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# NILPOTENT BY SUPERSOLVABLE M-GROUPS

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**1. Introduction.** A character of a finite group $G$ is *monomial* if it is induced from a linear (degree one) character of a subgroup of $G$. A group $G$ is an *$M$-group* if all its complex irreducible characters (the set $\mathrm{Irr}(G)$ ) are monomial.

In [**1**], Dade gave an example of an $M$-group with a normal subgroup which is itself not an $M$-group. In his group $G$, the supersolvable residual $N$ is an extra special 2-group and $G/N$ is supersolvable of even order. Moreover, the prime 2 is used in such a way that no analogous construction is possible in the case that $|N|$ or $|G:N|$ is odd. This led Isaacs in [**8**] and Dade in [**2**] to consider the effect of certain "oddness" hypotheses in the study of monomial characters.

Our main results are in the same spirit. Although our techniques seem to require a restrictive assumption on the supersolvable residual of the groups we consider, our theorems provide more evidence that under fairly general circumstances normal subgroups of $M$-groups should be $M$-groups.

THEOREM A. *Let $N \Delta G$ with $N$ nilpotent and $G/N$ a supersolvable group of odd order. Suppose that $G$ is an $M$-group. Then every subnormal subgroup of $G$ is an $M$-group.*

In [**8**], Isaacs asked whether a Hall subgroup of an $M$-group need be an $M$-group (this is known to be true if the Hall subgroup is normal).

THEOREM B. *Let $N \Delta G$ with $N$ nilpotent and $G/N$ supersolvable and of odd order. If $G$ is an $M$-group, then so is every Hall subgroup of $G$.*

In fact, under the hypothesis of Theorems A and B we will show that any subgroup of $G$ containing $N$ is an $M$-group (see Corollary 8.2). It is not true that all subgroups of these groups are $M$-groups: If $G$ is the regular wreath product of the quaternion group of order 8 with a cyclic group of order 3 then $O_2(G)$ has index 3 in $G$, and $G$ is an $M$-group, yet $G$ contains a subgroup isomorphic to the non-$M$-group $SL(2, 3)$.

In Section 3 of [**9**], Seitz investigated the supersolvable residual $N$ of an $M$-group $G$. Several of his results involve assuming $N$ to be an extra-special $p$-group of order $p^3$. We summarize Theorems 3.1, 3.3, 3.10 and Corollary 3.6 of [**9**].

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THEOREM (Seitz). *Let $N$ be the supersolvable residual of a group $G$ and suppose that $N$ is an extra-special $p$-group of order $p^3$. Put $C = C_G(N/Z(N))$. Then $G/C$ can be identified as a subgroup of $GL(2, p)$, and $G$ is an $M$-group if and only if $p > 3$ and $(G/C) \cap SL(2, p)$ is reducible.*

Our methods allow us to generalize this result in the case that $|G:N|$ is odd. We do not need to assume that $N$ is the residual or to restrict the structure of $N$ quite so severely. For $\theta \in \mathrm{Irr}(N)$ we identify $\ker(\theta)$ as in 2.20 of [7], and the inertia group $I_G(\theta)$ as in 6.10 of [7].

THEOREM C. *Let $N \Delta G$ with $N' \leqq Z(N)$ and $G/N$ supersolvable of odd order. Then $G$ is an $M$-group if and only if for every $\theta \in \mathrm{Irr}(N)$, there is a subgroup $A$ of $N$ containing $\ker(\theta)$, normalized by $I_G(\theta)$, and such that*

$$A = C_N(A/\ker\theta).$$

The relationship between Seitz' theorem and Theorem C is clearer in the following:

THEOREM D. *Let $N \Delta G$ with $N$ an extra-special $p$-group and $G/N$ supersolvable of odd order. Then $G$ is an $M$-group if and only if there is $A \leqq N$ with*

$$A \,\Delta\, C_G(Z(N)) \quad \text{and} \quad A = C_N(A).$$

In Section 2 we collect the facts about characters of nilpotent groups needed in Sections 3, 7, and 8. In Section 3 we prove a rather general result enabling us to find certain monomial characters of inertia groups. Sections 4 and 5 contain a generalization of Dade's hyperbolic modules, (found in [**2**]) which allows us to prove the key Theorem 6.4 in Section 6 and to obtain Theorems C and D. In Section 7 we introduce a process which uses a nilpotent normal subgroup of a group $G$ to control questions about which characters of $G$ are monomial. We are then able to obtain a characterization of $M$-groups of the type in Theorems A and B (see Theorem 7.4). We link Sections 6 and 7 together in Section 8 and prove Theorems A and B.

Throughout, we assume the notation of [7]. All groups considered are finite and all modules are right, unital, and finite dimensional. If $\psi$ is a character of $G$, we denote the set of its irreducible constituents by $\mathrm{Irr}(G|\psi)$.

Because of its fundamental importance we mention Clifford's theorem (6.11 of [7]). Let $N \Delta G$ and $\theta \in \mathrm{Irr}(N)$. Put $I = I_G(\theta)$. Then for each $\chi \in \mathrm{Irr}(G|\theta^G)$ there is a unique $\chi_\theta \in \mathrm{Irr}(I)$ such that $(\chi_\theta)_N$ is a multiple of $\theta$ and $(\chi_\theta)^G = \chi$. Also, if $\psi \in \mathrm{Irr}(I|\theta^I)$ then

$$\psi^G \in \mathrm{Irr}(G) \quad \text{and} \quad \psi = (\psi^G)_\theta.$$

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**2. Characters of nilpotent normal subgroups.** If $\theta \in \operatorname{Irr}(N)$ then as in Definition 2.26 of [7] we put

$$Z(\theta) = \{x \in N| \quad |\theta(x)| = \theta(1)\}.$$

By Lemma 2.27 of [7],

$$Z(\theta)/\ker(\theta) = Z(N/\ker(\theta))$$

and this factor group is cyclic.

The following well-known fact will be needed in Sections 3 and 6.

PROPOSITION 2.1. *Assume that* $N' \leqq Z(N)$. *Let* $\theta \in \operatorname{Irr}(N)$ *and put* $Z = Z(\theta)$. *Then there is a unique element* $\lambda$ *of* $\operatorname{Irr}(Z|\theta_Z)$ *and* $\theta$ *is the unique element of* $\operatorname{Irr}(N|\lambda^N)$.

In Sections 6 and 7 we will need to factor characters of nilpotent groups. If $N$ is a nilpotent group and $\theta \in \operatorname{Irr}(N)$ then in light of Definition 4.20 and Theorem 4.21 of [7] we have

PROPOSITION 2.2. *$\theta$ can be written uniquely of the form $\prod \theta_p$ where $p$ ranges over the prime divisors of $|N|$, $\theta_p \in \operatorname{Irr}(N)$, and $\ker(\theta_p)$ contains the normal $p$-complement of $N$. If $Q_p \in \operatorname{Syl}_p(N)$, then we also have that $\theta$ is a direct product $\prod \psi_p$ where $\psi_p \in \operatorname{Irr}(Q_p)$. In fact $\psi_p = (\theta_p)_{Q_p}$ and $\theta_p$ is determined by $\psi_p$. Finally,*

$$\ker(\theta) = \cap \ker(\theta_p) = \prod \ker(\psi_p).$$

PROPOSITION 2.3. *Let $\theta \in \operatorname{Irr}(N)$ and factor $\theta = \prod \theta_p$ as in Proposition 2.2. Let $A_p \leqq N$ with $\ker(\theta_p) \leqq A_p$. Suppose that $A = \cap A_p$. If $x \in N$ and $[A, x] \leqq \ker(\theta)$, then*

$$[A_p, x] \leqq \ker(\theta_p) \quad \textit{for all } p.$$

*Proof.* If $Q_p \in \operatorname{Syl}_p(A_p)$, then since $N/\ker(\theta_p)$ is a $p$-group, we have $A = \prod_p Q_p$, where $Q_p \in \operatorname{Syl}_p(A)$ for all $p$. Since $Q_p$ char $A$, $[A, x] \leqq \ker(\theta)$ implies that

$$[Q_p, x] \leqq (Q_p) \cap \ker(\theta) \leqq \ker(\theta_p)$$

where the last containment is furnished by Proposition 2.2.

Now $A_p = \ker(\theta_p) \cdot Q_p$ and $\ker(\theta_p)\ \Delta\ N$. Thus

$$[A_p, x] \leqq \ker(\theta_p) \quad \text{for all } p,$$

as needed.

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In what follows, $N$ is a nilpotent normal subgroup of a group $G$. Whenever a product sign or intersection sign occurs for objects with a subscript $p$ we understand that the product or intersection is taken over all primes $p$ dividing $|N|$. Similarly, the phrase "for each $p$" means "for each prime $p$ dividing $|N|$". Furthermore, if $B_p$ denotes a group and $\theta_p$ a character of $B_p$, we define

$$B = \cap B_p \quad \text{and} \quad \theta = \prod (\theta_p)_B.$$

PROPOSITION 2.4. *Let $A_p \leqq N$ for each $p$ and assume $\theta_p \in \mathrm{Irr}(A_p)$ and that $\ker(\theta_p)$ contains the normal $p$-complement of $N$. Then*

a) $\theta \in \mathrm{Irr}(A)$

b) $\ker(\theta) = \cap \ker(\theta_p)$

c) *if $x \in G$ then $A^x = A$ and $\theta^x = \theta$ if and only if $(A_p)^x = A_p$ and $(\theta_p)^x = \theta_p$ for all $p$.*

*Proof.* Let $Q_p \in \mathrm{Syl}_p(A_p)$ for each $p$. Then the hypothesis shows that $A = \prod Q_p$ (internal direct product). By Proposition 2.2 applied to each $\theta_p \in \mathrm{Irr}(A_p)$ we see if $\psi_p = (\theta_p)_{Q_p}$ then $\psi_p \in \mathrm{Irr}(Q_p)$ and $\theta_p$ is determined by $\psi_p$.

We see that

$$\theta = \prod (\theta_p)_A = \prod \psi_p,$$

and thus $\theta \in \mathrm{Irr}(A)$ by Theorem 4.21 of [7]. By Proposition 2.2,

$$\ker(\theta) = \prod \ker(\psi_p) = \cap \ker(\theta_p).$$

This proves (a) and (b).

Let $x \in G$ and suppose that $A^x = A$ and $\theta^x = \theta$. Since $Q_p$ char $A$ we find that $Q_p^x = Q_p$. By the uniqueness of factorization of $\theta$, $(\psi_p)^x = \psi_p$. Now $A_p = R_p \times Q_p$ where $R_p$ is the normal $p$-complement of $N$. Since $R_p$ char $N$, $(R_p)^x = R_p$ and hence $(A_p)^x = A_p$. Now $\psi_p$ determines $\theta_p = 1_{R_p} \times \psi_p$ and so $(\theta_p)^x = \theta_p$.

Conversely, if $x \in G$ and $(A_p)^x = A_p$ and $(\theta_p)^x = \theta_p$ for all $p$, then

$$A^x = \prod (A_p^x) = A \quad \text{and} \quad \theta^x = \prod (\theta_p^x)_A = \theta.$$

This proves (c).

