Canad. J. Math. Vol. 71 (3), 2019 pp. 557–578
http://dx.doi.org/10.4153/CJM-2017-035-3

Degrees of Regular Sequences With a Symmetric Group Action

Federico Galetto, Anthony Vito Geramita, and David Louis Wehlau

Abstract. We consider ideals in a polynomial ring that are generated by regular sequences of homogeneous polynomials and are stable under the action of the symmetric group permuting the variables. In previous work, we determined the possible isomorphism types for these ideals. Following up on that work, we now analyze the possible degrees of the elements in such regular sequences. For each case of our classification, we provide some criteria guaranteeing the existence of regular sequences in certain degrees.

1 Introduction

Consider the graded polynomial ring $R = \mathbb{C}[x_1, x_2, \dots, x_n]$. A set of n homogeneous polynomials $f_1, f_2, \dots, f_n$ is a maximal regular sequence in R if the only common zero of these n polynomials is the point $(0, 0, \dots, 0)$. A sequence $g_1, g_2, \dots, g_t$ is a regular sequence in R if it can be extended to a maximal regular sequence in R.

We suppose that G is a group acting linearly on R via an action that preserves the grading. The subring $R^G := \{f \in R : \forall \sigma \in G, \sigma \cdot f = f\}$ is called the *ring of invariants*. There has been some interest in determining the degrees $(d_1, d_2, \dots, d_t)$ for which there exists a regular sequence in R^G with $\deg(f_i) = d_i$. Dixmier [6] made a conjecture concerning this question for the classical case of the action of $\mathrm{SL}(2, \mathbb{C})$ on an irreducible representation. This conjecture has attracted some attention [1, 7, 17]. Recently, a few authors have taken up this question for the natural action of the symmetric group on R [4, 5, 14].

We consider a more general question. Our goal is to determine the degrees of a maximal regular sequence $f_1, f_2, \dots, f_n$ in R such that the ideal $I := (f_1, f_2, \dots, f_n)$ is stable under the group action. This is equivalent to the artinian quotient algebra R/I inheriting the action of the group.

We will also restrict our attention to the natural action of the symmetric group $\mathfrak{S}_n$ permuting the variables. In our earlier paper [10], we showed that there are four possible representation types for the action of $\mathfrak{S}_n$ on I (the notation follows that of [19]):

- the trivial representation $S^{(n)}$, given by all f_i being symmetric polynomials,
- the alternating representation $S^{(1^n)}$, given by one alternating polynomial, together with up to $n-1$ symmetric polynomials,

Received by the editors August 20, 2017.
Published electronically March 21, 2018.
The authors gratefully acknowledge the partial support of NSERC for this work.
AMS subject classification: **13A02**, 13A50, 20C30.
Keywords: complete intersection, symmetric group, regular sequence.

- the standard representation $S^{(n-1,1)}$, possibly together with one symmetric polynomial,
- the representation $S^{(2,2)}$, together with up to two symmetric polynomials (this only occurs when $n = 4$).

Our earlier paper showed examples of regular sequences corresponding to all four cases, but did not address the question of how often such regular sequences can appear or, more precisely, the degrees in which they can be realized. Here we give explicit answers showing in which degrees it is possible to find a regular sequence for each of the above four representation types for $n \leqslant 4$. We also derive a number of results for general values of n.

Note also that our results relating to the first case above actually apply to the degrees of regular sequences of homogeneous polynomials in the polynomial ring $\mathbb{C}[y_1, \dots, y_n]$, with the non-standard grading given by $\deg(y_i) = i$. This case corresponds geometrically to the homogeneous coordinate ring of a weighted projective space.

2 Regular Sequences of Symmetric Polynomials

We consider the polynomial ring $R = \mathbb{C}[x_1, x_2, \dots, x_n]$ in n indeterminates equipped with the standard grading. The symmetric group $\mathfrak{S}_n$ acts naturally on R by permuting the variables. It is well known that the invariant subring $R^{\mathfrak{S}_n}$ can be identified with the subalgebra $\mathbb{C}[e_1, e_2, \dots, e_n]$ generated by the elementary symmetric polynomials [11]. In particular, $R^{\mathfrak{S}_n}$ is a polynomial ring equipped with the non-standard grading $\deg(e_i) = i$.

2.1 Degree Sequences

We are concerned with the degrees of elements of regular sequences in $R^{\mathfrak{S}_n}$. All of the regular sequences we consider consist of homogeneous polynomials.

Definition 2.1 Let $(d_1, d_2, \dots, d_n)$ be an (unordered) sequence of n positive integers. If there exists a regular sequence $f_1, f_2, \dots, f_n \in R^{\mathfrak{S}_n}$ with $\deg(f_i) = d_i$, then we say that $(d_1, d_2, \dots, d_n)$ is a *regular degree sequence*.

Proposition 2.2 *Suppose $(d_1, d_2, \dots, d_n)$ is a regular degree sequence. We define $\beta_i := \#\{1 \leqslant j \leqslant n : i \mid d_j\}$, for $i = 1, 2, \dots, n$. Then*

$$\beta_i \geqslant \left\lfloor \frac{n}{i} \right\rfloor \quad \text{for all } i = 1, 2, \dots, n \tag{2.1}$$

In particular, $n! \mid \prod_{j=1}^{n} d_j$.

Proof If $(d_1, d_2, \dots, d_n)$ is a regular degree sequence, then there exists a regular sequence $f_1, f_2, \dots, f_n$ in $R^{\mathfrak{S}_n}$ with $\deg(f_i) = d_i$. The graded subring $A = \mathbb{C}[f_1, f_2, \dots, f_n]$ is a polynomial ring and $R^{\mathfrak{S}_n}$ is a free A-module: $R^{\mathfrak{S}_n} \cong \bigoplus_{\gamma \in \Gamma} A \cdot \gamma$ for some set of homogeneous elements $\Gamma \subset R^{\mathfrak{S}_n}$ [3, Lemma 6.4.13]. Thus the Hilbert series of $R^{\mathfrak{S}_n}$ and A are related by $\mathcal{H}(R^{\mathfrak{S}_n}, t) = \sum_{\gamma \in \Gamma} t^{\deg(\gamma)} \mathcal{H}(A, t)$. Since

$\mathcal{H}(R^{\mathfrak{S}_n}, t) = \prod_{i=1}^n (1-t^i)^{-1}$ and $\mathcal{H}(A, t) = \prod_{i=1}^n (1-t^{d_i})^{-1}$, we see that

$$\prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t^i} = \sum_{\gamma \in \Gamma} t^{\deg(\gamma)}$$

is a non-negative integer polynomial.

Working over $\mathbb{Q}$, all the irreducible factors of $(1-t^d)$ are cyclotomic polynomials. Specifically, $(1-t^d) = \prod_{i|d} \Phi_i(t)$, where Φ_i denotes the i-th cyclotomic polynomial. Since $\#\{1 \leqslant j \leqslant n : i \mid j\} = \lfloor n/i \rfloor$, we see that $\prod_{i=1}^n \frac{1-t^{d_i}}{1-t^i}$ is an integer polynomial if and only if $\beta_i \geqslant \lfloor n/i \rfloor$, for all $i = 1, 2, \dots n$.

