AUTOMATIC CONTINUITY AND SECOND ORDER COHOMOLOGY

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Abstract

Many Banach algebras A have the property that, although there are discontinuous homomorphisms from A into other Banach algebras, every homomorphism from A into another Banach algebra is automatically continuous on a dense subspace—preferably, a subalgebra—of A. Examples of such algebras are C^*-algebras and the group algebras $L^1(G)$, where G is a locally compact, abelian group. In this paper, we prove analogous results for $A = E \hat{\otimes} E^*$, where E is a Banach space, and $A = \ell^1(SL(2, \mathbb{R}))$. An important rôle is played by the second Hochschild cohomology group of $E \hat{\otimes} E^*$ and $\ell_0^1(SL(2, \mathbb{R}))$, respectively, with coefficients in the one-dimensional annihilator module. It vanishes in the first case and has linear dimension one in the second one.

Introduction

The automatic continuity problem for homomorphisms from a given Banach algebra A is to determine whether every homomorphism from A into another Banach algebra is continuous. There are Banach algebras for which this problem has a negative solution, that is there is a discontinuous homomorphism from A into another Banach algebra. However, even if A is the domain of a discontinuous homomorphism, its restriction to a 'large' subspace of A (preferably, a subalgebra) may be continuous.

The ancestor of results of this type is [BC, Theorem 4.1]: For a compact Hausdorff space Ω and a homomorphism θ from $\mathscr{C}(\Omega)$ into a Banach algebra, there is a dense

subalgebra $\mathscr{A}$ of $\mathscr{C}(\Omega)$ such that $\theta|_{\mathscr{A}}$ is automatically continuous. The techniques developed in [BC] apply to other (regular) commutative Banach algebras as well, among them to all group algebras $L^1(G)$, where G is a locally compact abelian group. In [Sin], Sinclair extended the Bade-Curtis theorem on homomorphisms from $\mathscr{C}(\Omega)$ to homomorphisms from arbitrary C^*-algebras, and in [Run1, Run2, Run3], we proved similar results for homomorphisms from $L^1(G)$ for certain, not necessarily abelian, locally compact groups G. Nevertheless, for an arbitrary locally compact group G, it is still unknown (and is likely to remain unknown for the foreseeable future) whether every homomorphism from $L^1(G)$ into a Banach algebra is continuous on a dense subalgebra ([Dal, Question 2.10(i)]). A particularly intriguing case is $G = SL(2, \mathbb{R})$ ([Run3, Question 2]).

Generally, an automatic continuity result of the above type is obtained via the following strategy (which, of course, has to be subjected to modifications depending on the Banach algebra under consideration).

Given a Banach algebra A and a homomorphism θ from A into another Banach algebra, we use the main boundedness theorem [DW, Theorem 1.3] (see [BC] for the very first version of this result) in order to obtain an ideal $\mathscr{I}$ of A such that $I := \mathrm{cl}\,\mathscr{I}$ has finite codimension in A and such that the following bilinear continuity assertion is true:

$$(1) \qquad\qquad \|\theta(ab)\| \leq C\|a\|\,\|b\| \quad (a, b \in \mathscr{I})$$

for some $C \geq 0$. From (1), we then wish to pass to a linear continuity assertion on $\mathscr{I}^2$, that is to conclude that the restriction of θ to $\mathscr{I}^2$, the linear span of all products ab with $a, b \in \mathscr{I}$, is automatically continuous. Obviously, $\theta|_{\mathscr{I}^2}$ is continuous with respect to the quotient norm on $\mathscr{I}^2$ induced by the multiplication map $\Delta : \mathscr{I} \otimes \mathscr{I} \to \mathscr{I}^2$. Hence, it suffices to prove that this quotient norm is equivalent to the given norm. Such a proof is easily accomplished in case I has a bounded left or right approximate identity (see [Sin] and also [Run3]). If no bounded approximate identity exists—this is the case, for example, if $A = L^1(G)$ with a non-amenable group G ([Rei, Wil1])—no general way exists to prove the equivalence of the two norms. In fact, they may be inequivalent, even if $I = \mathscr{I} = \mathscr{I}^2$ ([Dix]).

In the present paper, we seek to establish a condition on I which is weaker than the presence of a bounded approximate identity, but still enables us to conclude from (1) that $\theta|_{\mathscr{I}^2}$ is continuous. It turns out that a sufficiently general condition is that $\mathscr{H}^2(I, \mathbb{C}_{\mathrm{ann}})$, the second Hochschild cohomology group of I with coefficients in the one-dimensional annihilator module, vanishes. Since the Hochschild cohomology groups of a Banach algebra are in fact linear spaces and not only abelian groups, it makes sense to speak of their linear dimensions. We shall see that, if $\mathscr{H}^2(I, \mathbb{C}_{\mathrm{ann}}) = \{0\}$ is replaced by $\dim \mathscr{H}^2(I, \mathbb{C}_{\mathrm{ann}}) < \infty$, we can still show that θ is continuous, not necessarily on all of $\mathscr{I}^2$, but still on a dense subspace with finite-codimension. We

then apply these results to homomorphisms from $E \hat{\otimes} E^*$, where E is a Banach space, and from $\ell^1(SL(2, \mathbb{R}))$; in particular, we give a partial answer to [Run3, Question 2].