PROPOSITION 2.5. *Let $A_p \leqq B_p \leqq N$ for each $p$. Suppose*

$$\lambda_p \in \mathrm{Irr}(A_p), \quad \theta_p \in \mathrm{Irr}(B_p | (\lambda_p)^{B_p}),$$

*and that $\ker(\lambda_p)$ contains the normal $p$-complement of $N$ for each $p$. Then*

$$A \leqq B \quad \text{and} \quad \theta \in \mathrm{Irr}(B|\lambda^B).$$

*Proof.* It is clear that $A \leqq B$, and by Proposition 2.3, $\lambda \in \mathrm{Irr}(A)$ and $\theta \in \mathrm{Irr}(B)$.

For each $p$, let $Q_p \in \mathrm{Syl}_p(B_p)$, then

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$$A_p \cap Q_p \in \mathrm{Syl}_p(A_p).$$

Put

$$\psi_p = (\theta_p)_{Q_p} \quad \text{and} \quad \varphi_p = (\psi_p)_{A_p \cap Q_p}.$$

Then as in Proposition 2.4,

$$\theta = \prod \psi_p \quad \text{and} \quad \lambda = \prod \varphi_p.$$

Since $[\theta_p, (\lambda_p)^{B_p}] \neq 0$ we find that

$$[\psi_p, (\varphi_p)^{Q_p}] \neq 0 \quad \text{for each } p.$$

The proof of Theorem 4.21 of [7] shows that

$$[\prod \psi_p, (\prod \varphi_p)^B] = \prod [\psi_p, (\varphi_p)^{Q_p}] \neq 0.$$

Thus $[\theta, \lambda^B] \neq 0$.

Finally, we need information about a configuration which will arise in Section 7. Because we have all the notation at hand, it seems best to prove it now.

PROPOSITION 2.6. *Let $A_p \leqq B_p \Delta H_p \leqq G$ for each $p$ with $B_p \leqq N$ and $A_p \Delta H_p$. Let*

$$\lambda_p \in \mathrm{Irr}(A_p) \quad \textit{and} \quad \theta_p \in \mathrm{Irr}(B_p|\,(\lambda_p)^{B_p})$$

*and suppose that* $\ker(\lambda_p)$ *contains the normal $p$-complement of $N$ for all $p$. Put*

$$J_p = I_{H_p}(\theta_p), \quad K_p = I_{B_p}(\lambda_p), \quad \textit{and}$$
$$\varphi_p = (\theta_p)_{\lambda_p} \quad \textit{for each } p.$$

*Then $A \leqq B \Delta H$, $A \Delta H$, $J = I_H(\theta)$, $K = I_B(\lambda)$, and $\varphi = \theta_\lambda$.*

*Proof.* Observe that $K_p \leqq N$ and that $\ker(\varphi_p) \geqq \ker(\lambda_p)$ so that $\varphi \in \mathrm{Irr}(K)$ by Proposition 2.4. It is clear that $A$ and $B$ are normal in $H$. That $J = I_H(\theta)$ and $K = I_B(\lambda)$ follows from Proposition 2.4(c) applied respectively to $B_p$, $\theta_p$ and to $A_p$, $\lambda_p$.

Since $\theta_p \in \mathrm{Irr}(B_p|\,(\lambda_p)^{B_p})$, Proposition 2.5 shows that $\theta \in \mathrm{Irr}(B|\lambda^B)$ so that $\theta_\lambda$ makes sense. By the same proposition applied to $\theta_p \in \mathrm{Irr}(B_p|\,(\varphi_p)^{B_p})$ we conclude that

$$\theta \in \mathrm{Irr}(B|\varphi^B).$$

For the same reason, $\varphi_p \in \mathrm{Irr}(K_p|\,(\lambda_p)^{K_p})$ implies that

$$\varphi \in \mathrm{Irr}(K|\lambda^K).$$

By Clifford's theorem we must have that $\varphi = \theta_\lambda$. This completes the proof.

**3. Passing to inertia groups with monomial characters.** Let $N \Delta G$, and suppose that $\chi \in \mathrm{Irr}(G)$. Let $\theta \in \mathrm{Irr}(N|\chi_N)$, and put $I = I_G(\theta)$. By

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Clifford's theorem there is a unique element $\chi_\theta \in \mathrm{Irr}(I)$ such that $(\chi_\theta)_N$ is a multiple of $\theta$, and such that $(\chi_\theta)^G = \chi$. It is possible for $\chi$ and $\theta$ to be monomial, yet $\chi_\theta$ not; this situation is at the heart of many interesting examples in the theory of monomial characters (see [**1**], or see Example 6.4 of Berger in [**8**] ).

In various settings later on, we will need to assert that a certain $\chi_\theta$ is monomial. Theorem 3.1 shows that this is the case under a fairly general hypothesis.

We say a group is *Sylow-abelian by nilpotent* if the Sylow $p$-subgroups of its nilpotent residual are abelian for all primes $p$.

THEOREM 3.1. *Let $N \Delta G$ with $N$ solvable and Sylow abelian by nilpotent. Suppose $\chi \in \mathrm{Irr}(G)$ is monomial, and that $\theta \in \mathrm{Irr}(N|\chi_N)$. Then $\chi_\theta$ is monomial.*

A use of induction in the proof of Theorem 3.1 will reduce us to the case that all characteristic, abelian subgroups of $N$ are central in $N$. The following two lemmas allow us to handle this situation.

LEMMA 3.2. *Assume that $N$ is solvable, Sylow-abelian, and that all characteristic, abelian subgroups of $N$ are central in $N$. Then $N$ is abelian.*

*Proof.* Observe that the hypothesis is inherited by all characteristic subgroups of $N$. We induct on $|N|$.

Let $p$ be a prime divisor of $|N:N'|$ and let $A = O^p(N)$. Then $A < N$, and $A$ char $N$. By induction $A' = 1$, and thus, by hypothesis, $A \leqq Z(N)$.

If $P \in \mathrm{Syl}_p(N)$, then $AP = N$. Now $P$ is abelian and $A \leqq Z(N)$. It follows that $N$ is abelian.

LEMMA 3.3. *Assume that $N$ is solvable and Sylow-abelian by nilpotent. Suppose that all characteristic, abelian subgroups of $N$ are central in $N$. Then $N' \leqq Z(N)$.*

*Proof.* Let $A$ be the nilpotent residual of $N$. Since $A$ char $N$, Lemma 3.2 allows us to conclude that $A' = 1$, hence, by hypothesis, $A \leqq Z(N)$. Now $N$ is nilpotent, for $N/Z(N)$ is nilpotent.

If $K_i$ is the $i^{\text{th}}$ term of the lower central series for $N$, then by Satz III. 2.11 of [**6**], $K_i$ is abelian when $2i$ is greater than the nilpotence class of $N$. Such $K_i$ must be central in $N$ by hypothesis, hence $K_{i+1} = 1$. This situation forces that the class of $N$ is at most 2.

Our next result establishes Theorem 3.1 in the case where $\theta$ is linear, but without any special assumptions on $G$ or on $N$. This fairly easy and well known fact may be deduced from Lemma 4.1 of [**2**].

LEMMA 3.4. *Let $N \Delta G$. Suppose that $\chi \in \mathrm{Irr}(G)$ is monomial. Let $\theta \in \mathrm{Irr}(N|\chi_N)$ be linear. Then $\chi_\theta$ is monomial.*

*Proof of Theorem* 3.1. Among all quadruples $G$, $\chi$, $N$, $\theta$ satisfying the

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hypothesis but not the conclusion of Theorem 3.1, choose one so as to minimize $|G|$. We will derive a contradiction.

STEP 1. *All characteristic, abelian subgroups of N are central in G.*

*Proof.* Let $A \leqq N$, $A \,\Delta\, G$, and $A' = 1$. Choose $\lambda \in \mathrm{Irr}(A|\theta_A)$. Then $\lambda \in \mathrm{Irr}(A|\chi_A)$. We will show that $\lambda$ is invariant in $G$. Since by the minimality of $|G|$, $\chi$ is faithful; this will prove that $A \leqq Z(G)$ (see Lemma 2.27 of [7] ).

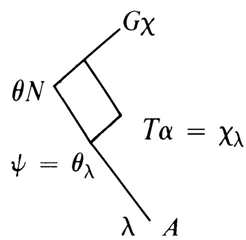
Assume, to the contrary, that $T = I_G(\lambda)$ is proper in $G$. Now $N \cap T = I_N(\lambda)$. By Clifford's theorem we find

$$\psi = \theta_\lambda \in \mathrm{Irr}(N \cap T)$$

with $\psi^N = \theta$ and $\psi_A$ a multiple of $\lambda$. Put

$$\alpha = \chi_\lambda \in \mathrm{Irr}(T),$$

so that $\alpha^G = \chi$ and $\alpha_A$ is a multiple of $\lambda$.




We claim that

$$\psi \in \mathrm{Irr}(N \cap T|\alpha_{N\cap T}).$$

Indeed, since $\psi^N = \theta$ and $\chi \in \mathrm{Irr}(G|\theta^G)$, we find that

$$\chi \in \mathrm{Irr}(G|\psi^G).$$

Thus $\chi_T$ and $\psi^T$ have a common irreducible constituent $\beta$. Clearly,

$$\beta \in \mathrm{Irr}(T|\chi_T).$$

Also, since $\psi_A$ is a multiple of $\lambda$, and since $\psi$ is a constituent of $\beta_{N\cap T}$, we find that

$$\lambda \in \mathrm{Irr}(A|\beta_A).$$

This proves that $\beta \in \mathrm{Irr}(T|\lambda^T)$. By Clifford's theorem, we must have $\beta = \chi_\lambda$, and so $\beta = \alpha$. Then

$$\psi \in \mathrm{Irr}(N \cap T|\alpha_{N\cap T}),$$

as claimed.

By Lemma 3.4, $\alpha$ is monomial. We want to apply the inductive hypothesis to

$$(N \cap T) \,\Delta\, T, \quad \alpha \in \mathrm{Irr}(T), \quad \text{and}$$

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$$\psi \in \mathrm{Irr}(N \cap I|\alpha_{N \cap T}).$$

Since $N \cap T \leqq N$, $N \cap T$ is Sylow-abelian by nilpotent. By the minimality of $|G|$, we conclude that $\alpha_{\psi}$ is monomial.

Put $J = I_T(\psi)$ and let $\beta = \alpha_{\psi}$, so then $\beta^T = \alpha$, and $\beta_{N \cap T}$ is a multiple of $\psi$. Put $I = I_G(\theta)$. We claim that $J \leqq I$. Indeed, if $x \in J$, then

$$\theta^x = (\psi^N)^x = (\psi^x)^N = \psi^N = \theta.$$

Thus $x \in I$, as needed.

Finally, we claim that $\beta^I = \chi_{\theta}$. This will be a contradiction, for $\beta^I$ is monomial, yet $\chi_{\theta}$ is not. Now

$$\beta^G = (\beta^T)^G = \alpha^G = \chi,$$

and thus $\beta^I \in \mathrm{Irr}(I|\chi_I)$. Since $\beta_{N \cap T}$ is a multiple of $\psi$, $(\beta^I)_{N \cap T}$ has $\psi$ as a constituent. Thus $(\beta^I)_N$ has $\psi^N = \theta$ as a constituent. We conclude that

$$\beta^I \in \mathrm{Irr}(I|\theta^I).$$

By Clifford's theorem, $\beta^I = \chi_{\theta}$. This completes step 1.