To prove the final assertion, we cancel the factors of $(1-t)$ from the numerator and denominator. Thus

$$\prod_{i=1}^{n} \frac{1+t+t^2+\cdots+t^{d_i}}{1+t+t^2+\cdots+t^i} = \sum_{\gamma\in\Gamma} t^{\deg(\gamma)}.$$

Evaluating at $t = 1$, we see that $\left(\prod_{i=1}^n d_i\right)/n! = |\Gamma| =$ rank of $R^{\mathfrak{S}_n}$ as an A-module. ∎

Remark 2.3 A number of authors have observed the restriction that the product of the d_i is divisible by $n!$, see for example [5, Lemma 2.8]. The inequality (2.1) was observed by Conca, Krattenthaler, and Watanabe for regular sequences of power sums [5, Lemma 2.6 (2)].

Suppose $(d_1, d_2, \dots, d_n)$ is a regular degree sequence. Since

$$\bigoplus_{d \leqslant i} R_d^{\mathfrak{S}_n} \subset \mathbb{C}[e_1, e_2, \dots, e_i]$$

and hence cannot contain a regular sequence with more than i terms, we deduce that $(d_1, d_2, \dots, d_n)$ must also satisfy the following condition

$$\#\{j : d_j \leqslant i\} \leqslant i \quad \text{for all } i = 1, 2, \dots, n. \tag{2.2}$$

Definition 2.4 Let $(d_1, d_2, \dots, d_n)$ be an (unordered) sequence of n positive integers. We say that $(d_1, d_2, \dots, d_n)$ is *permissible* if it satisfies the two conditions (2.1) and (2.2). Thus, every regular degree sequence is permissible.

Note that if there exists a *matching*, *i.e.*, a permutation $\pi \in \mathfrak{S}_n$ such that i divides $d_{\pi(i)}$ for all $i = 1, 2, \dots, n$, then $(d_1, d_2, \dots, d_n)$ is a regular degree sequence as is shown by the regular sequence of polynomials $(e_i)^{d_{\pi(i)}/i}$ for $i = 1, 2, \dots, n$. For example $(1, 4, 9)$ is a regular degree sequence since e_1, e_2^2, e_3^3 is a regular sequence.

2.2 Regular Degree Sequences for $n \leqslant 4$

Theorem 2.5 (i) *For $n = 2$, a degree sequence is regular if and only if it is permissible if and only if it satisfies (2.1).*

(ii) *For $n = 3$, a degree sequence is regular if and only if it is permissible.*

(iii) *For $n = 4$, degree sequences of the form $(1, 2, 5, 12\delta)$, $(2, 2, 5, 12\delta)$, $(2, 5, 5, 12\delta)$, where $\delta \in \mathbb{N}$, are permissible, but are not regular. Every other permissible degree sequence is regular when $n = 4$.*

https://doi.org/10.4153/CJM-2017-035-3 Published online by Cambridge University Press

Proof (i) If (d_1, d_2) satisfies (2.1), then at least one of d_1 or d_2 is even and so we have a matching.

(ii) Let $n = 3$ and suppose that (d_1, d_2, d_3) is permissible but has no matching. Then, without loss of generality, 6 divides d_3 while d_1 and d_2 are both odd numbers not divisible by 3 with $d_2 \geqslant d_1$. Now condition (2.2) implies that $d_1 = d_2 = 1$ is impossible, so $d_2 \geqslant 2$ and thus $d_2 \geqslant 5$. Therefore, we have a regular sequence $e_1^{d_1}$, $e_3 e_2^{(d_2-3)/2}$, $(e_2^3 + e_3^2)^{d_3/6}$ with degrees (d_1, d_2, d_3).

(iii) For $n = 4$, note that $R_1^{\mathfrak{S}_n} \oplus R_2^{\mathfrak{S}_n} \oplus R_5^{\mathfrak{S}_n}$ is contained in the ideal generated by e_1 and e_2. This implies that a regular degree sequence must satisfy

$$\#\{i : d_i \in \{1,2,5\}\} \leqslant 2.$$

This shows that the permissible degree sequences $(1,2,5,12\delta)$, $(2,2,5,12\delta)$, and $(2,5,5,12\delta)$ for $\delta \in \mathbb{N}$ are not regular.

Now suppose that (d_1, d_2, d_3, d_4) is a permissible degree sequence that is not of the form $(1,2,5,12\delta)$, $(2,2,5,12\delta)$, or $(2,5,5,12\delta)$. If (d_1, d_2, d_3, d_4) has a matching, then it is regular, and so we suppose that it has no matching. Condition (2.1) implies that two of the d_i are even, one is divisible by 3 and one is divisible by 4. Without loss of generality, d_3 and d_4 are both even, $d_4 = 4\delta$ is divisible by 4 and $d_2 \geqslant d_1$. Since there is no matching, neither, d_1 nor d_2 is divisible by 3. Thus either d_3 is a multiple of 6 or d_4 is a multiple of 12.

First we consider the case where $d_3 = 6\beta$ is a multiple of 6. Since there is no matching, both d_1 and d_2 are odd integers not divisible by 3. Thus (2.2) implies that $d_2 \geqslant 5$. Therefore $e_1^{d_1}, e_3 e_2^{(d_2-3)/2}, (e_2^3 + e_3^2)^{\beta}, e_4^{\delta}$ is a regular sequence of the required degrees.

Thus we may suppose that $d_4 = 12\delta$ is a multiple of 12. Now we adjust our labelling as follows. We suppose that d_3 is the largest of those elements of $\{d_1, d_2, d_3\}$ that are even. Further we assume that $d_2 \geqslant d_1$. Furthermore, since there is no matching, 3 divides neither d_1 nor d_2.

Since $d_2 \geqslant 2$, we may write $d_2 = 2p + 3q$, where p and q are non-negative integers. Suppose first that $d_3 \geqslant 4$ and define

$$f := \begin{cases} e_4^{d_3/4} & \text{if } d_3 \equiv 0 \pmod 4, \\ e_4^{(d_3-6)/4}(e_2^3 + e_3^2) & \text{if } d_3 \equiv 2 \pmod 4. \end{cases}$$

Then $e_1^{d_1}, e_2^p e_3^q, f, (e_2^6 + e_3^4 + e_4^3)^{\delta}$ is a regular sequence of degrees (d_1, d_2, d_3, d_4).

Finally we suppose that $d_3 = 2$. Then $d_1 \leqslant d_2$ and either $d_2 = 2$ or d_2 is odd. But the sequences $(1,2,2,12\delta)$ and $(2,2,2,12\delta)$ do not satisfy (2.2), so they are not permissible. Therefore d_2 must be odd. Since 3 does not divide d_2, we have $d_2 \geqslant 5$. If $d_2 = 5$, then $d_1 \in \{1,2,5\}$, which is again not possible since we have excluded sequences of the form $(1,5,2,12\delta)$, $(2,5,2,12\delta)$, and $(5,5,2,12\delta)$. Therefore $d_2 \geqslant 7$ if $d_3 = 2$. Thus we may write $d_2 = 3p + 4q$. Then $e_1^{d_1}, e_3^p e_4^q, e_2, (e_3^4 + e_4^3)^{\delta}$ is a regular sequence of the required degrees. ∎

Table 1 summarizes the regular sequences we have found when $n = 4$. The first row corresponds to a matching.

Note that we have in fact proved the following result.