Although we only discuss homomorphisms, as they seem to be the most interesting class of maps from Banach algebras for which automatic continuity results of the aforementioned type can be obtained, we could equally well have worked—as in [Run3]—in the more general context of intertwining maps. All results in this paper remain true for this larger class of maps.

1. Statement, proof and consequences of the main lemma

We state our main lemma, the technical centrepiece of this paper, in a context slightly more general than just outlined in the introduction. Cohomology groups do not show up yet in its formulation.

As is customary, for two Banach spaces E and F, let $E \hat{\otimes} F$ denote their completed projective tensor product.

LEMMA 1.1. *Let A be a Banach algebra, let E be a Banach A-left module, and let*

$$\Delta : A \hat{\otimes} E \to E, \quad a \otimes x \mapsto a \cdot x$$

and

$$\Gamma : A \hat{\otimes} A \hat{\otimes} E \to A \hat{\otimes} E, \quad a \otimes b \otimes x \mapsto a \otimes b \cdot x - ab \otimes x$$

be such that Δ is surjective, and such that there is $n \in \mathbb{N}_0$ with

$$\dim(\ker \Delta) \operatorname{cl} \Gamma(A \hat{\otimes} A \hat{\otimes} E) = n.$$

Further, let $\mathscr{A}$ be a dense subalgebra of A, let $\mathscr{E}$ be a dense $\mathscr{A}$-submodule of E, and let θ be a linear map from E into a Banach space such that

$$(2) \qquad\qquad \|\theta(a \cdot x)\| \le C \|a\| \|x\| \quad (a \in \mathscr{A}, x \in \mathscr{E})$$

for some constant $C \ge 0$. Then there is a dense subspace $\mathscr{F}$ of E with codimension at most n in $\mathscr{A} \cdot \mathscr{E}$ such that $\theta|_{\mathscr{F}}$ is continuous.

PROOF. Let $\phi_1, \ldots, \phi_n \in (A \hat{\otimes} E)^*$ be such that $\phi_1|_{\ker \Delta}, \ldots, \phi_n|_{\ker \Delta}$ are linearly independent, but $\phi_j \in \Gamma(A \hat{\otimes} A \hat{\otimes} E)^\circ$ for $j = 1, \ldots, n$ (such functionals can easily be found with the Hahn-Banach theorem). Since $\operatorname{cl} \Gamma(A \hat{\otimes} A \hat{\otimes} E)$ has codimension n in $\ker \Delta$, it follows that

$$(3) \qquad\qquad \ker \Delta \cap \bigcap_{j=1}^{n} \ker \phi_j = \operatorname{cl} \Gamma(A \hat{\otimes} A \hat{\otimes} E).$$

Let $X := \bigcap_{j=1}^{n} \ker \phi_j$. Assume that $\Delta(X)$ is a proper subspace of E. Then there is a non-zero, linear functional ψ on E which annihilates $\Delta(X)$, that is $\bigcap_{j=1}^{n} \ker \phi_j \subset \ker(\psi \circ \Delta)$. By [Rud, Lemma 3.9], there are $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that $\psi \circ \Delta = \lambda_1 \phi_1 + \cdots + \lambda_n \phi_n$; in particular,

$$\lambda_1 \phi_1|_{\ker \Delta} + \cdots + \lambda_n \phi_n|_{\ker \Delta} = 0.$$

Since $\phi_1|_{\ker \Delta}, \dots, \phi_n|_{\ker \Delta}$ are chosen to be linearly independent, it follows that $\lambda_1 = \cdots = \lambda_n = 0$, and consequently, that $\psi \circ \Delta = 0$. Since Δ is surjective by hypothesis, this means that $\psi = 0$, which contradicts the assumption that $\psi \neq 0$. Let $\mathscr{X} := (\mathscr{A} \otimes \mathscr{E}) \cap X$. Then $\mathscr{X}$ has codimension n in $\mathscr{A} \otimes \mathscr{E}$ and is dense in X. Set $\mathscr{F} := \Delta(\mathscr{X})$. Certainly, $\mathscr{F}$ is a dense subspace of E and contained in $\mathscr{A} \cdot \mathscr{E} = \Delta(\mathscr{A} \otimes \mathscr{E})$. Basic linear algebra yields

$$\dim(\mathscr{A} \cdot \mathscr{E})\Delta(\mathscr{X}) \leq \dim(\mathscr{A} \otimes \mathscr{E})\mathscr{X} = n.$$

