. *Astron. Nach.* **152** (1900), 33–46.
[4] L. Euler. De motu restilineo trium corporum se mutuo attrahentium. *Novi Comm. Acad. Sci. Imp. Petrop.* **11** (1767), 144–151.
[5] W. Greub. *Multilinear Algebra.* Springer Verlag: New York 1978, pp. 200–204.
[6] Hoppe. Erweiterung der bekannten Speziallosungen der Dreikorperproblem. *Archiv der Math. und Phys.* **64**, 218–223.

[7] J. L. Lagrange. *Ouvres*, vol. 6, 272-292. Paris 1873.
[8] R. Lehmann-Filhes. Über zwei fölle der vielkörpersproblem. *Astron. Nach.* **127** (1891).
[9] T. Levi-Civita. Sur la regularisation du problem des trois corps. *Acta Math.* **42** (1920), 99-144.
[10] W. R. Longley. Some particular solutions of the problem of n bodies. *Bull. Amer. Math. Soc.* **13** (1907), 324-335.
[11] W. D. MacMillan & W. Bartky. Permanent configurations in the problem of four bodies. *Trans. Amer. Math. Soc.* **34** (1932), 838-875.
[12] J. Milnor. On the Betti numbers of real varieties. *Proc. Amer. Math. Soc.* **15** (1964), 275-280.
[13] F. R. Moulton. The straight line solutions of the problem of N bodies. *Ann. of Math.* (2) **12** (1910), 1-17.
[14] D. Mumford. *Algebraic Geometry I, Complex Projective Varieties.* Springer-Verlag: Berlin, 1976, pp. 42-43.
[15] F. Pacella. Central configurations of the N-body problem via the equivariant Morse theory. To appear in *Archive for Rational Mechanics and Analysis.*
[16] J. I. Palmore. Classifying relative equilibria, I. *Bull. Amer. Math. Soc.* **79** (1973), 904-908; II, *Bull. Amer. Math. Soc.* **81** (1975)' 489-491; III, *Letters in Math. Physics* **1** (1975), 71-73.
[17] J. I. Palmore. Measure of degenerate relative equilibria, I. *Annals of Math.* **104** (1976), 421-429.
[18] L. M. Perko & E. L. Walter. Regular polygon solutions of the N-body problem. Preprint, 1984.
[19] J. Robbin. Relative equilibria in mechanical systems. In *Dynamical Systems.* Academic Press, New York, 1973.
[20] D. Saari. On the role and properties of central configurations. *Celestial Mechanics* **21**, No. 1 (1980), 9-20.
[21] C. L. Siegel & J. K. Moser. *Lectures on Celestial Mechanics.* Springer Verlag: New York, 1971.
[22] C. Simo. Relative equilibrium solutions in the four body problem. *Celestial Mechanics* **18** (1978), 165-184.
[23] M. Shub. Diagonals and relative equilibria. In *Manifolds-Amsterdam* 1970, *Lecture Notes in Math., vol.* 197, Springer: Berlin (1971), 199-201.
[24] S. Smale. Problems on the nature of relative equilibria in celestial mechanics. In *Manifolds-Amsterdam* 1970, *Lecture Notes in Math., Vol.* 197, Springer: Berlin (1971).
[25] S. Smale. Topology and mechanics, I. *Inventiones Math.* **10** (1970) 305-331; II, *Inventiones Math.* **11** (1970), 45-64.
[26] R. Thom. Sur l'homologie des variétés algebrique réeles. In *Differential and Combinational Topology.* Princeton Univ. Press (1965), pp 255-265.
[27] H. Whitney. Elementary structure of real algebraic varieties. *Annals of Math.* **66** (3) (1957), 545-556.
[28] W. L. Williams. Permanent configurations in the problem of five bodies. *Trans. Amer. Math. Soc.* **44** (1938), 563-579.
[29] A. Wintner. *The Analytical Foundations of Celestial Mechanics.* Princeton Math. series, vol. 5, Princeton Univ. Press, 1941.