Degrees				**Symmetric Polynomials**			
d_1	d_2	d_3	d_4	f_1	f_2	f_3	f_4
d_1	3β	2γ	4δ	$e_1^{d_1}$	e_3^{β}	e_2^{γ}	e_4^{δ}
d_1	$d_2 \geqslant 5$	6γ	4δ	$e_1^{d_1}$	$e_3 e_2^{(d_2-3)/2}$	$(e_2^3 + e_3^2)^{\gamma}$	e_4^{δ}
d_1	$d_2 \geqslant 2$	4β	12δ	$e_1^{d_1}$	$e_2^p e_3^q$	e_4^{β}	$(e_2^6 + e_3^4 + e_4^3)^{\delta}$
d_1	$d_2 \geqslant 2$	$4\beta + 2 \geqslant 6$	12δ	$e_1^{d_1}$	$e_2^p e_3^q$	$(e_2^3 + e_3^2)e_4^{\beta-1}$	$(e_2^6 + e_3^4 + e_4^3)^{\delta}$
d_1	$d_2 \geqslant 7$	2	12δ	$e_1^{d_1}$	$e_3^p e_4^q$	e_2	$(e_3^4 + e_4^3)^{\delta}$

Table 1: Regular sequences of symmetric polynomials for $n = 4$

Corollary 2.6 *Suppose that d_2, d_3, d_4 are three positive integers such that $4 \mid d_4$, d_3 is even, and $3 \mid d_2 d_3 d_4$. Then there exist three symmetric polynomials f_2, f_3, f_4 (as given in Table 1) of degrees d_2, d_3, d_4, respectively, such that e_1, f_2, f_3, f_4 is a regular sequence.*

Remark 2.7 Note that the degree sequence $(2,5,2,12)$ (which is not regular) has the property that

$$\frac{(1-t^2)(1-t^5)(1-t^2)(1-t^{12})}{(1-t)(1-t^2)(1-t^3)(1-t^4)} = 1 + t + t^3 + 2t^4 + 2t^7 + t^8 + t^{10} + t^{11}$$

is a non-negative integer polynomial.

For larger values of n little is known. The following statement was proved in [5, Proposition 2.9] using sequences of power sums and homogeneous symmetric polynomials.

Proposition 2.8 *For every positive integer a, the sequence of consecutive degrees $(a, a+1, a+2, \dots, a+n-1)$ is a regular degree sequence.*

2.3 Regular Sequences With an Alternating Polynomial

A polynomial $f \in R$ is said to be *alternating* if, for all $\sigma \in \mathfrak{S}_n$, $\sigma f = \pm f$, depending on the sign of σ. As an example, the Vandermonde determinant

$$\Delta := \det \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix} = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i) \in R$$

is clearly alternating. In fact, every homogeneous alternating polynomial in R is divisible by Δ, the quotient being a homogeneous symmetric polynomial.

https://doi.org/10.4153/CJM-2017-035-3 Published online by Cambridge University Press

As noted in [10, Proposition 2.5], there exist regular sequences $f_1, f_2, \dots, f_t, g\Delta$ in R with $f_1, f_2, \dots, f_t$, and g being symmetric polynomials. These sequences are closely related to sequences of symmetric polynomials.

Lemma 2.9 *Let $f_1, f_2, \dots, f_t, g, h \in R$ be homogeneous polynomials. Then the sequence $f_1, f_2, \dots, f_t, gh$ is regular if and only if both $f_1, f_2, \dots, f_t, g$ and $f_1, f_2, \dots, f_t, h$ are regular.*

Proof Suppose $f_1, f_2, \dots, f_t$ form a regular sequence. Then gh is not a zero-divisor modulo $(f_1, f_2, \dots, f_t)$ if and only if both g and h are not zero-divisors modulo $(f_1, f_2, \dots, f_t)$. ∎

The following is an immediate consequence of Lemma 2.9.

Proposition 2.10 *Let $f_1, f_2, \dots, f_t, g \in R$ be homogeneous symmetric polynomials. The sequence $f_1, f_2, \dots, f_t, g\Delta$ is regular if and only if both $f_1, f_2, \dots, f_t, g$ and $f_1, f_2, \dots, f_t, \Delta$ are regular.*

Proposition 2.10 allows ruling out the existence of regular sequences of certain degrees that contain an alternating polynomial.

Example 2.11 For $n = 4$, Δ has degree 6. By Theorem 2.5 (iii), there is no regular sequence of homogeneous symmetric polynomials f_1, f_2, f_3, g of degrees $1, 2, 5, 12\delta$. Therefore, Proposition 2.10 implies there is no regular sequence $f_1, f_2, f_3, g\Delta$ of degrees $1, 2, 5, 12\delta + 6$.

Remark 2.12 The polynomial Δ^{2k} is symmetric for all positive integers k. Moreover, the sequence $f_1, f_2, \dots, f_t, \Delta$ is regular if and only if $f_1, f_2, \dots, f_t, \Delta^{2k}$ is regular [8, Corollary 17.8 a]. As a consequence, we can exclude the existence of regular sequences in certain degrees. For example, there is no regular sequence of homogeneous polynomials f_1, f_2, f_3, Δ with f_1, f_2, f_3 symmetric of degrees $1, 2, 5$, because f_1, f_2, f_3, Δ^2 would violate Theorem 2.5 (iii).

3 Regular Sequences and the Standard Representation

We begin this section by recalling some basic facts about the representation theory of the symmetric group $\mathfrak{S}_n$ over a field of characteristic zero. We refer the reader to [19, Chapter 2] for the details.

We write $\lambda \vdash a$ to denote that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a partition of the integer a, *i.e.*, that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = a$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$. The irreducible representations of $\mathfrak{S}_n$ are in bijection with the partitions of n; for $\lambda \vdash n$, we denote by S^λ the corresponding irreducible. Every finite-dimensional representation of $\mathfrak{S}_n$ decomposes into a direct sum of copies of the S^λ.

The irreducible representation $S^{(n-1,1)}$ of $\mathfrak{S}_n$ is often called the *standard representation*. It can be described as the $\mathfrak{S}_n$-stable complement of the subspace spanned by e_1 inside the representation $R_1 = \langle x_1, x_2, \dots, x_n \rangle$. The polynomials $x_1 - x_n, x_2 - x_n, \dots, x_{n-1} - x_n$ give an explicit basis of the complement.

https://doi.org/10.4153/CJM-2017-035-3 Published online by Cambridge University Press

Let $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$ be the irrelevant maximal ideal of R. In this section, we study regular sequences $f_1, f_2, \ldots, f_t \in R$ such that the ideal $I = (f_1, f_2, \ldots, f_t)$ is stable under the action of $\mathfrak{S}_n$ and $I/\mathfrak{m}I$ contains a copy of the standard representation. As shown in [10, Proposition 2.5], there are two possibilities: $I/\mathfrak{m}I \cong S^{(n-1,1)}$ or $I/\mathfrak{m}I \cong S^{(n-1,1)} \oplus S^{(n)}$, where $S^{(n)}$ is the one-dimensional trivial representation.

3.1 Regular Sequences of Type $S^{(n-1,1)}$

Here we prove the existence of regular sequences of type $S^{(n-1,1)}$ in every positive degree.

Let $\mathcal{V}_d \subset \mathbb{A}^n$ denote the affine variety cut out by the $x_1^d - x_n^d, x_2^d - x_n^d, \ldots, x_{n-1}^d - x_n^d$, and $x_n = 1$, *i.e.*, $\mathcal{V}_d = \{(z_1, z_2, \ldots, z_n) \in \mathbb{A}^n : z_i^d = 1, z_n = 1\}$.