From step 1, it follows that all characteristic, abelian subgroups of $N$ are central in $N$. By Lemma 3.3, $N' \leqq Z(N)$. Put $Z = Z(N)$.

STEP 2. $Z = Z(\theta)$.

*Proof.* Certainly $Z \leqq Z(\theta)$. But

$$[N, Z(\theta)] \leqq N' \cap \ker(\theta).$$

Now $N' \leqq Z$, and since $N \neq 1$, we have $N'$ is a proper, characteristic, abelian subgroup of $N$. By step 1, $N' \leqq Z(G)$. Thus

$$N' \cap \ker(\theta)\ \Delta\ G.$$

Since $\chi$ is faithful, this proves that

$$N' \cap \ker(\theta) = 1$$

(see Corollary 6.7 of [7] ). Hence $Z(\theta) \leqq Z$.

By Proposition 2.1, if $\lambda \in \mathrm{Irr}(Z|\theta_Z)$, then $\theta$ is the unique element of $\mathrm{Irr}(N|\lambda^N)$. If $Z < N$, then by step 1, $Z \leqq Z(G)$, and so $\lambda$ is invariant in $G$. But then $\theta$ is invariant in $G$. Hence $\chi_{\theta} = \chi$, a contradiction.

Thus $Z = N$, and so $\theta$ is linear. By Lemma 3.4, $\chi_{\theta}$ is monomial, a contradiction. This proves Theorem 3.1

**4. Extensions of modules.** The goal of this section is to prove

THEOREM 4.1. *Let $M\ \Delta\ G$ with $|G:M| = q$, an odd prime. Suppose $U$ is an irreducible, $G$-invariant module for $M$ over a finite field $F$ of characteristic $p$, where $q$ divides $p(p - 1)$. Then there is to within $FG$-isomorphism at most one irreducible, self-dual $FG$-module lying over $U$.*

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LEMMA 4.2. *Assume the hypothesis of Theorem* 4.1. *Let $V$ be an irreducible $FG$-module lying over $U$. Then $V \cong U^G$ or $V_M \cong U$.*

*Proof.* We assume $U \subseteq V$. Let $x \in G\backslash M$ and let $n \geqq 1$ be minimal such that

$$(3) \qquad Ux^n \subseteq \sum_{i=0}^{n-1} Ux^i.$$

Then the sum on the right side of (3) is direct since $U$ is $M$-irreducible, and the object on the right side of (3) is a $G$-submodule, hence is $V$, for $V$ is $G$-irreducible. Because $|G:M| = q$, we have $n \leqq q$.

If $n = q$, then obviously $V \cong U^G$.

Now suppose that $n < q$. By Clifford's theorem $V_M$ is homogeneous. Put $E = \mathrm{End}_{FM}(U)$, and then the number of irreducible $M$-submodules of $V$ is

$$e = |E|^{n-1} + \ldots + |E| + 1$$

(see **[5]**, Lemma 2.2.3 for this result of Green).

We claim that $q$ does not divide $e$. Indeed, if $q = p$ then $e \equiv 1 \bmod q$. If, on the other hand, $q|p - 1$ then $|E| \equiv 1 \bmod q$, for $E$ is a finite field of characteristic $p$. Since $n < q$, $q \nmid e$ in this case either.

The previous paragraph shows that some irreducible $M$-submodule of $V$ is a $G$-submodule, for the orbits of $M$-submodules under right translation by elements of $G$ have size $q$ or 1. Since $V$ is irreducible, this proves that $n = 1$, as needed.

*Proof of Theorem* 4.1. Let $V$, $W$ be irreducible, self-dual $FG$-modules lying over $U$. We can assume that $U$ is an $FM$-submodule of $V$ and of $W$.

By a standard result, both $V$ and $W$ are homomorphic images of $U^G$. If either $V$ or $W$ is isomorphic to $U^G$, then so is the other, and we are done.

By Lemma 4.2, the only possibility left is that $V_M = U = W_M$. The rest of the proof is meant to show that $W$ is isomorphic to the tensor product of $V$ with a 1-dimensional module. To this end, for $g \in G$, denote by $\circ g$ the endomorphism of $U$ induced by the action of $g$ on $V$, and by $\cdot g$, the endomorphism induced by the action on $W$. We have

$$(1) \qquad \circ g = \cdot g \quad \text{for all } g \in M.$$

Put $E = \mathrm{End}_{FM}(U)$. Since $F$ is finite, and $U$ is irreducible, $E$ is a finite field. We view $F \subseteq E$.

Choose $x \in G - M$ and define $\sigma \in \mathrm{Aut}(E)$ by

$$\delta^\sigma = \circ x^{-1} \delta \circ x \quad \text{for } \delta \in E.$$

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Next define $N_\sigma : E \to E$ by

(2) $\quad N_\sigma(\delta) = \delta^{\sigma^{q-1}} \ldots \delta^\sigma \delta \quad$ for $\delta \in E$.

Observe that

$$N_\sigma(\delta) = \circ x^{-q}(\circ x \delta)^q.$$

Finally observe that $\circ x^{-1} \cdot x$ is an element of $E$, call it $\alpha$. Then

$$N_\sigma(\alpha) = \circ x^{-q}(\circ x \alpha)^q = \circ x^{-q} \cdot x^q = 1,$$

using (1).

Because $x^q \in M$, we have $\sigma^q = 1$.

Case 1. $\sigma = 1$. Then $1 = N_\sigma(\alpha) = \alpha^q$ and since $q$ divides $p(p - 1)$ we have $\alpha \in F$. From the definition of $\alpha$, we conclude that $W$ is $FG$-isomorphic to $V \otimes R$ where $R$ is the 1-dimensional $FG$-module on which $x$ has eigenvalue $\alpha$. Since $V$ and $W$ are both self-dual and $q$ is odd, it follows easily that $R$ is trivial ($\alpha = 1$) and therefore $V \cong W$.

Case 2. $\sigma \neq 1$. Then $N_\sigma$ is the norm map from $E$ to the fixed field of $\sigma$. Since $N_\sigma(\alpha) = 1$, Hilbert's theorem 90 forces there to be $\beta \in E^\times$ satisfying $\beta\alpha = \beta^\sigma$. It is then routine to check that the map sending $u \in U$ to $u\beta$ is an $FG$-isomorphism from $W$ to $V$.

**5. Forms on abelian groups.** Let $V$ be an abelian group and let $Z$ be a cyclic group. We will write the group operation on $V$ and on $Z$ as $+$. A *$Z$-form on $V$* is a map $[,]: V \times V \to Z$ satisfying:

$$[u, v + w] = [u, v] + [u, w]$$

$$[u + v, w] = [u, w] + [v, w]$$

$$[u, u] = 0 \text{ for all } u, v, w \in V.$$

From these conditions, it is clear that $[u, v] = -[v, u]$.

This is a straightforward generalization of the idea of an alternating bilinear form on a vector space. Our preliminary work is essentially to generalize certain elementary facts from vector spaces to arbitrary abelian groups.

For $U \subseteq V$, put

$$U^\perp = \{v \in V \mid [u, v] = 0 \text{ for all } u \in U\}, \quad \text{then}$$

$$U^\perp = \{v \in V \mid [v, u] = 0 \text{ for all } u \in U\}$$

as well, and so $(U^\perp)^\perp \supseteq U$. If $U \subseteq U^\perp$ we say $U$ is *isotropic* and define the *factor form* on $U^\perp/U$ by

$$(U + v, U + w) = [v, w].$$

This is clearly a well defined $Z$-form on $U^\perp/U$. If $U \cap U^\perp = 0$, we say $U$ is *nondegenerate.*

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Throughout this section, $V$ is an abelian $p$-group for some prime $p$ and $G$ acts on $V$ so as to stabilize the $Z$-form [,] on $V$

$$[ug, vg] = [u, v] \quad \text{for all } u, v \in V, g \in G.$$

We will say that $V$ is *$G$-hyperbolic* if there is a $G$ invariant $U \subseteq V$ such that $U = U^{\perp}$. The goal of this section is to prove the following.

THEOREM 5.1. *Let $G$ be a group of odd order. Suppose $M \Delta G$ with the property that if $q$ is a prime divisor of $|G:M|$, then $q$ divides $p(p - 1)$. Assumes that $V$ is $M$-hyperbolic. Then $V$ is $G$-hyperbolic.*

In Theorem 3.2 of **[2]**, Dade proved that if $V$ is elementary abelian and $|G:M|$ is a power of $p$, then $V$ is $M$-hyperbolic implies $V$ is $G$-hyperbolic, without assuming that $M \Delta G$. Our result was inspired by Dade's theorem.

PROPOSITION 5.2. *Let $U$ be an abelian group. Then*

$$|\mathrm{Hom}(U, Z)| \leqq |U|.$$

*Proof.* Write $U$ as a direct product $\langle x_1 \rangle \dot{+} \ldots \dot{+} \langle x_n \rangle$. An element $\sigma \in \mathrm{Hom}(U, Z)$ is determined by the values $x_i^{\sigma}$. Since $Z$ is cyclic, there are $n_i \leqq O(x_i)$ possible choices for $x_i^{\sigma}$. Thus

$$|\mathrm{Hom}(U, Z)| = \prod n_i \leqq \prod O(x_i) = |U|.$$

LEMMA 5.3. *Let $U \subseteq V$ be nondegenerate. Then $V = U \dot{+} U^{\perp}$ (internal direct product).*

*Proof.* We can map $V$ into $\mathrm{Hom}(U, Z)$ by sending $v \in V$ to $[\cdot, v]$ considered as a function on $U$. Clearly, the kernel of this map is $U^{\perp}$. Because $U$ is nondegenerate, the map is an injection when restricted to $U$. Thus

$$|U| \leqq |\mathrm{Hom}(U, Z)|.$$

By Proposition 5.2, we must have

$$|U| = |\mathrm{Hom}(U, Z)|.$$

Thus $U$ is mapped bijectively to $\mathrm{Hom}(U, Z)$. It follows that $V = U \dot{+} U^{\perp}$.

Given Lemma 5.3, the proof of the following is virtually identical to Proposition 1.5 of **[2]**.

LEMMA 5.4. *Let $X$, $Y$ be maximal, $G$-invariant, isotropic subgroups of $V$. Then $X^{\perp}/X \cong Y^{\perp}/Y$ as $G$-groups.*

LEMMA 5.5. *Suppose that every $G$-subgroup of $V$ is nondegenerate. Then $V$ is elementary and so is a vector space over a field $F$ of order $p$. Furthermore,* the *$Z$-form* [,] *is equivalent to a $G$-invariant, $F$-bilinear, alternating form on*

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*$V \times V$. Moreover, $V = U_1 \dotplus \dots \dotplus U_n$ where $U_i$ are irreducible $FG$-submodules and $U_i \subseteq U_j^\perp$ if $i \neq j$.*

*Proof.* We induct on $|V|$. Let $U > 0$ be a minimal $G$-subgroup. Then $U$ has exponent $p$ and so the values $[w, v]$ for $w, v \in U$ lie in the unique subgroup of $Z$ of order $p$. Thus $U$ may be viewed as an $FG$-module and $[,]$ restricted to $U$ as an $F$-linear form.