Because of (2), $\theta|_{\mathscr{F}}$ is continuous with respect to the quotient norm on $\mathscr{F}$ induced by $\Delta|_{\mathscr{X}}$. It is therefore sufficient to show that this quotient norm is equivalent to the given norm. As we have previously seen, Δ maps X onto E. From the open mapping theorem it follows that the given norm on E and the quotient norm induced by $\Delta|_X$ are equivalent. Therefore, we must show that $\mathscr{X} \cap \ker \Delta$ is dense in $X \cap \ker \Delta$. Since $\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{E}$ is dense in $A \hat{\otimes} A \hat{\otimes} E$, the continuity of Γ yields

$$\mathrm{cl}\, \Gamma(\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{E}) = \mathrm{cl}\, \Gamma(A \hat{\otimes} A \hat{\otimes} E).$$

Hence, we have

$$\mathrm{cl}\, (\mathscr{X} \cap \ker \Delta) \supset \mathrm{cl}\, \Gamma(\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{E}) = \mathrm{cl}\, \Gamma(A \hat{\otimes} A \hat{\otimes} E) = X \cap \ker \Delta, \qquad \text{by (3)}.$$

Since the converse inclusion holds trivially, this completes the proof. $\square$

Admittedly, the hypotheses of Lemma 1.1 are often not easily verifiable. In case $E = A$, however, there is a connection with (bounded) Hochschild cohomology (see [Joh2] or [Hel] for general background on cohomology of Banach algebras). Let $\mathbb{C}_{\mathrm{ann}}$ denote the one-dimensional annihilator bimodule over A, that is $\mathbb{C}_{\mathrm{ann}} = \mathbb{C}$ with the module operation

$$a \cdot \lambda := 0 \quad \text{and} \quad \lambda \cdot a := 0 \qquad (a \in A, \, \lambda \in \mathbb{C}).$$

Let Δ and Γ be defined as in Lemma 1.1 (with $E = A$), and let $A \hat{\otimes}_A A$ be defined as in [Hel], that is $A \hat{\otimes}_A A = (A \hat{\otimes} A) \, \mathrm{cl}\, \Gamma(A \hat{\otimes} A \hat{\otimes} A)$. If Δ is surjective, [Hel, Corollary II.3.20] yields

$$\mathscr{H}^2(A, \mathbb{C}_{\mathrm{ann}}) \cong (A \hat{\otimes}_A A)^* \Delta^*(A^*)$$
$$= \Gamma(A \hat{\otimes} A \hat{\otimes} A)^{\circ}/(\ker \Delta)^{\circ} \cong ((\ker \Delta)/\mathrm{cl}\, \Gamma(A \hat{\otimes} A \hat{\otimes} A))^*.$$

Consequently, if $\mathscr{H}^2(A, \mathbb{C}_{ann})$ is finite-dimensional, we have

$$\dim \mathscr{H}^2(A, \mathbb{C}_{ann}) = \dim(\ker \Delta)/\operatorname{cl} \Gamma(A \hat{\otimes} A \hat{\otimes} A).$$

The following corollary is now an immediate consequence of Lemma 1.1.

COROLLARY 1.2. *Let A be a Banach algebra such that*

$$\Delta : A \hat{\otimes} A \to A, \quad a \otimes b \mapsto ab$$

is surjective, and such that there is $n \in \mathbb{N}_0$ with $\dim \mathscr{H}^2(A, \mathbb{C}_{ann}) = n$. Further, let $\mathscr{A}$ be a dense subalgebra of A, and let θ be a linear map from A into a Banach space such that

$$\|\theta(ab)\| \le C\|a\|\|b\| \quad (a, b \in \mathscr{A})$$

for some constant $C \ge 0$. Then there is a dense subspace $\mathscr{F}$ of A with codimension at most n in $\mathscr{A}^2$ such that $\theta|_{\mathscr{F}}$ is continuous.

A particularly pleasant case of Corollary 1.2 occurs whenever $n = 0$. In this case θ is continuous on all of $\mathscr{A}^2$, which is a subalgebra of A. From the discussion preceding Corollary 1.2, it is clear that $\mathscr{H}^2(A, \mathbb{C}_{ann}) = \{0\}$ and the surjectivity of Δ are equivalent to $A \hat{\otimes}_A A \cong A$, where the isomorphism is induced via Δ.

COROLLARY 1.3. *Let A be a Banach algebra such that $A \hat{\otimes}_A A \cong A$, let $\mathscr{A}$ be a dense subalgebra of A, and let θ be a linear map from A into a Banach space such that*

$$\|\theta(ab)\| \le C\|a\|\|b\| \quad (a, b \in \mathscr{A})$$

for some constant $C \ge 0$. Then $\theta|_{\mathscr{A}^2}$ is continuous.

REMARK 1.4. Banach algebras A such that $A \hat{\otimes}_A A \cong A$ were termed *self-induced* by Grønbæk in [Grø]. Self-induced Banach algebras seem to be the largest class of Banach algebras for which a sensible theory of Morita equivalence can be developed (see [Grø]). Every Banach algebra with a (one-sided) bounded approximate identity is self-induced.

2. Homomorphisms from $E \hat{\otimes} E^*$

Let E be a Banach space. Then there are several Banach algebras naturally associated with it.