Theorem 3.1 *Let d be a positive integer. The polynomials*

$$x_1^d - x_n^d, x_2^d - x_n^d, \ldots, x_{n-1}^d - x_n^d$$

form a regular sequence of type $S^{(n-1,1)}$.

Proof The polynomials in question form a basis of the $\mathfrak{S}_n$-stable complement of the one-dimensional invariant subspace spanned by $x_1^d + x_2^d + \cdots + x_n^d$ inside $\langle x_1^d, x_2^d, \ldots, x_n^d \rangle$. It is clear from the comments at the beginning of the section that this complement is isomorphic to $S^{(n-1,1)}$.

To prove $x_1^d - x_n^d, x_2^d - x_n^d, \ldots, x_{n-1}^d - x_n^d$ form a regular sequence, we extend it by adding the polynomial x_n^d. It is clear that the two ideals

$$(x_1^d - x_n^d, x_2^d - x_n^d, \ldots, x_{n-1}^d - x_n^d, x_n^d) \quad \text{and} \quad (x_1^d, x_2^d, \ldots, x_{n-1}^d, x_n^d)$$

are equal and that the latter is generated by a regular sequence. Thus the extended sequence, and so also the original, is a regular sequence. ∎

3.2 Regular Sequences of Type $S^{(n-1,1)} \oplus S^{(n)}$

Let $I \subseteq R$ be an $\mathfrak{S}_n$-stable homogeneous ideal such that $I/\mathfrak{m}I \cong S^{(n-1,1)} \oplus S^{(n)}$. Then I admits a generating set $g_1, g_2, \ldots, g_{n-1}, f$ such that

- $\deg(g_i) = d$ for $i = 1, 2, \ldots, n-1$ and the vector space spanned by $g_1, g_2, \ldots, g_{n-1}$ is a representation of $\mathfrak{S}_n$ isomorphic to $S^{(n-1,1)}$;
- $\deg(f) = a$ and $f \in R^{\mathfrak{S}_n}$.

We are interested in understanding the possible choices of degrees d and a for which such an ideal I can be generated by a regular sequence. For simplicity, we restrict to the case $g_i = x_i^d - x_n^d$ for $i = 1, 2, \ldots, n-1$. This is the instance of regular sequence described in Theorem 3.1. Therefore our main question becomes: when can a symmetric polynomial f of degree a be chosen so that $x_1^d - x_n^d, x_2^d - x_n^d, \ldots, x_{n-1}^d - x_n^d, f$ is a regular sequence? Such a regular sequence containing n elements is always maximal.

Definition 3.2 Let n, d, a be three positive integers. We say the triple (n, d, a) is *good* if there exists $f \in R_a^{\mathfrak{S}_n}$ such that $x_1^d - x_n^d, x_2^d - x_n^d, \ldots, x_{n-1}^d - x_n^d, f$ is a regular sequence. Otherwise (n, d, a) is called *bad*.

Remark 3.3 Clearly, if (n,d,a) is good, then there exists a regular sequence of type $S^{(n-1,1)} \oplus S^{(n)}$ with $S^{(n-1,1)}$ in degree d and $S^{(n)}$ in degree a. However, the converse is not true in general. For example, the triple $(5,6,1)$ is bad because $x_1^6 - x_5^6, x_2^6 - x_5^6, x_3^6 - x_5^6, x_4^6 - x_5^6, e_1$ is not a regular sequence. However, if we set

$$g_i = \sum_{j=2}^{5} e_j (x_i^{6-j} - x_5^{6-j}) \quad \text{for } i = 1,2,3,4,$$

then g_1, g_2, g_3, g_4, e_1 is a regular sequence. The assertions about these sequences of polynomials can be verified computationally using the software Macaulay2 [13], and the code provided in Appendix A.

Observe that, if $f \in R$ is homogeneous, then $x_1^d - x_n^d, x_2^d - x_n^d, \dots, x_{n-1}^d - x_n^d, f$ is a regular sequence if and only if f does not vanish on $\mathcal{V}_d$.

For a positive integer a, the *power sum* $\mathcal{P}_a = x_1^a + x_2^a + \cdots + x_n^a$ is a homogeneous symmetric polynomial of degree a. Furthermore, given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of a, we write $\mathcal{P}_\lambda$ for the symmetric polynomial $\prod_{t=1}^{r} \mathcal{P}_{\lambda_t}$ of degree a. The set of $\mathcal{P}_\lambda$ where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a partition of a whose parts λ_i do not exceed n is a basis of $R_a^{\mathfrak{S}_n}$ as a complex vector space [20, Proposition 7.8.2]).

Lemma 3.4 *The triple (n,d,a) is bad if and only if there exists a point $Q \in \mathcal{V}_d$ such that $\mathcal{P}_\lambda(Q) = 0$ for every partition $\lambda \vdash a$.*

Proof If such a point Q exists, then it is clear that (n,d,a) is bad. Suppose then that (n,d,a) is bad. Enumerate the partitions $\lambda \vdash a$ whose parts do not exceed n, and denote them by $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(t)}$. Introduce the homogeneous symmetric polynomial $f := \sum_{i=1}^{t} \pi^i \mathcal{P}_{\lambda^{(i)}}$ of degree a. Since (n,d,a) is bad, there exists $Q \in \mathcal{V}_d$ such that

$$0 = f(Q) = \sum_{i=1}^{t} \pi^i \mathcal{P}_{\lambda^{(i)}}(Q).$$

Since the coordinates of Q are algebraic numbers, $\mathcal{P}_{\lambda^{(i)}}(Q)$ is algebraic for all $i = 1,2,\dots,t$. Then $f(Q) = 0$ implies $\mathcal{P}_{\lambda^{(i)}}(Q) = 0$ for all $i = 1,2,\dots,t$, because π is transcendental. The result follows. ∎

The following is an immediate consequence of Lemma 3.4.

Corollary 3.5 *The triple (n,d,a) is bad if and only if there exists a point $Q \in \mathcal{V}_d$ such that $f(Q) = 0$ for every $f \in R_a^{\mathfrak{S}_n}$.*

Lemma 3.4 suggests it might be useful to understand the vanishing of power sums at roots of unity. The following result is due to Lam and Leung [15, Theorem 5.2].

Theorem 3.6 *Let d be a positive integer and let $\Gamma(d)$ denote the numerical semi-group generated by the prime divisors of d. Then there exist d-th roots of unity $z_1, z_2, \dots, z_n$ (not necessarily distinct) such that $z_1 + z_2 + \cdots + z_n = 0$ if and only if $n \in \Gamma(d)$.*

Note that $\Gamma(1) := \{0\}$ here.

Corollary 3.7 *Let a, d be positive integers and let $g := \gcd(a, d)$. Then there exist d-th roots of unity $z_1, z_2, \dots, z_n$ (not necessarily distinct) such that $\mathcal{P}_a(z_1, z_2, \dots, z_n) = 0$ if and only if $n \in \Gamma(d/g)$.*

Proof Assume there exist d-th roots of unity $z_1, z_2, \dots, z_n$ such that

$$\mathcal{P}_a(z_1, z_2, \dots, z_n) = 0.$$

Note that z_i^a is a (d/g)-th root of unity. Then $z_1^a + z_2^a + \cdots + z_n^a = \mathcal{P}_a(z_1, z_2, \dots, z_n) = 0$ implies $n \in \Gamma(d/g)$ by Theorem 3.6.