By Lemma 5.3, $V = U \dotplus U^\perp$. By induction, $U^\perp$ has exponent $p$ and can be decomposed as required.

We remind the reader that $|G|$ is odd in the following lemma. This result follows from the work of Isaacs in [**8**] and it is stated as Corollary 2.10 of [**2**], together with Proposition 1.10 of [**2**].

LEMMA 5.6. *Assume $F$ is a finite field and that $V$ is an $FG$-module with a nondegenerate, $G$-invariant, alternating $F$-bilinear form $[,]$. Suppose $V = U_1 \dotplus U_2$ where $U_i$ are nondegenerate isomorphic irreducible $G$-submodules and $U_1 \subseteq U_2^\perp$. Then $V$ contains a nonzero, isotropic $G$-submodule.*

*Proof of Theorem* 5.1. Choose $(G, M, V)$ among all counterexamples to Theorem 5.1 to minimize first $|G|$, then $|G:M|$, and then $|V|$. Because $G/M$ is solvable, the minimality of $G$ forces that $|G:M| = q$, a prime. (Then $q | p(p - 1)$.)

STEP 1. *If $U \subseteq V$ is a $G$-subgroup then $U$ is nondegenerate.*

*Proof.* Assume $0 < U$ is irreducible and not nondegenerate. Then $U$ is isotropic. Let $U \subseteq X$ a maximal, $M$-invariant isotropic subgroup of $V$. Let $Y \subseteq V$ be $M$-invariant with $Y = Y^\perp$. By Lemma 5.4,

$$X^\perp / X \cong Y^\perp / Y = 0.$$

Thus $X = X^\perp$; and this shows that $U^\perp / U$ with the factor form is $M$-hyperbolic: $X/U = (X/U)^\perp$.

By the minimality of $V$, $U^\perp / U$ is $G$-hyperbolic and so we find $G$-invariant $W$ with

$$U \subseteq W \subseteq U^\perp \quad \text{and} \quad W/U = (W/U)^\perp.$$

Then $W = W^\perp$ and so $V$ is $G$-hyperbolic, a contradiction.

Now by Lemma 5.5, we view $V$ as an $FG$-module where $F$ is a field of order $p$. By Lemma 5.5, $V$ is completely reducible; thus, by Clifford's theorem, $V_M$ is completely reducible.

STEP 2. *Let $V = X \dotplus Y$ where $X$, $Y$ are nonzero, $G$-submodules with $X \subseteq Y^\perp$. Then there is $X_1 \subseteq X$, $Y_1 \subseteq Y$ with $X_1$, $Y_1$ nonzero $M$-submodules and $X_1 \cong Y_1$.*

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*Proof*. Otherwise $X$ and $Y$ have no common $M$-composition factor. It follows that if $W \subseteq V$ is any $M$-submodule, then

$$W = (W \cap X) + (W \cap Y).$$

In particular, if $W = W^{\perp}$ then $W \cap X$ and $W \cap Y$ are their own perpendicular subspaces in $X$ and $Y$ respectively.

This proves that $X$ and $Y$ are $M$-hyperbolic. By the minimality of $V$, $X$ and $Y$ are $G$-hyperbolic, hence so is $V$, which is not the case.

The group $G$ acts on the homogeneous components of $V_M$ by right translation. The sum of the elements of any $G$ orbit is a $G$-submodule $X$ and $X$ is nondegenerate by Step 1. By Lemma 5.3, $V = X \dotplus X^{\perp}$, but this contradicts Step 2 unless $X = V$.

STEP 3. *$V_M$ is homogeneous.*

*Proof*. Otherwise, since $|G:M| = q$, there are exactly $q$ homogeneous components to $V_M$, say $W_1, \ldots, W_q$. Then $V = W_1 \dotplus \ldots \dotplus W_q$. Since $q$ is odd and since the $W_i$ are all $G$-translates of $W_1$, each $W_i$ is nondegenerate and orthogonal to the others.

By hypothesis there is $A \subseteq V$, an $M$-submodule with $A = A^{\perp}$. Then

$$A = \sum_i (A \cap W_i),$$

and so

$$(W_1 \cap A)^{\perp} \cap W_1 \subseteq A.$$

Choose $x \in G - M$, and we can arrange that $W_1 x^i = W_{i+1}$. Put

$$X = \sum_{i=1}^{q} (W_1 \cap A)x^i$$

and it follows that $X = X^{\perp}$ is a $G$-submodule of $V$, a contradiction.

Write $V = U_1 \dotplus \ldots \dotplus U_n$ where $U_i$ are irreducible, pairwise orthogonal (hence self-dual) $G$-submodules (Lemma 5.5).

The form [,] is alternating, and so, since $|G|$ is odd, if $U_i \cong U_j$ say with $i \neq j$ then by Lemma 5.6, $U_i + U_j$ has an isotropic $G$-submodule, contradicting Step 1. Thus the $U_i$ are pairwise nonisomorphic.

STEP 4. *$V$ is irreducible.*

*Proof*. Let $n > 1$. By Step 3, $U_1$ and $U_2$ have irreducible $M$-submodules $X_1$, $X_2$ respectively which are $FM$-isomorphic. By Theorem 4.1, $U_1 \cong U_2$, which is not so.

Because $V$ is $M$-hyperbolic, $V_M$ is not irreducible. By Lemma 4.2

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$V \cong X^G$ where $X$ is an irreducible $FM$-submodule of $V$. Thus the composition length of $V_M$ is $q$ which is odd.

Now let $A \subseteq V$ with $A = A^{\perp}$ and $A$ an $M$-submodule. Because $V_M$ is completely reducible $V = A \dot{+} B$ for some $M$-submodule $B$. Since the composition length of $V_M$ is odd and $V_M$ is homogeneous, we cannot have $|A| = |B|$. But since $V$ is a vector space with a nondegenerate, alternating form we cannot have $A = A^{\perp}$ unless $|A| = |B|$. This contradiction completes the proof of Theorem 5.1.

Finally, we indicate a natural setting where a $G$-invariant $Z$-form on an abelian group $V$ can arise, and we interpret the condition of being hyperbolic in this setting.

PROPOSITION 5.7. *Let $N \Delta G$ with $N' \leqq Z(N) \leqq Z(G)$. Assume that $Z = Z(N)$ is cyclic. Put $V = N/Z$ and let $G$ act on $V$ by conjugation. Then the map*

$$(Zx, Zy) = x^{-1}y^{-1}xy$$

*is a well-defined, $G/N$-invariant $Z$-form on $V$.*

*Proof.* Because $N' \leqq Z$, $[g, h] = g^{-1}h^{-1}gh$ is independent of the choice of $g \in Zx$ and $h \in Zy$. Thus (,) is well-defined. The usual commutator identities show that (,) is a $Z$-form.

Since $Z \subseteq Z(G)$, (,) is $G$-invariant:

$$\begin{aligned}((Zx)^g, (Zy)^g) &= (Zx^g, Zy^g) = [x^g, y^g] \\ &= [x, y]^g = [x, y] = (Zx, Zy)\end{aligned}$$

for all $x, y \in N$ and $g \in G$. Since $N$ acts trivially on $V$, $G/N$ acts on $V$.

PROPOSITION 5.8. *Assume the hypothesis of Proposition 5.7. Then $V$ is $G/N$ hyperbolic if and only if there is $A \leqq N$ with $A \Delta G$ and $A = C_N(A)$.*

*Proof.* By definition of the $Z$-form, a subgroup $B/Z$ of $V$ is isotropic if and only if $B$ is abelian ($B \leqq C_N(B)$). Also

$$(B/Z)^{\perp} = C_N(B)/Z.$$

The result is now clear.

**6. Class 2 by supersolvable groups.** We want to prove Theorems C and D as well as to develop some further machinery relevant to Theorems A and B.

THEOREM 6.1. *Let $N \Delta G$ with $N' \leqq Z(N)$ and $G/N$ a supersolvable group. Suppose $\theta \in \mathrm{Irr}(N)$ is invariant in $G$ and that there is $A \leqq N$ with*

$$A \Delta G,\ \ker(\theta) \leqq A, \quad \textit{and} \quad A = C_N(A/\ker(\theta)).$$

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*Then all elements of* $\mathrm{Irr}(G|\theta^G)$ are monomial.

Theorem 6.1 provides one direction of Theorem C. To prove it we need two simple facts.

LEMMA 6.2. *Assume the hypothesis of Theorem* 6.1. *If* $\lambda \in \mathrm{Irr}(A|\theta_A)$, *then* $\lambda(1) = 1$ *and* $A = I_N(\lambda)$.

*Proof.* Clearly, $A/\ker(\theta)$ is abelian, and so $\lambda(1) = 1$. Clearly

$$Z(N) \leqq C_N(A/\ker(\theta))$$

so that the hypothesis forces that $N' \leqq A$. It follows that $I_N(\lambda)\ \Delta\ N$.

Assume $A < I_N(\lambda)$ and choose $B \leqq I_N(\lambda)$ with $B\ \Delta\ N$ and $|B:A|$ a prime. By Corollary 6.20 of [**7**], $\lambda$ extends to $\mu \in \mathrm{Irr}(B)$, and by Gallagher's theorem (Corollary 6.17 of [**7**]) we may suppose $\mu \in \mathrm{Irr}(B|\theta_B)$.

Then $\mu(1) = \lambda(1) = 1$ and so

$$B' \leqq \ker(\mu) \cap Z(N).$$

Since also, $B'\ \Delta\ N$, we see that

$$\ker(\theta) \geqq \ker(\mu^N) \geqq B'.$$

Thus $[A, B] \leqq \ker(\theta)$. This contradicts that

$$A = C_N(A/\ker(\theta))$$

and proves the lemma.

LEMMA 6.3. *Let* $A\ \Delta\ G$ *with* $G/A$ *supersolvable. Suppose* $\lambda \in \mathrm{Irr}(A)$ *is linear. Then every element of* $\mathrm{Irr}(G|\lambda^G)$ *is monomial.*

*Proof.* If $\chi \in \mathrm{Irr}(G|\lambda^G)$ then

$$\ker \chi \geqq \ker(\lambda^G) \geqq A'.$$

It follows that $\chi$ can be viewed as a character of $G/A'$. Then $\chi$ is monomial by Huppert's theorem (Satz $V$ 18.4 of [**6**]).

*Proof of Theorem* 6.1. Let $\chi \in \mathrm{Irr}(G|\theta^G)$. If $\lambda \in \mathrm{Irr}(A|\chi_A)$ then $\lambda \in \mathrm{Irr}(A|\theta_A)$. Thus $\lambda(1) = 1$ and $A = I_N(\lambda)$ by Lemma 6.2. Put $I = I_G(\lambda)$.

Now $N \cap I = I_N(\lambda) = A$. Let $\psi = \chi_\lambda$. Then $\psi \in \mathrm{Irr}(I|\lambda^I)$ and $I/A = I/(N \cap I)$ is isomorphic to a subgroup of $G/N$, hence is supersolvable. By Lemma 6.3, $\psi$ is monomial, and thus $\chi = \psi^G$ is monomial. This proves Theorem 6.1.

Our next major step is to establish the converse of Theorem 6.1 in the case that $N$ is a $p$-group and $|G:N|$ is odd.