Let $\mathscr{B}(E)$ denote the Banach algebra of all bounded linear operators on E. In [Joh1], Johnson showed that the automatic continuity problem for homomorphisms from $\mathscr{B}(E)$ has a positive solution provided $E \cong E \oplus E$. Analogous results for some Banach spaces E such that $E \not\cong E \oplus E$ were obtained much later by Ogden ([Ogd]) and Willis ([Wil3]). On the other hand, there are Banach spaces E such that there is a discontinuous homomorphism from $\mathscr{B}(E)$ into a Banach algebra ([DLW]). Another Banach algebra naturally associated with E is $\mathscr{A}(E)$, the operator norm closure of the bounded, finite rank operators. In [DJ], Dales and Jarchow proved that, if E has the bounded approximation property and satisfies $E \cong E \oplus E$, the automatic continuity problem for homomorphisms from $\mathscr{A}(E)$ has a positive solution whereas there are examples of Banach spaces E for which the solution is negative.

In in this section we wish to consider yet another Banach algebra associated with E. Recall that for any Banach space E, the projective tensor product $E \hat{\otimes} E^*$ becomes a Banach algebra through

$$(x \otimes \phi)(y \otimes \psi) := \langle y, \phi \rangle x \otimes \psi \qquad (x, y \in E, \phi, \psi \in E^*).$$

There is a natural epimorphism from $E \hat{\otimes} E^*$ onto $\mathscr{N}(E)$, the algebra of nuclear operators on E, which becomes a Banach algebra if it is equipped with the corresponding quotient norm, the so-called nuclear norm. In [DJ], Dales and Jarchow construct a discontinuous homomorphism from $\mathscr{N}(E)$ whenever E is infinite-dimensional. Via composition with the canonical epimorphism $E \hat{\otimes} E^* \to \mathscr{N}(E)$ we thus obtain a discontinuous homomorphism from $E \hat{\otimes} E^*$ for each infinite-dimensional E. As we shall see, however, for a large class of Banach spaces E, every homomorphism from $E \hat{\otimes} E^*$ is continuous on $(E \hat{\otimes} E^*)^2$.

Before we prove our first result, note that $E \hat{\otimes} E^*$ is also a (unital) Banach $\mathscr{B}(E)$-bimodule via

$$T \cdot (x \otimes \phi) := Tx \otimes \phi \quad \text{and} \quad (x \otimes \phi) \cdot T := x \otimes T^*\phi$$
$$(x \in E, \phi \in E^*, T \in \mathscr{B}(E)).$$

In the terminology of [Run3], $E \hat{\otimes} E^*$ is a compatible Banach $\mathscr{B}(E)$-bimodule.

THEOREM 2.1. *Let E be a Banach space such that $E \cong E \oplus E$, let B be a Banach algebra, and let $\theta : E \hat{\otimes} E^* \to B$ be a homomorphism. Then $\theta|_{(E \hat{\otimes} E^*)^2}$ is continuous.*

PROOF. Let

(4) $\quad \mathscr{J} := \{T \in \mathscr{B}(E) : (E \hat{\otimes} E^*) \times (E \hat{\otimes} E^*) \ni (x, y) \mapsto \theta(x \cdot T \cdot y) \text{ is continuous}\}.$

Then, obviously, $\mathscr{J}$ is an ideal of $\mathscr{B}(E)$. A standard application of the main boundedness theorem [DW, Theorem 1.3] shows that for every two sequences $(R_n)_{n=1}^{\infty}$ and

$(S_n)_{n=1}^\infty$ in $\mathscr{B}(E)$ such that $R_n S_m = 0$ for $n \neq m$, there is $N \in \mathbb{N}$ such that $R_n S_n \in \mathscr{J}$ for $n \geq N$. Utilizing the hypothesis that $E \cong E \oplus E$ as in [Joh1], we obtain sequences $(P_n)_{n=1}^\infty$ and $(Q_n)_{n=1}^\infty$ of projections in $\mathscr{B}(E)$ such that $\mathrm{id}_E = P_1 + Q_1$, and $P_n Q_n = Q_n P_n = 0$, $P_n = P_{n+1} + Q_{n+1}$ and $\mathscr{B}(E)P_n \mathscr{B}(E) = \mathscr{B}(E)Q_n \mathscr{B}(E)$ for $n \in \mathbb{N}$. It is easily seen that $Q_n Q_m = 0$ for $n \neq m$, and hence, by the foregoing, there is $N \in \mathbb{N}$ such that $Q_n \in \mathscr{J}$ for all $n \geq N$. Let N be minimal with this property, and assume that $N \geq 2$. Since