Conversely, if $n \in \Gamma(d/g)$, then Theorem 3.6 implies the existence of (d/g)-th roots of unity $w_1, w_2, \dots, w_n$ such that $w_1 + w_2 + \cdots + w_n = 0$. Since $g = \gcd(a, d)$, we have $1 = \gcd(a, d/g)$. By Bezout's identity [16, Proposition 5.1], there exist integers u, v such that $au + (d/g)v = 1$. Note that $z_i = w_i^u$ is a d-th root of unity. Therefore we get

$$0 = \sum_{i=1}^n w_i = \sum_{i=1}^n w_i^{au+(d/g)v} = \sum_{i=1}^n (w_i^u)^a (w_i^{d/g})^v = \mathcal{P}_a(z_1, z_2, \dots, z_n). \qquad \blacksquare$$

Remark 3.8 Let ζ_d be a primitive d-th root of unity. The Galois group of the cyclotomic field $\mathbb{Q}(\zeta_d)$ is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^\times$, the group of units modulo d. An element of $(\mathbb{Z}/d\mathbb{Z})^\times$ is represented by the class of an integer s coprime to d. Let γ_s denote the corresponding Galois automorphism of $\mathbb{Q}(\zeta_d)$, which is defined by fixing $\mathbb{Q}$ and sending ζ_d to ζ_d^s. If z is a d-th root of unity, then z is a power of ζ_d, therefore $\gamma_s(z) = z^s$.

Now let $Q = (z_1, z_2, \dots, z_n) \in \mathcal{V}_d$. We have that

$$\begin{aligned}\mathcal{P}_s(Q) &= z_1^s + z_2^s + \cdots + z_n^s = \gamma_s(z_1) + \gamma_s(z_2) + \cdots + \gamma_s(z_n)\\ &= \gamma_s(z_1 + z_2 + \cdots + z_n) = \gamma_s(\mathcal{P}_1(Q)).\end{aligned}$$

Therefore $\mathcal{P}_s(Q) = 0$ if and only if $\mathcal{P}_1(Q) = 0$.

3.3 Numerical Criteria for Good and Bad Triples

Throughout the rest of this section (n, d, a) is intended to be a triple of positive integers. We present criteria to decide whether (n, d, a) is good or bad in the sense of Definition 3.2.

Proposition 3.9 *Let $g := \gcd(a, d)$. If $n \notin \Gamma(d/g)$, then (n, d, a) is good. In particular, if $n \notin \Gamma(d)$, then (n, d, a) is good for every a.*

Proof If $n \notin \Gamma(d/g)$, then $\mathcal{P}_a$ does not vanish on $\mathcal{V}_d$ by Corollary 3.7, thus (n, d, a) is good. The second assertion follows from the fact that $\Gamma(d/g) \subseteq \Gamma(d)$ for any divisor g of d. $\blacksquare$

Remark 3.10 The proof of Proposition 3.9 uses a power sum as the symmetric polynomial of degree a. It seems that we might be able to use Theorem 3.6 to handle more cases by using some other symmetric polynomial f. While it is possible that $n \in \Gamma(d)$ and $f \in R^{\mathfrak{S}_n}$ is homogeneous having m terms with $m \notin \Gamma(d)$, this only happens in two cases.

The first case is $f = e_n$, the n-th elementary symmetric polynomial, which consists of a single term and does not vanish on $\mathcal{V}_d$. In particular, this shows that if n divides a, then (n, d, a) is good.

The second case is essentially when d is a power of a prime. See Corollaries 3.20 and 3.21 below. In fact, suppose two distinct primes p, q divide d, $n \geqslant p + q$, $n \in \Gamma(d)$ and let f be a non-constant symmetric polynomial having m terms. Then $n \geqslant p + q$ implies that $\binom{n}{2} \geqslant (p-1)(q-1)$. Thus, if $m \geqslant \binom{n}{2}$, then $m \geqslant (p-1)(q-1)$, which implies $m \in \Gamma(pq)$ [18, Theorem. 2.1.1]. Since $\Gamma(pq) \subseteq \Gamma(d)$, we deduce that $m \geqslant \binom{n}{2}$ implies $m \in \Gamma(d)$. Therefore, if $m \notin \Gamma(d)$, then $m < \binom{n}{2}$. Since we are assuming $n \in \Gamma(d)$, this implies that $f = \lambda e_n$ for some scalar λ.

Proposition 3.11 *Define $S := \{q : q \mid d, n \notin \Gamma(d/q)\}$. If a lies in the numerical semi-group generated by S, then the triple (n, d, a) is good.*

Proof By the hypothesis, we can write $a = \sum_{i=1}^{r} \lambda_i$, where $\lambda_i \in S$ for $i = 1, 2, \dots, r$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$. Then $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a partition of a and $\mathcal{P}_\lambda$ is a symmetric polynomial of degree a.

Since $\lambda_i \in S$, we have that $\lambda_i \mid d$, hence $\gcd(\lambda_i, d) = \lambda_i$. Moreover, $n \notin \Gamma(d/\lambda_i)$. Therefore Corollary 3.7 implies that $\mathcal{P}_{\lambda_i}$ does not vanish on $\mathcal{V}_d$. Since this holds for all indices $i = 1, 2, \dots, r$, we conclude that $\mathcal{P}_\lambda(Q)$ does not vanish on $\mathcal{V}_d$. Therefore (n, d, a) is good. ∎

Remark 3.12 Note that $d \in S$ always. Furthermore, if $d = p_1^{b_1} p_2^{b_2} \cdots p_t^{b_t}$ is the prime factorization of d, then the set $\{d/p_i^{b_i} : p_i \nmid n\}$ is a subset of S.

Remark 3.13 Proposition 3.11 remains true if we use $S \cup \{n\}$ instead of S. In fact, if a lies in the numerical semi-group generated by $S \cup n$, then $a = b + cn$, where b, c are positive integers and b lies in the numerical semi-group generated by S. By the proof of Proposition 3.11, there exists $\lambda \vdash b$ such that $\mathcal{P}_\lambda$ does not vanish on $\mathcal{V}_d$. At the same time, the elementary symmetric polynomial e_n does not vanish on $\mathcal{V}_d$. Therefore $\mathcal{P}_\lambda e_n^c$ is a homogeneous symmetric polynomial of degree a that does not vanish on $\mathcal{V}_d$.

Proposition 3.14 *Suppose that $n \in \Gamma(d)$ and $a \notin \Gamma(d)$. Then (n, d, a) is bad.*

Proof Since $n \in \Gamma(d)$, there exists $Q \in \mathcal{V}_d$ such that $\mathcal{P}_1(Q) = 0$ by Theorem 3.6. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \vdash a$, then some part λ_t is coprime to d since $a \notin \Gamma(d)$. Hence, by Remark 3.8, we have $\mathcal{P}_{\lambda_t}(Q) = 0$ and thus $\mathcal{P}_\lambda(Q) = 0$. The reasoning holds for all $\lambda \vdash a$. Therefore (n, d, a) is bad by Lemma 3.4. ∎

Proposition 3.15 *Let $g := \gcd(d, n)$. If $g \nmid a$, then (n, d, a) is bad.*

Proof Let ω be a primitive g-th root of unity and define $Q = (\omega, \omega^2, \dots, \omega^n) \in \mathcal{V}_d$. Observe that $\omega^i = \omega^{i+gj}$ for all $i, j \in \mathbb{Z}$. Hence, using the auxiliary variable y, we have

$$\prod_{i=1}^{n} (y - \omega^i) = \Big[\prod_{i=1}^{g} (y - \omega^i) \Big]^{n/g} = (y^g - 1)^{n/g}.$$

On the other hand $\prod_{i=1}^{n}(y-\omega^i) = \sum_{j=0}^{n}(-1)^j e_j(Q)y^{n-j}$. By comparing the two expressions, we deduce that $e_j(Q) = 0$ whenever $g \nmid j$. Thus the only symmetric polynomials potentially not vanishing at Q are the ones in the subring $\mathbb{C}[e_j : g \mid j]$. Note how the degree of any element in this subring is divisible g. Since $g \nmid a$, (n,d,a) is bad by Corollary 3.5. ■

Proposition 3.16 *Let $g := \gcd(d,n)$ and assume that $a \geqslant \frac{(n-g)(d-g)}{g}$. Then (n,d,a) is bad if and only if $g \nmid a$.*

Proof If $g \nmid a$, then the triple is bad by Proposition 3.15.