THEOREM 6.4. *Let* $N\ \Delta\ G$ *with* $N' \leqq Z(N)$, $N$ *a* $p$*-group for some prime* $p$, *and* $G/N$ *a supersolvable group of odd order. Let* $\theta \in \mathrm{Irr}(N)$ *be invariant in*

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*G and assume that all elements of* $\mathrm{Irr}(G|\theta^G)$ *are monomial. Then there is* $A \leqq N$ *with*

$$A \Delta G, \quad \ker(\theta) \leqq A, \quad \text{and} \quad A = C_N(A/\ker(\theta)).$$

Because $\theta$ is invariant in $G$, $\ker(\theta) \Delta G$. Since we are seeking $A \geqq \ker(\theta)$, there is no loss of generality in assuming that $\ker(\theta) = 1$. Then by Lemma 2.27 of [**7**], $Z(N) = Z(\theta)$ is cyclic. Put $Z = Z(N)$. Then $\theta$ being invariant in $G$ leads to $Z \leqq Z(G)$. By Proposition 5.7 commutation induces a $G/N$-invariant $Z$-form on $N/Z$. By Proposition 5.8 the existence of $A \leqq N$ with $A \Delta G$ and $A = C_N(A)$ is equivalent to $N/Z$ being $G/N$-hyperbolic.

We begin with a host of preliminaries. The first is an easy consequence of the fact that supersolvable groups have Sylow towers.

LEMMA 6.5. *Let S be a supersolvable group and p a prime. Then there are normal Hall subgroups of S,* $T \leqq U$ *with* $U/T$ *a p-group.*

LEMMA 6.6 *Let S be a supersolvable group and* $P \in \mathrm{Syl}_p(S)$ *with* $P \Delta S$. *Suppose* $Q \in \mathrm{Syl}_q(S)$ *and* $q \nmid p(p-1)$. *Then Q centralizes P.*

*Proof.* If $M \Delta S$ with $M \leqq P$, then $|M| = p$ and so $|S:C_S(M)|$ divides $p - 1$. Hence $Q \leqq C_S(M)$. By induction applied to $S/M$, we find that $[P, Q] \leqq M$. Then

$$[P, Q, Q] \leqq [M, Q] = 1.$$

By coprime action of $Q$ on $P$, we see that $[P, Q] = 1$.

Lemmas 6.5 and 6.6 allow us to obtain useful subgroups of $G/N$ in the proof of Theorem 6.4. The subgroups provided by 6.5 are Hall subgroups, and our next three facts establish that certain of their characters are monomial. Theorem 6.7 is due to Gallagher and found in [**3**]. Theorem 6.8 is a trivial consequence of Lemma 3.4 of [**9**].

THEOREM 6.7. *Let* $M \Delta G$ *be a Hall subgroup of G. Let* $\theta \in \mathrm{Irr}(M)$ *be invariant in G. Then* $\theta$ *extends to an element of* $\mathrm{Irr}(G)$.

THEOREM 6.8. *Let* $M \Delta G$ *be a Hall subgroup of G. Let* $\theta \in \mathrm{Irr}(M)$ *and assume that all elements of* $\mathrm{Irr}(G|\theta^G)$ *are monomial. Then there is a monomial extension of* $\theta$ *to an element of* $\mathrm{Irr}(I_G(\theta))$.

LEMMA 6.9. *Let* $M \Delta G$ *and let* $M \leqq J \leqq G$. *Let* $\chi \in \mathrm{Irr}(G)$ *be monomial and assume that* $\chi_M \in \mathrm{Irr}(M)$. *Then* $\chi_J$ *is monomial.*

*Proof.* Let $H \leqq G$ and $\lambda \in \mathrm{Irr}(H)$ with $\lambda(1) = 1$ and $\lambda^G = \chi$. Then $(\lambda^{MH})_M$ has $\chi_M$ as a constituent, hence $\lambda^{MH}(1) \geqq \chi(1)$, that is to say

$$|MH:H| \geqq |G:H|.$$

It follows that $MH = G$.

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By Mackey's theorem

$$\chi_J = (\lambda^G)_J = (\lambda^{MH})_J = (\lambda_{J \cap H})^J.$$

Thus $\chi_J$ is monomial.

LEMMA 6.10. *Let $H, K \leqq G$ with $G = HK$. Let $\lambda$ be a character of $H$ and $\mu$ a character of $K$. Then*

$$\lambda^G \cdot \mu^G = (\lambda_{H \cap K} \cdot \mu_{H \cap K})^G.$$

*In particular if $\alpha$, $\beta$ are monomial characters of $G$ of coprime degree, then $\alpha\beta$ is monomial.*

*Proof.* We put $J = H \cap K$ and compute

$$\begin{aligned}(\lambda_J\mu_J)^G &= (\,(\lambda_J\mu_J)^K)^G = (\,(\lambda_J)^K\mu)^G \\ &= (\,(\lambda^G)_K\mu)^G = \lambda^G \cdot \mu^G,\end{aligned}$$

proving the first statement.

For the second, let $\alpha = \lambda^G$ where $\lambda \in \mathrm{Irr}(H)$ for $H \leqq G$ and $\beta = \mu^G$ where $\mu \in \mathrm{Irr}(K)$ for $K \leqq G$. Then $|G:H|$ and $|G:K|$ are coprime, hence $HK = G$, and we are done by the first part.

At a crucial point in the proof of Theorem 6.4, we will need to quote Dade's theorem $O$ of [**2**].

THEOREM 6.11. *Let $G$ be a $p$-solvable group for an odd prime $p$ and let $\chi \in \mathrm{Irr}(G)$ with $\chi(1)$ a power of $p$ and $\chi$ monomial. Let $M \Delta\Delta\ G$ and $\theta \in \mathrm{Irr}(M|\chi_M)$. Then $\theta$ is monomial.*

*Proof of Theorem* 6.4. Put $Z = Z(\theta)$. As remarked following the statement of Theorem 6.4 we can assume $\ker(\theta) = 1$ and we can view our task as to show that $N/Z$ is $G/N$-hyperbolic with respect to the form defined as in Proposition 5.7. We will use induction on $|G|$.

By Lemma 6.5 applied to the supersolvable group $G/N$ and prime $p$ ($N$ is a $p$-group), we find $M \leqq L \leqq G$ with $N \leqq M$, $M \Delta G$, $L \Delta G$, $M/N$ and $L/N$ Hall subgroups of $G/N$, and $L/M$ a $p$-group. Since $N$ is a $p$-group, it follows that $L$ is a normal, Hall subgroup of $G$.

STEP 1. *Let $q$ be a prime with $q \nmid p(p - 1)$ and choose $Q \in \mathrm{Syl}_q(G)$. Then $Q \leqq C_G(L/M)$.*

*Proof.* We have that $QM/M \in \mathrm{Syl}_q(G/M)$. By Lemma 6.6,

$$[L/M, QM/M] = 1.$$

Thus $[L, Q] \leqq M$, as needed.

Put $K = L \cdot C_G(L/M)$. By Step 1, if $q||G:K|$ and $q$ is a prime, then $q|p - 1$.

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STEP 2. *All elements of* $\mathrm{Irr}(K|\theta^K)$ *are monomial.*

*Proof.* Let $\alpha \in \mathrm{Irr}(K|\theta^K)$ and let $\beta \in \mathrm{Irr}(L|\alpha_L)$ so then $\beta \in \mathrm{Irr}(L|\theta^L)$. Put $I = I_G(\beta)$. Then $L$ is a normal, Hall subgroup of $I$. Because all elements of $\mathrm{Irr}(G|\beta^G)$ are elements of $\mathrm{Irr}(G|\theta^G)$ and all elements of the second set are monomial, Theorem 6.8 produces a monomial $\psi \in \mathrm{Irr}(I)$ such that $\psi_L = \beta$.

Thus if $J = I \cap K = I_K(\beta)$, then by Lemma 6.9 $\psi_J$ is monomial. Now $\psi_L = \beta$ and so by Corollary 6.17 of [7], every element of $\mathrm{Irr}(J|\beta^J)$ has the form $\psi_J \cdot \delta$ for $\delta \in \mathrm{Irr}(J/L)$. In particular, $\alpha_\beta = \psi_J \cdot \delta$ for some such $\delta$. We will show that $\alpha_\beta$ is monomial. Since $\alpha = (\alpha_\beta)^K$, this will prove that $\alpha$ is monomial.

We have already remarked that $\psi_J$ is monomial and irreducible. The character $\delta$ is monomial, being an irreducible character of the supersolvable group $J/L$. Because $L$ is a Hall subgroup,

$$(\delta(1), |L|) = 1, \quad \text{for } \delta(1)\,||J:L|.$$

Now $\psi_L = \beta$ and so $(\psi(1), \delta(1)) = 1$ since $\psi(1)\,||L|$. By Lemma 6.10, $\psi_j \cdot \delta$ is monomial, and this completes Step 2.

STEP 3. *We are done in the case* $K < G$.

*Proof.* If $K < G$, then by Step 2, induction applies to the group $K$ and we conclude that $N/Z$ is $K/N$-hyperbolic. By Step 1, if $q||G:K|$ then $q|p - 1$. Also, $|G:N|$ is odd and thus by Theorem 5.1, $N/Z$ is $G/N$-hyperbolic and we are done in this case.

We are left to assume that $K = G$. By the Schur-Zassenhaus theorem, there is a complement $J/M$ for $L/M$ in $G/M$. We have $[L, J] \leqq M$ and thus $J \Delta G$. Because $M/N$ is a $p'$-group, $N$ is a normal, Hall subgroup of $J$.

By Theorem 6.7, there is an extension $\varphi$ of $\theta$ to $J$.

STEP 4. $\varphi$ *is monomial.*

*Proof.* If $J = G$, then the conclusion is obvious. Otherwise, $p||G:J|$ and so $p$ is odd.

Let $\chi \in \mathrm{Irr}(G|\varphi^G)$, then $\chi \in \mathrm{Irr}(G|\theta^G)$. Furthermore $\chi(1)/\varphi(1)$ divides $|G:J|$ (by Corollary 11.29 of [7]) and thus is a power of $p$. But $\varphi(1) = \theta(1)$ is also a power of $p$ and we now see that $\chi(1)$ is a power of $p$. By hypothesis $\chi$ is monomial, and thus Theorem 6.11 shows that $\varphi$ is monomial.

STEP 5. $N/Z$ *is* $J/Z$-*hyperbolic.*

*Proof.* By Step 4 we can find $H \leqq J$ and $\lambda \in \mathrm{Irr}(H)$ with $\lambda(1) = 1$ and $\lambda^J = \varphi$.

Then since $\theta$ is invariant in $J$, $(\lambda^{NH})_N$ has $\theta$ as a constituent. Thus

$$|NH:H| = \lambda^{NH}(1) \geqq \theta(1) = \varphi(1) = |J:H|.$$

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Now $NH = J$ and then $(\lambda^{NH})_N = \theta$.

By Mackey's theorem,

$$\theta = (\lambda^{NH})_N = (\lambda_{N \cap H})^N.$$

Put $A = N \cap H$. Then since $(\lambda_A)^N$ is irreducible, we must have $Z \leqq A$, so then $A \Delta N$. Also $A = N \cap H \Delta H$ and thus $A \Delta NH = J$. In fact, since $(\lambda_A)^N = \theta$, we have $A \geqq C_N(A)$. We can finish Step 5 by showing that $A$ is abelian.