(5) $$P_N \in \mathscr{B}(E)P_N \mathscr{B}(E) = \mathscr{B}(E)Q_N \mathscr{B}(E) \subset \mathscr{J},$$

we have $P_{N-1} = P_N + Q_N \in \mathscr{J}$ as well. The same argument as in (5) then yields $Q_{N-1} \in \mathscr{J}$, contradicting the minimality of N. Consequently, $Q_1 \in \mathscr{J}$. Again the argument in (5) can be applied and yields $P_1 \in \mathscr{J}$. Therefore, $\mathrm{id}_E = P_1 + Q_1 \in \mathscr{J}$, that is there is $C \geq 0$ such that

(6) $$\|\theta(xy)\| \leq C\|x\|\|y\| \quad (x, y \in E \hat{\otimes} E^*).$$

By [Gro, 4.6(i)], we have $(E \hat{\otimes} E^*) \hat{\otimes}_{E \hat{\otimes} E^*} (E \hat{\otimes} E^*) \cong E \hat{\otimes} E^*$. By Corollary 1.3, this means that the restriction of θ to $(E \hat{\otimes} E^*)^2$ is continuous. $\qquad\square$

REMARK 2.2.   1. Note that with the help of Corollary 1.3, we were able to conclude from (6) that $\theta|_{(E \hat{\otimes} E^*)^2}$ is continuous although $E \hat{\otimes} E^*$ in general does not have a bounded left or right approximate identity. Suppose that $E \hat{\otimes} E^*$ has a bounded left or right approximate identity. Since $\mathscr{N}(E)$ is a quotient of $E \hat{\otimes} E^*$, it follows that $\mathscr{N}(E)$ has a bounded approximate identity of the same type. As Dales and Jarchow point out in [DJ], this is possible only if $\dim E < \infty$.

2. Let θ be a discontinuous homomorphism from $\mathscr{N}(E)$ into a Banach algebra, and let $\pi : E \hat{\otimes} E^* \to \mathscr{N}(E)$ be the canonical epimorphism. Then $\theta \circ \pi$ is a discontinuous homomorphism from $E \hat{\otimes} E^*$ whose restriction to $(E \hat{\otimes} E^*)^2$ is continuous. We see, however, no way to conclude from there that $\theta|_{\mathscr{N}(E)^2}$ is continuous.

As we already mentioned, there are Banach spaces E for which fail to satisfy $E \cong E \oplus E$, but such that the automatic continuity for homomorphisms from $\mathscr{B}(E)$ still has a positive solution: these are the so-called James space ([Wil3]) and the space $\mathscr{C}([0, \omega_\eta])$, where η is a non-zero ordinal and ω_η is the smallest ordinal with cardinality $\aleph_\eta$ ([Ogd]).

In order to tackle those cases, we require the following lemma.

LEMMA 2.3. *Let E be a Banach space, let A be a Banach subalgebra of $\mathscr{B}(E)$ which contains the finite rank operators and has bounded left or right approximate identity, and let θ be a linear map from $E \hat{\otimes} E^*$ into another Banach space such that, for each $a \in A$, the bilinear map*

$$(E \hat{\otimes} E^*) \times (E \hat{\otimes} E^*) \ni (x, y) \mapsto \theta(x \cdot a \cdot y)$$

is continuous. Then the bilinear map

$$(E \hat{\otimes} E^*) \times (E \hat{\otimes} E^*) \ni (x, y) \mapsto \theta(xy)$$

is also continuous.

PROOF. We treat the case where A has a bounded left approximate identity first. Let F be the target space of θ, and let $\mathcal{B}^2(E \hat{\otimes} E^*, F)$ denote the Banach space of all bounded, bilinear maps from $E \hat{\otimes} E^*$ into F. We claim that the map

$$(7) \quad A \to \mathcal{B}^2(E \hat{\otimes} E^*, F), \quad a \mapsto ((E \hat{\otimes} E^*) \times (E \hat{\otimes} E^*) \ni (x, y) \mapsto \theta(x \cdot a \cdot y))$$

is bounded. To prove this, we assume that (7) is unbounded. Applying the uniform boundedness theorem twice, we obtain $x_0, y_0 \in E \hat{\otimes} E^*$ such that $A \ni a \mapsto \theta(x_0 \cdot a \cdot y_0)$ is unbounded; in particular, there is a sequence $(a_n)_{n=1}^{\infty}$ in A such that $a_n \to 0$ and $\|\theta(x_0 \cdot a_n \cdot y_0)\| \to \infty$. By a variant of Cohen's factorization theorem ([HR2, (32.23) Theorem]), there is $b \in A$ and a sequence $(c_n)_{n=1}^{\infty}$ in A such that $c_n \to 0$ and $a_n = bc_n$ for $n \in \mathbb{N}$. Since by hypothesis the map $(E \hat{\otimes} E^*) \times (E \hat{\otimes} E^*) \ni (x, y) \mapsto \theta(x \cdot b \cdot y)$ is continuous, and since $c_n \cdot y_0 \to 0$, we see that

$$\theta(x_0 a_n y_0) = \theta(x_0 b \cdot (c_n \cdot y_0)) \to 0,$$

which contradicts our choice of the sequence $(a_n)_{n=1}^{\infty}$. Consequently, there is $C \geq 0$ such that

$$\|\theta(x \cdot a \cdot y)\| \leq C \|x\| \, \|a\| \, \|y\| \qquad (a \in A, \, x, y \in E \hat{\otimes} E^*).$$