Assume $g \mid a$ and let $a' = a/g$, $n' = n/g$, and $d' = d/g$. The inequality in the assumption gives $a' = \frac{a}{g} \geqslant \frac{n-g}{g}\frac{d-g}{g} = (n'-1)(d'-1)$. By [18, Theorem 2.1.1], a' belongs to the numerical semi-group generated by d' and n'. Thus we can write $a' = sd' + tn'$, for some non-negative integers s and t. Multiplying by g, we obtain $a = sd + tn$. This equality implies that the homogeneous symmetric polynomial $f := \mathcal{P}_d^s e_n^t$ has degree a. For all $Q \in \mathcal{V}_d$, we have $\mathcal{P}_d(Q) = n \neq 0$. Moreover, e_n does not vanish on $\mathcal{V}_d$. Therefore f does not vanish on $\mathcal{V}_d$ and the triple (n,d,a) is good. ■

3.4 Triples and Prime Factors

Here we analyze the property of a triple (n,d,a) being good or bad in relation to certain prime factors of n, d, and a. We begin by developing some technical results.

Let $z_1, z_2, \dots, z_n$ be d-th roots of unity and consider the point $Q = (z_1, z_2, \dots, z_n) \in \mathbb{A}^n$. For an integer ν, we say that Q is *ν-symmetric* if, given a primitive ν-th root of unity ϵ, there exists $\tau \in \mathfrak{S}_n$ such that $(\epsilon z_1, \epsilon z_2, \dots, \epsilon z_n) = (z_{\tau(1)}, z_{\tau(2)}, \dots, z_{\tau(n)})$. In other words, Q is ν-symmetric if rotating each of the complex coordinates z_i by $2\pi/\nu$ radians produces a point in the $\mathfrak{S}_n$-orbit of Q. Note that $\nu \mid d$ because $1 = z_{\tau(1)}^d = \epsilon^d z_1^d = \epsilon^d$ and ϵ is primitive.

Lemma 3.17 *The point $Q \in \mathcal{V}_d \subset \mathbb{A}^n$ is ν-symmetric if and only if $\nu \mid n$ and $e_j(Q) = 0$, for all j such that $\nu \nmid j$.*

Proof First suppose that Q is ν-symmetric. The coordinates of Q split into orbits under the cyclic group of order ν acting on the complex plane by rotation. Since $Q \in \mathcal{V}_d$, we have $z_i \neq 0$ for all i. Therefore all of the above orbits have cardinality ν and $\nu \mid n$.

Since Q is p-symmetric, there is a primitive ν-th root of unity ϵ such that, up to reordering, we may write $z_{j\nu+i} = \epsilon^i \omega_j$ for $1 \leqslant i \leqslant \nu$, $1 \leqslant j \leqslant n/\nu$, and for some d-th roots of unity ω_j. Using the auxiliary variable y, we have

$$\sum_{j=1}^{n}(-1)^j e_j(Q)y^j = \prod_{i=1}^{n}(y-z_i) = \prod_{j=1}^{n/\nu}\prod_{i=1}^{\nu}(y-\epsilon^i\omega_j) = \prod_{j=1}^{n/\nu}\omega_j^\nu\prod_{i=1}^{\nu}(y/\omega_j-\epsilon^i)$$
$$= \prod_{j=1}^{n/\nu}\omega_j^\nu((y/\omega_j)^\nu - 1) = \prod_{j=1}^{n/\nu}(y^\nu - \omega_j^\nu).$$

Thus $e_j(Q) = 0$ whenever $\nu \nmid j$.

https://doi.org/10.4153/CJM-2017-035-3 Published online by Cambridge University Press

Conversely, suppose that $v \mid n$, $Q \in \mathcal{V}_d$, and $e_j(Q) = 0$ whenever $j \nmid v$. We have $\prod_{i=1}^{n}(y - z_i) = \sum_{j=1}^{n}(-1)^j e_j(Q) y^j = f(y^v)$, where f is a polynomial in one variable. At the same time $\prod_{i=1}^{n}(y - \epsilon z_i) = \epsilon^n \prod_{i=1}^{n}(y/\epsilon - z_i) = \epsilon^n f((y/\epsilon)^v) = f(y^v)$. Therefore, comparing factors, we deduce that Q is symmetric. ∎

Lemma 3.18 *Suppose $Q = (z_1, z_2, \ldots, z_n) \in \mathcal{V}_d$ is v^m-symmetric and*

$$(z_1^{v^m}, z_2^{v^m}, \ldots, z_n^{v^m})$$

is v-symmetric. Then Q is v^{m+1}-symmetric.

Proof Proceeding as in the proof of Lemma 3.17, Q being v^m-symmetric implies the existence of a primitive (v^m)-th root of unity ϵ such that, up to reordering, we may write $z_{jv^m+i} = \epsilon^i \omega_j$ for $1 \leqslant i \leqslant v^m$, $1 \leqslant j \leqslant n/v^m$, and for some d-th roots of unity ω_j. Using the auxiliary variable y, we have

$$\prod_{i=1}^{n}(y - z_i^{v^m}) = \prod_{j=1}^{n/v^m}\prod_{i=1}^{v^m}(y - \omega_j^{v^m}) = \prod_{j=1}^{n/v^m}(y - \omega_j^{v^m})^{v^m} = \Big(\prod_{j=1}^{n/v^m}(y - \omega_j^{v^m})\Big)^{v^m}. \tag{3.1}$$

Since $(z_1^{v^m}, z_2^{v^m}, \ldots, z_n^{v^m})$ is v-symmetric, Lemma 3.17 implies

$$\prod_{i=1}^{n}(y - z_i^{v^m}) = f(y^v), \tag{3.2}$$

for some polynomial f in one variable. The only way to reconcile equations (3.1) and (3.2) is if $\prod_{j=1}^{n/v^m}(y - \omega_j^{v^m}) = g(y^v)$, for some polynomial g in one variable. Therefore we must have

$$\begin{aligned}\prod_{i=1}^{n}(y - z_i) &= \prod_{j=1}^{n/v^m}\prod_{i=1}^{v^m}(y - z_{jv^m+i}) = \prod_{j=1}^{n/v^m}\prod_{i=1}^{v^m}(y - \epsilon^i \omega_j)\\ &= \prod_{j=1}^{n/v^m}(y^{v^m} - \omega_j^{v^m}) = g((y^{v^m})^v) = g(y^{v^{m+1}}).\end{aligned}$$

Using Lemma 3.17 again, we conclude that Q is v^{m+1}-symmetric. ∎

Proposition 3.19 *Let p be prime and suppose that all points $Q \in \mathcal{V}_d \subseteq \mathbb{A}^n$ with $\mathcal{P}_1(Q) = 0$ are p-symmetric. Let $g := \gcd(d, n)$ and assume $p \mid g$. Then (n, d, a) is bad if and only if $g \nmid a$.*

Proof If $g \nmid a$, then (n, d, a) is bad by Proposition 3.15.