Now since $(\lambda_A)^N = \theta$ we see that

$$A' \leqq \ker(\lambda^N) \leqq \ker(\theta) = 1.$$

This completes Step 5.

Since $p$ is odd and $G/J$ is a $p$-group, Theorem 5.1 shows that $N/Z$ is $G/N$-hyperbolic. This completes the proof of Theorem 6.4.

We can now prove Theorem C, which was stated in Section 1.

*Proof of Theorem* C. First assume for every nonlinear $\theta \in \text{Irr}(N)$, that $I_G(\theta)$ has a subgroup $A$ as indicated in the statement of Theorem C.

Let $\theta \in \text{Irr}(N)$. If $\theta$ is linear, then all elements of $\text{Irr}(G|\theta^G)$ are monomial by Lemma 6.3. If, on the other hand, $\theta(1) > 1$, put $J = I_G(\theta)$ and get $A \leqq N$ with

$$\ker(\theta) \leqq A, \quad A \Delta J, \quad \text{and} \quad A = C_N(A/\ker(\theta)).$$

By Theorem 6.1, all elements of $\text{Irr}(J|\theta^J)$ are monomial. By Clifford's theorem, this proves that all elements of $\text{Irr}(G|\theta^G)$ are monomial.

If $\chi \in \text{Irr}(G)$ then $\chi \in \text{Irr}(G|\theta^G)$ for some $\theta \in \text{Irr}(N)$. We have now shown that $G$ is an $M$-group.

Now assume that $G$ is an $M$-group. Let $\theta \in \text{Irr}(N)$. We will find $A \leqq N$ with

$$\ker(\theta) \leqq A, \quad A \Delta I_G(\theta), \quad \text{and} \quad A = C_N(A/\ker(\theta)).$$

As in Proposition 2.2, $\theta$ can be factored $\theta = \prod \theta_p$ where $p$ ranges over the prime divisors of $|N|$ and $N/\ker(\theta_p)$ is a $p$-group. Put $J_p = I_G(\theta_p)$ and suppose, for each $p$, we can find $A_p \leqq N$ with

$$A_p \Delta J_p, \quad \ker(\theta_p) \leqq A_p, \quad \text{and} \quad A_p = C_N(A_p/\ker(\theta_p)).$$

We claim that then $A = \cap A_p$ meets our requirements.

Indeed, by Proposition 2.4(c), $I_G(\theta) = \cap J_p$ and thus $A \Delta I_G(\theta)$. By Proposition 2.2,

$$\ker(\theta) = \cap \ker(\theta_p)$$

and so $\ker(\theta) \leqq A$. By Proposition 2.3, if $x \in N$ and $[A, x] \leqq \ker(\theta)$ we have

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$$[A_p, x] \leqq \ker(\theta_p) \quad \text{for all } p.$$

Thus

$$C_N(A/\ker(\theta)) \leqq \cap C_N(A_p/\ker(\theta_p)) = \cap A_p = A.$$

Also,

$$A' \leqq \cap A'_p \leqq \cap \ker(\theta_p) = \ker(\theta),$$

so then

$$A \leqq C_N(A/\ker(\theta)).$$

This shows that such $A$ would have the required properties.

Thus, to finish the proof of Theorem C, it suffices to consider $\theta \in \mathrm{Irr}(N)$ where $N/\ker(\theta)$ is a $p$-group. Put $T = I_G(\theta)$. By Theorem 3.1, every element of $\mathrm{Irr}(T|\theta^T)$ is monomial. Because $\theta$ is invariant in $T$, if $J = T/\ker(\theta)$, then $\theta$ can be viewed as a character of $J$ and the sets $\mathrm{Irr}(T|\theta^T)$ and $\mathrm{Irr}(J|\theta^J)$ can be identified.

Then $(N/\ker(\theta)) \Delta J$ satisfies the hypothesis of Theorem 6.4. Hence there is a subgroup $A/\ker(\theta) \leqq N/\ker(\theta)$ with

$$(A/\ker(\theta)) \Delta J \quad \text{and} \quad A/\ker(\theta) = C_{N/\ker(\theta)}(A/\ker(\theta)).$$

Clearly,

$$A \Delta T, \ker(\theta) \leqq A \leqq N, \quad \text{and} \quad A = C_N(A/\ker(\theta)).$$

This completes the proof of Theorem C.

Theorem D is a straightforward consequence of Theorem C.

*Proof of Theorem* D. The group $N$ is assumed to be an extra-special $p$-group. Put $Z = Z(N)$. If $\theta \in \mathrm{Irr}(N)$ is nonlinear, then there is a unique $\lambda \in \mathrm{Irr}(Z|\theta_Z)$, and $\theta$ is the unique element of $\mathrm{Irr}(N|\lambda^N)$. It follows that $I_G(\lambda) = I_G(\theta)$.

Because $|Z| = p$, we find that $\theta(1) > 1$ forces that $\lambda$ is faithful. Hence $I_G(\lambda) = C_G(Z)$. Theorem D is now seen to be a direct corollary of Theorem C.

**7. Searching.** In this section we develop the tool which allows us to use Theorem 6.4 to prove Theorems A and B.

Let $\chi \in \mathrm{Irr}(G)$ be monomial. Then there is $H \leqq G$ and $\lambda \in \mathrm{Irr}(G)$ with $\lambda(1) = 1$ and $\lambda^G = \chi$. Part of the difficulty in obtaining information about the subgroups of $M$-groups is that there is no canonical method of producing such an $H$ and $\lambda$. Let $N \Delta G$ with $N$ nilpotent and let $\theta \in \mathrm{Irr}(N)$. We will describe a process which will produce $K \leqq G$ and $\varphi \in \mathrm{Irr}(K \cap N)$ such that:

a) $(N \cap K)/\ker \varphi$ is nilpotent of class at most 2.

b) Character induction yields a surjection from $\mathrm{Irr}(K|\varphi^K)$ to

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$\mathrm{Irr}(G|\theta^G)$.

c) If $\psi \in \mathrm{Irr}(K|\varphi^K)$ then $\psi$ is monomial if and only if $\psi^G$ is monomial.

d) If $N \leqq L \leqq G$ then (a)-(c) holds with $L$ replacing $G$ and $L \cap K$ replacing $K$.

The surjection of (b) will actually be a bijection and we will have

$$[\psi, \varphi^K] = [\psi^G, \theta^G] \text{ for all } \psi \in \mathrm{Irr}(K|\varphi^K),$$

but we will not need (or prove) these stronger facts.

Throughout this section $N \Delta G$ with $N$ nilpotent and $\theta \in \mathrm{Irr}(N)$. To begin our description of the process mentioned above we choose $A \leqq \ker(\theta)$ with $A \Delta G$ and define the quintuple $(G, N, \theta, A, 1_A)$ to be an *initial step for* $(G, N, \theta)$.

Now suppose $S = (H, M, \varphi, B, \lambda)$ has been defined with the following properties (which are obviously satisfied by an initial step):

7.1a) $B \leqq M \Delta H \leqq G$, with $B \Delta H$.

7.1b) $\varphi \in \mathrm{Irr}(M)$

7.1c) $\lambda \in \mathrm{Irr}(B|\varphi_B)$ and $\lambda(1) = 1$.

Put $J = I_H(\varphi)$ and choose $C \leqq M$ satisfying

7.2a) $B \leqq C \Delta J$

7.2b) $\varphi_C$ has a linear constituent.

Choose $\mu \in \mathrm{Irr}(C|\varphi_C)$. Put $I = I_J(\mu)$, $L = M \cap I$, $\alpha = \varphi_\mu$, and $T = (I, L, \alpha, C, \mu)$. We say $T$ is *constructed from S for* $(G, N, \theta)$ *using C and* $\mu$. Observe that with the notation interpreted appropriately, $T$ satisfies all of 7.1.

We call such $S$ as above a *final step* for $(G, N, \theta)$ if

7.3a) $\varphi$ is invariant in $H$.

7.3b) there is no $D \Delta H$ with $B < D \leqq M$ and such that $\varphi_D$ has a linear constituent.

A sequence $S_1, \dots, S_n$ where $S_i = (G_i, N_i, \theta_i, A_i, \lambda_i)$ is a $(G, N, \theta)$ *search* provided $S_1$ is an initial step for $(G, N, \theta)$, $S_{i+1}$ is constructed from $S_i$ for $(G, N, \theta)$ using $A_{i+1}$ and $\lambda_{i+1}$ for $1 \leqq i < n$, and $S_n$ is a final step for $(G, N, \theta)$. The $S_i$ are called *terms*. If, in addition, we have $N_n = A_n$ then we say that the search is *successful*. One of our main goals is the following characterization.

THEOREM 7.4. *Let $N \Delta G$ with $N$ nilpotent and $G/N$ a supersolvable group of odd order. Then $G$ is an $M$-group if and only if for every $\theta \in \mathrm{Irr}(N)$ there is a successful $(G, N\ \theta)$ search.*

We remark that given $\theta \in \mathrm{Irr}(N)$, a $(G, N, \theta)$ search always exists. If in the choice of $C$ for 7.2a and 7.2b in the construction of each term, we also require that $|C|$ be as large as possible with the desired properties, then we will necessarily reach a final step after a finite number of constructions.

It is apparent that if $S$ is a final step and $T$ is constructed from $S$, then $S = T$. Thus, given a $(G, N, \theta)$ search $S_1, \dots, S_n$, we can always append

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copies of $S_n$ to obtain a search of longer length. This will be useful in Proposition 7.9 below.

We will prove Theorem 7.4 in Section 8. For now we collect auxiliary facts.

PROPOSITION 7.5 *Let* $T = (I, L, \alpha, C, \mu)$ *be a term of a* $(G, N, \theta)$ *search, where* $N \Delta G$ *with* $N$ *nilpotent. Then*

a) $L = I \cap N$ *so then* $L \Delta I$

b) $C \Delta I$

c) $\mu$ *is invariant in* $I$

d) *Character induction defines a surjection from* $\mathrm{Irr}(I|\alpha^I)$ *to* $\mathrm{Irr}(G|\theta^G)$.

e) *If* $\psi \in \mathrm{Irr}(I|\alpha^I)$ *then* $\psi$ *is monomial if and only if* $\psi^G$ *is monomial.*

*Proof.* The properties are clear for an initial step. Assume $T$ is constructed from $S = (H, M, \varphi, B, \lambda)$ using $C$ and $\mu$, and assume (by induction) that, with the notation interpreted appropriately, $S$ satisfies (a)-(e) of Proposition 7.5.

STEP 1. (a)-(c) *hold for* $T$.

*Proof.* By definition $L = M \cap I$. By (a) applied to $S$, $M = H \cap N$. Thus

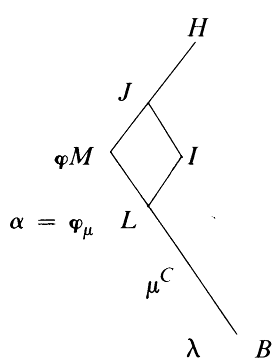
$$L = H \cap N \cap I = N \cap I,$$

since $I \leqq H$. By 7.2a $C \Delta I_H(\varphi)$. Thus $C \Delta I$, for we have $I \leqq I_H(\varphi)$. By definition of $I$, $\mu$ is invariant in $I$.