Let $(e_\alpha)_\alpha$ be a left approximate identity for A bounded by $\kappa \geq 0$. Since A contains the finite rank operators, it follows immediately that $e_\alpha x \to x$ for all $x \in E$. From the definition of the left module action of $\mathcal{B}(E)$—and hence of A—on $E \hat{\otimes} E^*$, we see that $(e_\alpha)_\alpha$ is also a bounded left approximate identity for $E \hat{\otimes} E^*$. Let $x, y \in E \hat{\otimes} E^*$, and let $\epsilon > 0$. Then another variant of Cohen's factorization theorem ([HR2, (32.22) Theorem]) yields $a \in A$ with $\|a\| \leq \kappa$ and $z \in E \hat{\otimes} E^*$ with $\|y - z\| < \epsilon$ and $y = a \cdot z$. Then we have

$$\|\theta(xy)\| = \|\theta(x \cdot a \cdot z)\| \leq C \|x\| \, \|a\| \, \|z\| \leq C\kappa \|x\| (\|y\| + \epsilon).$$

Since $\epsilon > 0$ is arbitrary, this means that

$$\|\theta(xy)\| \leq C\kappa \|x\| \, \|y\| \qquad (x, y \in E \hat{\otimes} E^*)$$

as claimed.

We now sketch the proof in the case where A has a bounded right approximate identity. As in the first case, we see that (7) is bounded. Let $(e_\alpha)_\alpha$ be a right approximate identity for A, and let $A^\dagger := \{a^* : a \in A\}$. Then $A^\dagger$ is a Banach subalgebra of $\mathscr{B}(E^*)$, and it is easily seen that $(e_\alpha^*)_\alpha$ is a bounded left approximate identity for $A^\dagger$. Although, unless E is reflexive, $A^\dagger$ need not contain all of the finite rank operators on E^*, it does contain all operators of the form $E^* \ni \psi \mapsto \langle x, \psi \rangle \phi$; this is enough to ascertain that $e_\alpha^* \phi \to \phi$ for all $\phi \in E^*$. From the definition of the right module action of A on $E \hat\otimes E^*$, we see that $(e_\alpha)_\alpha$ is also a bounded right approximate identity for $E \hat\otimes E^*$. The remainder of the proof is exactly as in the first case.   □

For the definition of the so-called James space, see [Wil3].

THEOREM 2.4. *Let E be the James space, or let $E = \mathscr{C}([0, \omega_\eta])$, where η is a non-zero ordinal, let B be a Banach algebra, and let $\theta : E \hat\otimes E^* \to B$ be a homomorphism. Then $\theta|_{(E \hat\otimes E^*)^2}$ is continuous.*

PROOF (sketch). Define $\mathscr{J}$ as in (4), and use the main boundedness theorem as in the proofs of [Wil3, Proposition 7] and [Ogd, Theorem 6.18], respectively, to show that $\mathscr{J}$ contains a closed subalgebra A of $\mathscr{B}(E)$ which contains the finite rank operators and has a bounded right approximative identity. From Lemma 2.3, we conclude that there is $C \geq 0$ such that

$$\|\theta(xy)\| \leq C\|x\| \, \|y\| (x, y \in E \hat\otimes E^*).$$

As in the proof of Theorem 2.1, we conclude with the help of Corollary 1.3 that that $\theta|_{(E \hat\otimes E^*)^2}$ is continuous.   □

Since both the James space and each space $\mathscr{C}([0, \omega_\eta])$ has the approximation property, the canonical epimorphism $E \hat\otimes E^* \to \mathscr{N}(E)$ is in fact an isomorphism under the hypotheses of Theorem 2.4. Hence, we could have formulated Theorem 2.4 equally well in terms of $\mathscr{N}(E)$.

3. Homomorphisms from $\ell^1(SL(2, \mathbb{R}))$

In [Run3], we proved that, for certain factorizable, locally compact groups G, every homomorphism from $L^1(G)$ into a Banach algebra is continuous on a dense subalgebra of $L^1(G)$ ([Run3, Theorem 4.2]). One serious limitation of our technique was that the group G had to be amenable. As a consequence, $G = SL(2, \mathbb{R})$ both with its usual topology and as a discrete group was beyond our reach.

In this section we apply Corollary 1.2, to give at least a partial answer to [Run3, Question 2].

THEOREM 3.1. *Let B be a Banach algebra, and let $\theta : \ell^1(SL(2, \mathbb{R})) \to B$ be a homomorphism. Then there is a dense subspace $\mathscr{E}$ of $\ell^1(SL(2, \mathbb{R}))$ with codimension at most one in a dense subalgebra of $\ell^1(SL(2, \mathbb{R}))$ such that $\theta|_{\mathscr{E}}$ is continuous.*

PROOF. The first half of the proof is very similar to that of [Run3, Theorem 4.2].