We prove the other implication by contradiction. So suppose that $g \mid a$. Let $n = p^r n'$, $d = p^s d'$, and $a = p^t a'$, where $\gcd(p, n') = \gcd(p, d') = \gcd(p, a') = 1$. Set $k = \min\{r, s\}$. Since $p^k \mid g$, the condition $g \mid a$ implies $p^k \mid a$ and therefore $k \leqslant t$.

The hypothesis $p \mid g$ implies $s \geqslant 1$; hence $p \in \Gamma(d)$. At the same time, $p \mid g$ also implies $r \geqslant 1$; hence $n \in \Gamma(d)$. Thus, by Theorem 3.6, there exists $Q \in \mathcal{V}_d \subseteq \mathbb{A}^n$ such that $\mathcal{P}_1(Q) = 0$. By the hypothesis, Q is p-symmetric. However, Q is not p^{k+1}-symmetric because either $p^{k+1} \nmid n$ or $p^{k+1} \nmid d$. Therefore, there is an integer m, with $1 \leqslant m \leqslant k$, such that Q is p^m-symmetric, but not p^{m+1}-symmetric.

Now suppose that $\mathcal{P}_{p^m}(Q) = 0$. Then we would have

$$\mathcal{P}_1(z_1^{p^m}, z_2^{p^m}, \dots, z_n^{p^m}) = \mathcal{P}_{p^m}(Q) = 0.$$

Our hypothesis would imply that $(z_1^{p^m}, z_2^{p^m}, \dots, z_n^{p^m})$ is p-symmetric. However, Lemma 3.18 would give that Q is p^{m+1}-symmetric, contradicting our choice of m. Therefore $\mathcal{P}_{p^m}(Q) \neq 0$. Thus the homogeneous polynomial $(\mathcal{P}_{p^m})^{a'p^{t-m}}$ has degree a and does not vanish at Q. We conclude that (n, d, a) is good by Corollary 3.5. ∎

In [15], Lam and Leung consider sequences $z_1, z_2, \dots, z_n$ with each z_i a d-th root of unity and whose sum is 0, that is, points $Q = (z_1, z_2, \dots, z_n) \in \mathcal{V}_d$ such that $\mathcal{P}_1(Q) = 0$. They showed that if $d = p^r$ is a prime power, then Q must be p-symmetric [15, Corollary 3.4]. This yields the following corollary of Proposition 3.19.

Corollary 3.20 *Suppose $d = p^s$ for some prime p and positive integer s. Let $g := \gcd(d, n)$. Then (n, d, a) is bad if and only if $g \nmid a$.*

Proof If $p \mid g$, then the result follows from [15, Corollary 3.4] and Proposition 3.19.

Assume $p \nmid g$. In this case, $g = 1 \mid a$, so we must show that (n, d, a) is good. Note that $p \nmid g$ implies $p \nmid n$. Hence $n \notin \Gamma(d) = \langle p \rangle$. Therefore (n, d, a) is good by Proposition 3.9. ∎

Lam and Leung also showed that if $(z_1, z_2, \dots, z_n)$ is not p-symmetric for all primes p dividing d, then $n \geqslant p_1(p_2 - 1) + p_3 - p_2$, where $p_1 < p_2 < p_3$ are the three smallest primes dividing d [15, Theorem. 4.8]. This yields the following corollary of Proposition 3.19.

Corollary 3.21 *Suppose that at least two distinct primes divide d and that $n < p + q$, where p and q are the smallest two distinct primes dividing d. Let $g := \gcd(d, n)$. Then (n, d, a) is bad if and only if $g \nmid a$.*

Proof Let $d = p^s \prod_{i=1}^m q_i^{s_i}$ be the prime factorization of d, where $p < q_1 < q_2 < \cdots < q_m$. Suppose that $Q = (z_1, z_2, \dots, z_n) \in \mathcal{V}_d$ is such that $\mathcal{P}_1(Q) = 0$. Since

$$p(q_1 - 1) + q_2 - q_1 \geqslant 2(q_1 - 1) + q_2 - q_1 = (q_2 - 1) + q_1 > p + q_1 > n,$$

[15, Theorem. 4.8] implies that every non-empty minimal subset $I \subset \{1, 2, \dots, n\}$ such that $\sum_{i \in I} z_i = 0$ corresponds to a ν-symmetric point $(z_i : i \in I)$, where ν is a prime dividing d. Moreover, ν divides the cardinality of I. Clearly, we can partition $\{1, 2, \dots, n\}$ into a disjoint union $I_1 \sqcup I_2 \sqcup \cdots \sqcup I_t$ of such minimal subsets. Thus $n = \#I_1 + \#I_2 + \cdots + \#I_t$. Since the cardinality of each I_j is either p or some q_i, the hypothesis $n < p + q_1$ implies we must have either $t = 1$ and $n = \#I_1 = q_i$ for some i, or else $\#I_j = p$ for all j and $n = tp$.

Thus, there are two possibilities: either $n = q_i$ for some q_i or else $n = pt$. In the former case, $q_i \mid g$ and every $Q \in \mathcal{V}_d$ with $\mathcal{P}_1(Q) = 0$ is q_i-symmetric. In the latter case, $p \mid g$ and every $Q \in \mathcal{V}_d$ with $\mathcal{P}_1(Q) = 0$ is p-symmetric. Thus the hypotheses of Proposition 3.19 are satisfied (either with the prime q_i or with p). ∎

3.5 Generating Good and Bad Triples

We illustrate how to obtain more good and bad triples from the ones already at our disposal.

Proposition 3.22 *Let k be a positive integer.*

(i) *If (n,d,a) is bad, then (n,kd,a) is also bad.*
(ii) *If (n,d,a) is bad, then (kn,d,a) is also bad.*
(iii) *If (n,d,a) is bad, then (kn,kd,ka) is also bad.*

Proof Suppose that (n,d,a) is bad. By Corollary 3.5, there is a point

$$Q=(z_1,z_2,\dots,z_n)\in\mathcal{V}_d\subset\mathbb{A}^n$$

such that $f(Q)=0$ for all $f\in R_a^{\mathfrak{S}_n}$. Assertion (i) follows immediately since $\mathcal{V}_d\subset\mathcal{V}_{kd}$.

For the second assertion, choose a point $Q=(z_1,z_2,\dots,z_n)\in\mathcal{V}_d$. Define the point $Q'=(z'_1,z'_2,\dots,z'_{kn})\in\mathcal{V}_{kd}\subset\mathbb{A}^{kn}$ by $z'_{i+n(j-1)}:=z_i$ for $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant k$. Assume, by way of contradiction, that there exists $f'\in\mathbb{C}[x_1,x_2,\dots,x_{kn}]_a^{\mathfrak{S}_{kn}}$ such that $f'(Q')\neq 0$. The polynomials $\mathcal{P}_\lambda$ where λ is a partition of a whose parts do not exceed kn form a basis of $\mathbb{C}[x_1,x_2,\dots,x_{kn}]_a^{\mathfrak{S}_{kn}}$. Then $f'(Q')\neq 0$ implies that there exists a partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)\vdash a$ with $\mathcal{P}_\lambda(Q')\neq 0$. Hence $\mathcal{P}_{\lambda_t}(Q')\neq 0$ for all $t=1,2,\dots,r$. Since $\mathcal{P}_{\lambda_t}(Q')=kz_1^{\lambda_t}+kz_2^{\lambda_t}+\dots+kz_n^{\lambda_t}=k\mathcal{P}_{\lambda_t}(Q)$, we have $\mathcal{P}_{\lambda_t}(Q)\neq 0$ for all $t=1,2,\dots,r$, and therefore $\mathcal{P}_\lambda(Q)\neq 0$. Because $Q\in\mathcal{V}_d$ is arbitrary, Lemma 3.4 shows (n,d,a) is not bad. This contradicts the assumption, thus proving (ii).