As in the definition of the construction, let $J = I_H(\varphi)$.

STEP 2. $MI = J$.

*Proof.* Since $\varphi$ is invariant in $J$ and $C \Delta J$, $J$ acts on the set $\mathrm{Irr}(C|\varphi_C)$ by conjugation. By Clifford's theorem, $M$ acts transitively on this set. By definition, $I$ is the stabilizer of the point $\mu$. Thus $MI = J$.




STEP 3. (d) *holds for* $T$.

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*Proof.* Because (d) applies to $S$, it suffices to show that character induction yields a surjection from $\mathrm{Irr}(I|\alpha^I)$ to $\mathrm{Irr}(H|\varphi^H)$.

By Clifford's theorem, since $\alpha = \varphi_\mu$, $\alpha_C$ is a multiple of $\mu$. Thus, by Frobenius reciprocity,

$$\mathrm{Irr}(I|\alpha^I) \subseteq \mathrm{Irr}(I|\mu^I).$$

By the definition of $I = I_J(\mu)$ and by Clifford's theorem, character induction provides a surjection from $\mathrm{Irr}(I|\mu^I)$ to $\mathrm{Irr}(J|\mu^J)$. Thus if $\psi \in \mathrm{Irr}(I|\alpha^I)$, then $\psi^J \in \mathrm{Irr}(J|\mu^J)$. Moreover, by Mackey's theorem, $(\psi^J)_M = (\psi_L)^M$ so then $(\psi^J)_M$ has $\varphi = \alpha^M$ as a constituent. Thus $\psi^J \in \mathrm{Irr}(J|\varphi^J)$. We conclude that character induction maps $\mathrm{Irr}(I|\alpha^I)$ into $\mathrm{Irr}(J|\varphi^J)$. We claim that this map is onto. Indeed, if $\beta \in \mathrm{Irr}(J|\varphi^J)$ then since $\mu \in \mathrm{Irr}(C|\varphi_C)$ we see that $\beta \in \mathrm{Irr}(J|\mu^J)$, hence $\beta = \psi^J$ for some $\psi \in \mathrm{Irr}(I|\mu^I)$ by Clifford's theorem. Also

$$\beta_M = (\psi^J)_M = (\psi_L)^M$$

and thus $\psi_L$ and $\varphi_L$ have a common irreducible constituent $\alpha'$. Since $\psi \in \mathrm{Irr}(I|\mu^I)$ and since $\mu$ is invariant in $I$, $\alpha' \in \mathrm{Irr}(L|\mu^L)$. Also $L = M \cap I = I_M(\mu)$, and so by Clifford's theorem $\alpha' = \varphi_\mu = \alpha$. Thus $\psi \in \mathrm{Irr}(I|\alpha^I)$, and the claim holds.

The last two paragraphs establish that character induction defines a surjection from $\mathrm{Irr}(I|\alpha^I)$ to $\mathrm{Irr}(J|\varphi^J)$. By definition $J = I_H(\varphi)$ and thus Clifford's theorem implies that induction gives a surjection from $\mathrm{Irr}(J|\varphi^J)$ to $\mathrm{Irr}(H|\varphi^H)$. By the transitivity of induction we have a surjection from $\mathrm{Irr}(I|\alpha^I)$ to $\mathrm{Irr}(H|\varphi^H)$. As remarked above, this completes Step 3.

STEP 4. (e) *holds for* $T$.

*Proof.* Let $\psi \in \mathrm{Irr}(I|\alpha^I)$. If $\psi$ is monomial, then clearly $\psi^G$ is monomial.

Let $\psi^G$ be monomial. By Step 3,

$$\psi^G \in \mathrm{Irr}(G|\theta^G) \quad \text{and} \quad \psi^H \in \mathrm{Irr}(H|\varphi^H).$$

Since (3) holds for $S$ we find that $\psi^H$ is monomial.

Now $M \leqq N$ (by (a) ) so then $M$ is nilpotent. Theorem 3.1 shows that $\psi^J = (\psi^H)_\varphi$ is monomial. But then $C \leqq N$ so then $C$ is nilpotent and Theorem 3.1 forces that $\psi = (\psi^J)_\mu$ is monomial. This completes Step 4 and Proposition 7.5.

COROLLARY 7.6. *If* $(H, M, \varphi, B, \lambda)$ *is any term of a* $(G, N, \theta)$ *search then* $H/M$ *is isomorphic to a subgroup of* $G/N$.

*Proof.* By Proposition 7.5a, $M = N \cap H$.

Next we show that at a final step, the structure of $M/\ker(\varphi)$ is limited.

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PROPOSITION 7.7. *Let* $S = (H, M, \varphi, B, \lambda)$ *be a final step of a* $(G, N, \theta)$ *search. Then* $\ker(\varphi) \leqq B$. *Put* $B^* = B/\ker(\varphi)$ *and* $M^* = M/\ker(\varphi)$. *Then* $B^* = Z(M^*)$ *and* $(M^*)' \leqq Z(M^*)$.

*Proof.* By 7.3a $\varphi$ is invariant in $H$, and hence $Z(\varphi)\ \Delta\ H$. Then $B \cdot Z(\varphi)\ \Delta\ H$ and $\varphi$ restricted to $B \cdot Z(\varphi)$ has a linear constituent, for $\varphi_B$ does. By the maximality condition 7.3b on $B$, $B \cdot Z(\varphi) = B$, and so $Z(\varphi) \leqq B$. In particular $\ker(\varphi) \leqq B$.

Because $\lambda$ is invariant in $H$ (Proposition 7.5c), we see that $\ker(\lambda) = \ker(\varphi_B)$ and that $B \leqq Z(\varphi)$. Thus $B = Z(\varphi)$, and so $B^* = Z(M^*)$.

We have left to show that $(M^*)' \leqq Z(M^*)$. We will apply Lemma 3.3 to $M^*$, which is a nilpotent group, since $M \leqq N$.

Let $C$ char $M^*$ with $C' = 1$. Pull $C$ back to $D \leqq M$. We claim that $D\ \Delta\ H$. Indeed, since $\ker(\varphi)\ \Delta\ H$, $H$ acts on $M^*$ by automorphisms, and so $H$ leaves $C$ invariant, hence $D\ \Delta\ H$. Because $C' = 1$ and $B = Z(\varphi)$, $\varphi_{DB}$ has a linear constituent. By the maximality of $B$, $DB = B$, hence $C \leqq B^* = Z(M^*)$. Now Lemma 3.3 applies to show that $(M^*)' \leqq Z(M^*)$.

Now we show successful searches pass to subgroups.

PROPOSITION 7.8. *Let* $S_i = (G_i, N_i, \theta_i, A_i, \lambda_i)$, $1 \leqq i \leqq n$, *be a successful* $(G, N, \theta)$ *search. Suppose* $N \leqq K \leqq G$. *Then* $S_i' = (K \cap G_i, N_i, \theta_i, A_i, \lambda_i)$ *is a successful* $(K, N, \theta)$ *search.*

*Proof.* It is clear that $S_1'$ is an initial step for $(K, N, \theta)$. We show that $S_{i+1}'$ is constructed from $S_i'$ for $(K, N, \theta)$ using $A_{i+1}$ and $\lambda_{i+1}$.

Indeed, because $N \leqq K$, 7.1 holds for $S_i'$. Put

$$J = I_{G_i}(\theta_i).$$

Then

$$I_{K \cap G_i}(\theta) = K \cap I_{G_i}(\theta_i) = K \cap J.$$

Because $A_{i+1}$ satisfies 7.2a and 7.2b for $J$, it satisfies 7.2a and 7.2b for $K \cap J$. By the definition of the construction,

$$G_{i+1} = I_J(\lambda_{i+1}).$$

To show that $S_{i+1}'$ is constructed from $S_i'$, we need to show that

$$K \cap G_{i+1} = I_{K \cap J}(\lambda_{i+1}).$$

But

$$I_{K \cap J}(\lambda_{i+1}) = K \cap I_J(\lambda_{i+1}) = K \cap G_{i+1},$$

as needed.

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Now we show that $S_n'$ is a final step for $(K, N, \theta)$. Since $S_n$ is a final step, by 7.3a, $\theta_n$ is invariant in $G_n$. Thus $\theta_n$ is invariant in $K \cap G_n$. Because the $(G, N, \theta)$ search is successful, $A_n = N_n$ and so there is no subgroup $D \leqq N_n$ with $A_n < D$. Thus 7.3b holds for $S_n'$ and we see that $S_n'$ is a final step. Since $N_n = A_n$, it is clear that the $(K, N, \theta)$ search is successful.

Finally, we need to show that searches can be built from the Sylow subgroups of $N$.

PROPOSITION 7.9. *Let $N \Delta G$ with $N$ nilpotent and let $\theta \in \mathrm{Irr}(N)$. Factor $\theta = \prod \theta_p$ as in Proposition 2.2. Suppose for each $p||N|$, that there is a successful $(G, N, \theta_p)$ search. Then there is a successful $(G, N, \theta)$ search.*

*Proof.* For each $p||N|$, let

$$S_i^{(p)} = (G_i^{(p)}, N_i^{(p)}, \theta_i^{(p)}, A_i^{(p)}, \lambda_i^{(p)}), \quad 1 \leqq i \leqq n_p$$

be the terms of a successful $(G, N, \theta_p)$ search. By a remark which followed the statement of Theorem 7.4, we can assume that all the $n_p$ are equal to a common $n$. Also, as is apparent from the construction, since the normal $p$-complement $R_p$ of $N$ is contained in $\ker(\theta_p)$, we can assume that

$$R_p \leqq \ker(\theta_i^{(p)})$$

and that

$$R_p \leqq \ker(\lambda_i^{(p)}) \quad \text{for all } i \text{ and } p.$$

Taking intersection and product signs over $p||N|$, we put

$$G_i = \cap G_i^{(p)}, \quad N_i = \cap N_i^{(p)}, \quad \theta_i = \prod \theta_i^{(p)},$$

$$A_i = \cap A_i^{(p)}, \quad \lambda_i = \prod (\lambda_i^{(p)})_{A_i}, \quad \text{and}$$

$$S_i = (G_i, N_i, \theta_i, A_i, \lambda_i) \quad \text{for } 1 \leqq i \leqq n.$$

We claim that $S_1, \ldots, S_n$ is a successful $(G, N, \theta)$ search.

First of all, by Proposition 2.2, $\ker(\theta) = \cap \ker(\theta_p)$ hence $A_1 \Delta G$ and $A_1 \leqq \ker(\theta)$. It is now clear that $S_1$ is an initial step for $(G, N, \theta)$.