As an immediate consequence of its Iwasawa decomposition (see, for example, [Lan]), $SL(2, \mathbb{R})$ is factorizable, that is there are abelian subgroups $H_1, \ldots , H_n$ of $SL(2, \mathbb{R})$ such that $SL(2, \mathbb{R}) = H_1 \cdots H_n$ (in fact, $n = 3$ will do). For $j = 1, \ldots , n$, let

$$\mathscr{J}_j := \{\phi \in \ell^1(H_j) : \ell^1(SL(2, \mathbb{R})) \times \ell^1(SL(2, \mathbb{R}))$$
$$\ni (f, g) \mapsto \theta(f * \phi * g) \text{ is continuous}\}.$$

For $j = 1, \ldots , n$, let S_j be the hull of $\mathscr{J}_j$ in $\Phi_{\ell^1(H_j)}$, the Gelfand space of $\ell^1(H_j)$; a routine application of the main boundedness theorem ascertains that S_j is finite. Let

$$I(S_j) := \{\phi \in \ell^1(H_j) : \hat{\phi}|_{S_j} \equiv 0\} \qquad (j = 1, \ldots , n).$$

By [Wil2, Lemma 2.1], there are $x_1, \ldots , x_m \in SL(2, \mathbb{R})$ such that the right ideal

$$I := \sum_{k=1}^{m} \sum_{j=1}^{n} \delta_{x_k} * I(S_j) * \delta_{x_k^{-1}} * \ell^1(SL(2, \mathbb{R}))$$

is closed and has finite codimension in $\ell^1(SL(2, \mathbb{R}))$. For $j = 1, \ldots , n$, let

$$J(S_j) := \{\phi \in \ell^1(H_j) : \text{supp}(\hat{\phi}) \cap S_j = \varnothing\}.$$

As in the proof of [Run3, Theorem 4.2], we see that there is a constant $C \geq 0$ such that

(8) $$\|\theta(f * \phi * g)\| \leq C\|f\|_1\|\phi * g\|_1$$
$$(f, g \in \ell^1(SL(2, \mathbb{R})), \ \phi \in \delta_{x_k} * J(S_j) * \delta_{x_k^{-1}}, \ j = 1, \ldots , n, \ k = 1, \ldots , m).$$

Let

$$X := \bigoplus_{k=1}^{m} \bigoplus_{j=1}^{n} \delta_{x_k} * I(S_j) * \delta_{x_k^{-1}} * \ell^1(SL(2, \mathbb{R}))$$

be equipped with the ℓ^1-norm, and define

$$\Pi : X \to \ell^1(SL(2, \mathbb{R})), \quad (f_1, \ldots , f_{n,m}) \mapsto f_1 + \cdots + f_{n,m}.$$

Note that $\Pi(X) = I$. Let

$$\mathscr{X} := \bigoplus_{k=1}^{m} \bigoplus_{j=1}^{n} \delta_{x_k} * J(S_j) * \delta_{x_k^{-1}} * \ell^1(SL(2, \mathbb{R})).$$

Obviously, $\mathscr{X}$ is dense in X. Define a bilinear map

$$\Theta : \ell^1(SL(2, \mathbb{R})) \times \mathscr{X} \to B, \quad (f, x) \mapsto \theta(f * \Pi(x)).$$

Due to (8), Θ is bounded and thus extends to a bounded bilinear map $\ell^1(SL(2, \mathbb{R})) \times X \to B$, which we shall denote by Θ as well. We claim that Θ drops to a bounded, bilinear map $\tilde{\Theta} : I \times I \to B$. Let $f \in I$, and let $x, y \in X$ such that $\Pi(x) = f$ and $\Pi(y) = 0$. Further, let $(x_k)_{k=1}^{\infty}$ be a sequence in $\mathscr{X}$ such that $x_k \to x$, and note that, for each $k \in \mathbb{N}$, the bilinear map $\ell^1(SL(2, \mathbb{R})) \times \ell^1(SL(2, \mathbb{R})) \ni (g, h) \mapsto \theta(g * \Pi(x_k) * h)$ is continuous. Hence, we have

$$\Theta(f, y) = \Theta(\delta_1 * \Pi(x), y) = \lim_{k \to \infty} \Theta(\delta_1 * \Pi(x_k), y)$$
$$= \lim_{k \to \infty} \theta(\delta_1 * \Pi(x_k) * \Pi(y)) = 0,$$

where δ_1 is the point mass at the group identity. With $\mathscr{I} := \Pi(\mathscr{X})$, we thus have

$$(9) \qquad \|\theta(f * g)\| = \|\tilde{\theta}(f, g)\| \le C'\|f\|\|g\| \qquad (f, g \in \mathscr{I})$$

with some constant $C' \ge 0$.