We now show that $S_{i+1}$ is constructed from $S_i$ for $(G, N, \theta)$ using $A_{i+1}$, $\lambda_{i+1}$. By induction on $i$, we assume that $S_i$ satisfies 7.1. Then $\theta_i \in \mathrm{Irr}(N_i)$ by 7.1b and $N_i \Delta G_i$ by 7.1a, and so we can put $J = I_{G_i}(\theta_i)$ as in the definition of the construction. Now if

$$J^{(p)} = I_{G_i^{(p)}}(\theta_i^{(p)})$$

then

$$A_i^{(p)} \leqq A_{i+1}^{(p)} \Delta J^{(p)}.$$

Using the notation and conclusion of Proposition 2.6, $J = \cap J^{(p)}$ and also $A_i \leqq A_{i+1} \Delta J$ so that 7.2a holds for $A_{i+1}$. By Proposition 2.4a,

$$\lambda_{i+1} \in \mathrm{Irr}(A_{i+1})$$

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and since

$$\theta_i^{(p)} \in \mathrm{Irr}(N_i^{(p)} | (\lambda_{i+1}^{(p)})^{N_i^{(p)}},$$

Proposition 2.5 shows that $\lambda_{i+1}$ is a constituent of $(\theta_i)_{A_{i+1}}$. By definition, $\lambda_{i+1}$ is linear and now 7.2b holds for $A_{i+1}$ and $\lambda_{i+1}$.

By Proposition 2.6,

$$G_{i+1} = I_J(\lambda_{i+1}) \quad \text{and} \quad \theta_{i+1} = (\theta_i)_{\lambda_{i+1}}.$$

Clearly

$$N_{i+1} = \cap(N_i^{(p)} \cap G_{i+1}^{(p)}) = N_i \cap G_{i+1}.$$

This completes the proof that $S_{i+1}$ is constructed from $S_i$.

Because the given $(G, N, \theta_p)$ searches are successful, we have that $N_n^{(p)} = A_n^{(p)}$ for all $p$ and that $\theta_n^{(p)}$ is invariant in $G_n^{(p)}$. Thus $\theta_n$ is invariant in $G_n$ by Proposition 2.4c, and $N_n = A_n$. This proves that $S_n$ is a final step for $(G, N, \theta)$, and that this search is successful.

**8. Theorems A and B.** We begin with a result which, in conjunction with Proposition 7.9, yields one direction of Theorem 7.4. This result is also relevant to Theorem A.

THEOREM 8.1. *Let $N \Delta G$ with $N$ a $p$-group for some prime $p$ and $G/N$ supersolvable of odd order. Let $\theta \in \mathrm{Irr}(N)$ and assume all elements of $\mathrm{Irr}(G|\theta^G)$ are monomial. Then there is a successful $(G, N, \theta)$ search.*

*Proof.* As remarked following the statement of Theorem 7.4, there is always some $(G, N, \theta)$ search $S_1, \ldots, S_n$. Let

$$S_n = (H, M, \varphi, B, \lambda).$$

By 7.3, $\varphi$ is invariant in $H$, hence $\ker(\varphi) \Delta H$. We put

$$H^* = H/\ker(\varphi).$$

By Proposition 7.7, $\ker(\varphi) \leqq B$, and if $B^* = B/\ker(\varphi)$ and $M^* = M/\ker(\varphi)$ then $(M^*)' \leqq Z(M^*) = B^*$.

We can view $\varphi$ as a character of $M^*$, and we can identify the sets $\mathrm{Irr}(H|\varphi^H)$ and $\mathrm{Irr}(H^*|\varphi^{H^*})$. By Proposition 7.5e, all elements of $\mathrm{Irr}(H^*|\varphi^{H^*})$ are monomial. By Corollary 7.6, $H/M \cong H^*/M^*$ is a supersolvable group of odd order. Clearly, $M^*$ is a $p$-group. Thus, by Theorem 6.4, there is $A \leqq M^*$ with $A \Delta H^*$ and $A = C_{M^*}(A)$. Clearly

$$B^* = Z(M^*) \leqq A.$$

Pull $A$ back to $C \leqq M$ with $B \leqq C \Delta H$. Then $C' \leqq \ker(\varphi)$ and so $\varphi_C$ has a linear constituent. By 7.3b, we must have $B = C$. But then $B^* = A$ and so

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$$M^* = C_{M^*}(B^*) = C_{M^*}(A) = A.$$

Thus $B^* = M^*$, so then $B = M$ and the search is successful.

*Proof of Theorem* 7.4. First assume that $G$ is an $M$-group and choose $\theta \in \mathrm{Irr}(N)$. Factor $\theta = \prod \theta_p$ as in Proposition 2.2. For each $p \| |N|$, all the elements of $\mathrm{Irr}(G| (\theta_p)^G)$ are monomial. If $R_p$ is the normal $p$-complement of $N$, then $R_p \Delta G$ and we can view $\theta_p$ as a character of $N/R_p$ and we can identify $\mathrm{Irr}(G| (\theta_p)^G)$ and $\mathrm{Irr}(G/R_p| (\theta_p)^{G/R_p})$. By Theorem 8.1, there is a successful $(G/R_p, N/R_p, \theta_p)$ search, which can be viewed as a $(G, N, \theta_p)$ search. Since this holds for each $p \| |N|$, Proposition 7.9 shows that there is a successful $(G, N, \theta)$ search.

Now assume for each $\theta \in \mathrm{Irr}(N)$ that there is a successful $(G, N, \theta)$ search. Let $\chi \in \mathrm{Irr}(G)$ and choose $\theta \in \mathrm{Irr}(N|\chi_N)$ and a successful $(G, N, \theta)$ search whose last term is $(H, M, \varphi, B, \lambda)$. Then $B = M$ and $\varphi = \lambda$ is linear. By Corollary 7.6, $H/B$ is supersolvable. By Lemma 6.3, every element of $\mathrm{Irr}(H|\lambda^H)$ is monomial. Then by Proposition 7.5d and 7.5e, all elements of $\mathrm{Irr}(G|\theta^G)$ (including $\chi$) are monomial. This proves that $G$ is an $M$-group.

COROLLARY 8.2. *Let $N \Delta G$ with $N$ nilpotent and $G/N$ supersolvable and of odd order. Assume that $G$ is an $M$-group. Then every subgroup of $G$ containing $N$ is an $M$-group.*

*Proof.* Let $N \leqq K \leqq G$. For every $\theta \in \mathrm{Irr}(N)$ there is a successful $(G, N, \theta)$ search, by Theorem 7.4. By Proposition 7.8 there is a successful $(K, N, \theta)$ search. Theorem 7.4 implies that $K$ is an $M$-group.

Theorem B is almost instantaneous.

*Proof of Theorem* B. We use induction on $|G|$. Let $H$ be a Hall subgroup of $G$ and we must show that $H$ is an $M$-group.

Let $p$ be a prime divisor of $|N|$ and $P \in \mathrm{Syl}_p(N)$ so then $P \Delta G$. We have $P \leqq H$ or $p \nmid |H|$. If $p \nmid |H|$ for some such $p$, then $H \cong HP/P$. Now $HP/P$ is a Hall subgroup of $G/P$ and $|G/P| < |G|$. Since $G$ is an $M$-group, $G/P$ is too. By induction, $HP/P$ is an $M$-group, hence $H$ is too.

We are left to assume that $N \leqq H$. But then $H$ is an $M$-group by Corollary 8.2.

To prove Theorem A we need only slightly more work.

LEMMA 8.3. *Let $N$, $K \Delta G$ with $|N:N \cap K|$ coprime to $|K:N \cap K|$. Let $\theta \in \mathrm{Irr}(N \cap K)$ be invariant in $N$. Then all elements of $\mathrm{Irr}(K|\theta^K)$ are invariant in $NK$.*

*Proof.* We can assume that $G = NK$. Observe that $I_K(\theta)$ is normalized by $N$. Thus the hypothesis holds with $I_K(\theta)$ replacing $K$. If $I_K(\theta) < K$, then an inductive argument shows that all elements of $\mathrm{Irr}(I_K(\theta))$ lying over $\theta$

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are $N$-invariant. By Clifford's theorem all elements of $\mathrm{Irr}(K|\theta^K)$ are $N$-invariant, as needed. Thus we can assume that $\theta$ is invariant in $NK$.

Now the theory of projective representations (see Chapter 11 of [7]) allows us to assume that $\theta(1) = 1$ and that $N \cap K$ is central in $G$. Now the coprimeness forces that if $\Pi$ is the set of prime divisors of $|N:N \cap K|$ then $G = G_\Pi \times G_{\Pi'}$ for a normal Hall $\Pi$-subgroup $G_\Pi$ and Hall $\Pi'$-subgroup $G_{\Pi'}$. We have

$$N = G_\Pi(N \cap K).$$

The result is now trivial.

*Proof of Theorem* A. If $R$ is the supersolvable residual of $G$ then the group $R \cdot O^{2'}(G)$, which is contained in $N$, has the same properties in $G$ that $N$ does. We assume $N = R \cdot O^{2'}(G)$.

Let $K \Delta\Delta G$ and we will show that $K$ is an $M$-group by induction on $|G:K|$. We can assume that $K \Delta G$ and that $|G:K| = q$ is a prime. If $N \leqq K$, then $K$ is an $M$-group by Corollary 8.2, and so we can assume that $N \nleqq K$.

Now $G/K$ is cyclic and so $R \leqq K$. Since $N \nleqq K$, we must have $O^{2'}(G) \nleqq K$ and thus $|G:K| = 2$.

Suppose $K$ is not an $M$-group. By Theorem 7.4, there is $\theta \in \mathrm{Irr}(N \cap K)$ such that there is no successful $(K, N \cap K, \theta)$ search. Factor $\theta = \prod \theta_p$ as in Proposition 2.2, and then by Proposition 7.9, there is a prime $p \,\|\, |N \cap K|$ such that there is no successful $(K, N \cap K, \theta_p)$ search. Because $(N \cap K)/\ker(\theta_p)$ is a $p$-group, Theorem 8.1 finds $\chi \in \mathrm{Irr}(K|\,(\theta_p)^K)$ which is not monomial. We change the notation and put $\theta = \theta_p$.

CLAIM 1. *$\theta$ is not invariant in $N$.*

*Proof.* Now $N \nleqq K$ and so $NK = G$. Then $|N:N \cap K| = 2$ and $G/N$ is a supersolvable $2'$ group. If $\theta$ is invariant in $N$, then by Lemma 8.3, all elements of $\mathrm{Irr}(K|\theta^K)$ ($\chi$ for instance) are invariant in $NK = G$. By Corollary 6.19 of [7], $\chi$ extends to $\psi \in \mathrm{Irr}(G)$. Since $\psi$ is monomial, Lemma 6.9 shows that $\chi$ is monomial, a contradiction.

CLAIM 2. *$\theta$ is invariant in $K$.*

*Proof.* Put $J = I_K(\theta)$ and assume $J < K$. By Corollary 8.2, $NJ$ is an $M$-group. Now $[N, K] \leqq N \cap K$ and so $J \Delta NJ$. By induction ($NJ < G$) $J$ is an $M$-group, thus $\chi_\theta$ is monomial. But then $\chi = (\chi_\theta)^K$ is monomial, which is not so.

By Claims 1 and 2, $K = I_G(\theta)$ thus

$$\chi^G \in \mathrm{Irr}(G) \quad \text{and} \quad \chi = (\chi^G)_\theta.$$

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Observe that $N \cap K \Delta G$ and $N \cap K$ is nilpotent. Since every element of $\mathrm{Irr}(G|\theta^G)$ is monomial, Theorem 3.1 forces that $\chi = (\chi^G)_\theta$ is monomial, a contradiction. This proves Theorem A.

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