Let

$$\ell_0^1(SL(2, \mathbb{R})) := \left\{ f = \sum_{x \in SL(2,\mathbb{R})} \lambda_x \delta_x : \sum_{x \in SL(2,\mathbb{R})} \lambda_x = 0 \right\}$$

be the so-called augmentation ideal of $\ell^1(SL(2, \mathbb{R}))$. Obviously, $\ell_0^1(SL(2, \mathbb{R}))$ is a closed ideal of $\ell^1(SL(2, \mathbb{R}))$ with codimension one. Moreover, it follows from [HR1, (22.22)(h)] that $\ell_0^1(SL(2, \mathbb{R}))$ is the only proper closed ideal of $\ell^1(SL(2, \mathbb{R}))$ with finite codimension. Since I has finite codimension, the closed, two-sided ideal

$$\{f \in \ell^1(SL(2, \mathbb{R})) : \ell^1(SL(2, \mathbb{R})) * f \subset I\} \subset I$$

has finite codimension as well, and consequently contains $\ell_0^1(SL(2, \mathbb{R}))$. Hence, we have either $I = \ell^1(SL(2, \mathbb{R}))$ or $I = \ell_0^1(SL(2, \mathbb{R}))$. Suppose first that $I = \ell^1(SL(2, \mathbb{R}))$. Then $\mathscr{I}$ is a dense right ideal of $\ell^1(SL(2, \mathbb{R}))$ and therefore equals $\ell^1(SL(2, \mathbb{R}))$. In this case, (9) immediately yields the continuity of θ on all of $\ell^1(SL(2, \mathbb{R}))$. We may therefore assume without loss of generality that $I = \ell_0^1(SL(2, \mathbb{R}))$.

We wish to apply Corollary 1.2. First, observe that by [Wil2, Lemma 2.2] the multiplication map $\Delta : I \hat{\otimes} I \to I$ is onto. To finish the proof, we have to compute $\mathscr{H}^2(\ell_0^1(SL(2, \mathbb{R})), \mathbb{C}_{ann})$. From well known facts about Hochschild cohomology, it follows that

$$\mathscr{H}^2(\ell_0^1(SL(2, \mathbb{R})), \mathbb{C}_{ann}) \cong \mathscr{H}^2(\ell^1(SL(2, \mathbb{R})), \mathbb{C}),$$

where the module actions of $\ell^1(SL(2, \mathbb{R}))$ on $\mathbb{C}$ are induced through the trivial $SL(2, \mathbb{R})$-module action

(10) $\qquad x \cdot \lambda := \lambda \quad \text{and} \quad \lambda \cdot x := \lambda \qquad (x \in SL(2, \mathbb{R}), \lambda \in \mathbb{C}).$

As pointed out on [Joh2, page 28] (see also [Gri]), we have

$$\mathscr{H}^2(\ell^1(SL(2, \mathbb{R})), \mathbb{C}) \cong H_b^{(2)}(SL(2, \mathbb{R}), \mathbb{C}),$$

where our notation for bounded group cohomology is as in [Gri]. As pointed out on [Gri, page 121], $H_b^{(2)}(SL(2, \mathbb{R}), \mathbb{C})$ is the complexification of $H_b^{(2)}(SL(2, \mathbb{R}), \mathbb{R})$, where the module actions of $SL(2, \mathbb{R})$ on $\mathbb{R}$ are defined as in (10). Finally, Matsumoto and Morita compute in [MM] that $H_b^{(2)}(SL(2, \mathbb{R}), \mathbb{R}) \cong \mathbb{R}$. In view of the foregoing, we obtain

$$\mathscr{H}^2(\ell_0^1(SL(2, \mathbb{R})), \mathbb{C}_{\text{ann}}) \cong \mathbb{C}.$$

Corollary 1.2 then yields a dense subspace $\mathscr{F}$ of I with codimension at most one in $\mathscr{I}^2$ such that $\theta|_{\mathscr{F}}$ is continuous. Let $\mathscr{E} := \mathbb{C}\delta_1 + \mathscr{F}$. Then $\theta|_{\mathscr{E}}$ is continuous, and $\mathscr{E}$ has codimension at most one in $\mathbb{C}\delta_1 + \mathscr{I}^2$, which is a dense subalgebra of $\ell^1(SL(2, \mathbb{R}))$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

REMARK 3.2. 1. Certainly, Theorem 3.1 would be much more appealing if we could prove the continuity of θ on a dense subalgebra of $\ell^1(SL(2, \mathbb{R}))$. In fact, if our conjecture on the existence of discontinuous homomorphisms from group algebras stated in [Run1] is correct, θ should be continuous on all of $\ell^1(SL(2, \mathbb{R}))$. However, we are unable to prove these stronger assertions.

2. It would be desirable to have an analogue of Theorem 3.1 for homomorphisms from $L^1(SL(2, \mathbb{R}))$. In fact, most of the proof can be modified to fit the non-discrete situation (see the proof of [Run3, Theorem 4.2]). The problem here is that we do not know how to compute $\mathscr{H}^2(L_0^1(SL(2, \mathbb{R}), \mathbb{C}_{\text{ann}})$.

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