Now we prove part (iii). By contradiction, assume (kn,kd,ka) is not bad. Given $Q\in\mathcal{V}_d$, we will construct $f\in R_a^{\mathfrak{S}_n}$ such that $f(Q)\neq 0$, which will prove (n,d,a) is not bad. Consider the primitive d-th root of unity $\zeta:=e^{2\pi i/d}$. We have $Q=(\zeta^{b_1},\zeta^{b_2},\dots,\zeta^{b_n})$ for some positive integers $b_1,b_2,\dots,b_n$. Let $\omega:=e^{2\pi i/(kd)}$; observe that ω is a (kd)-th root of unity and $\omega^k=\zeta$. Define the point $Q'=(z'_1,z'_2,\dots,z'_{kn})\in\mathcal{V}_{kd}\subset\mathbb{A}^{kn}$ by $z'_{k(j-1)+i}:=\omega^{b_j+id}$ for $1\leqslant i\leqslant k$ and $1\leqslant j\leqslant n$. Since we have assumed that (kn,kd,ka) is not bad, by Lemma 3.4, there exists a partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)\vdash ka$ such that $\mathcal{P}_\lambda(Q')\neq 0$. In particular, $\mathcal{P}_{\lambda_t}(Q')\neq 0$ for all $t=1,2,\dots,r$.

Using the auxiliary variable y, we can write

$$\begin{aligned}\prod_{t=1}^{kn}(y-z'_t)&=\prod_{j=1}^{n}\prod_{i=1}^{k}(y-\omega^{b_j+id})=\prod_{j=1}^{n}\prod_{i=1}^{k}\omega^{b_j}(y/\omega^{b_j}-\omega^{id})\\&=\prod_{j=1}^{n}\omega^{kb_j}\prod_{i=1}^{k}(y/\omega^{b_j}-(\omega^d)^i).\end{aligned}$$

Since ω^d is a primitive k-th root of unity, the k elements $(\omega^d)^1,(\omega^d)^2,\dots,(\omega^d)^k$ are all the k-th roots of unity. Therefore we get $\prod_{i=1}^{k}(y/\omega^{b_j}-(\omega^d)^i)=(y/\omega^{b_j})^k-1$. Combining the two previous equations, we obtain

$$\prod_{t=1}^{kn}(y-z'_t)=\prod_{j=1}^{n}\omega^{kb_j}[(y/\omega^{b_j})^k-1]=\prod_{j=1}^{n}(y^k-\zeta^{b_j}).$$

On the other hand, we have $\prod_{t=1}^{kn}(y - z'_t) = \sum_{j=0}^{kn}(-1)^j e_j(Q')y^{kn-j}$. By comparing these expressions, we deduce that $e_j(Q') = 0$ whenever $k \nmid j$. This implies that every homogeneous polynomial in $\mathbb{C}[x_1, x_2, \dots, x_{kn}]^{\mathfrak{S}_{kn}}$ whose degree is not divisible by k vanishes at Q'.

Thus the above integers $\lambda_1, \lambda_2, \dots, \lambda_r$ are all divisible by k, and we set $c_t := \lambda_t/k$ for all $i = 1, 2, \dots, r$. We have

$$\begin{aligned}\mathcal{P}_{\lambda_t}(Q') = \mathcal{P}_{kc_t}(Q') &= \sum_{s=1}^{kn}(z'_s)^{kc_t} = \sum_{j=1}^{n}\sum_{i=1}^{k}(\omega^{b_j+id})^{kc_t} = \sum_{i=1}^{k}\sum_{j=1}^{n}(\omega^k)^{(b_j+id)c_t}\\ &= \sum_{i=1}^{k}\sum_{j=1}^{n}\zeta^{(b_j+id)c_t} = \sum_{i=1}^{k}\sum_{j=1}^{n}(\zeta^{b_j})^{c_t} = \sum_{i=1}^{k}\mathcal{P}_{c_t}(Q) = k\mathcal{P}_{c_t}(Q).\end{aligned}$$

We deduce that $\mathcal{P}_{c_t}(Q) \neq 0$ for all $i = 1, 2, \dots, r$. Define $f := \prod_{t=1}^{r}\mathcal{P}_{\lambda_t/k} \in R$ and observe that f is an element of $R_a^{\mathfrak{S}_n}$ with $f(Q) \neq 0$. This concludes the proof. ∎

As the following example illustrates, (n, d, a) being bad does not imply that any one of (n, d, ka), (n, kd, ka), or (kn, d, ka) is bad.

Example 3.23 Consider $(n, d, a) = (8, 15, 4)$. Since $8 = 5 + 3 \in \Gamma(d) = \Gamma(15)$ and $4 \notin \Gamma(d)$, we see that $(8, 15, 4)$ is bad by Proposition 3.14.

Let $k = 2$. The triples $(n, d, ka) = (8, 15, 8)$ and $(n, kd, ka) = (8, 30, 8)$ are good because e_8 clearly does not vanish on $\mathcal{V}_{15}$ nor on $\mathcal{V}_{30}$.

Now consider the triple $(kn, d, ka) = (16, 15, 8)$. Observe that

$$S = \{q : q \mid 15, 16 \notin \Gamma(15/q)\} = \{3, 5, 15\},$$

hence the numerical semi-group $\langle 3, 5\rangle$ generated by S contains $ka = 8$. Therefore, $(16, 15, 8)$ is good by Proposition 3.11.

Remark 3.24 Consider the triple (n, d, a) and let $g := \gcd(n, d)$. By Proposition 3.15, (n, d, a) is bad if $g \nmid a$. Thus we suppose that $g \mid a$. By Proposition 3.22 (iii), if (n, d, a) is good, then $(n/g, d/g, a/g)$ is also good.

Proposition 3.25 *Let k be a positive integer.*

(i) *If (n, d, a) is good, then (n, d, ka) is also good.*
(ii) *If (n, d, a) is good, then (n, kd, ka) is also good.*

Proof Suppose that (n, d, a) is good. This implies that there exists $f \in R_a^{\mathfrak{S}_n}$ that does not vanish on $\mathcal{V}_d$. Assertion (i) now follows since $f^k \in R_{ka}^{\mathfrak{S}_n}$ also does not vanish on $\mathcal{V}_d$.

To prove (ii), define

$$f'(x_1, x_2, \dots, x_n) := f(x_1^k, x_2^k, \dots, x_n^k) \in R_{ka}^{\mathfrak{S}_n}.$$

For every point $Q' = (z_1, z_2, \dots, z_n) \in \mathcal{V}_{kd}$, the point $Q = (z_1^k, z_2^k, \dots, z_n^k)$ lies in $\mathcal{V}_d$; moreover, $f'(Q') = f(Q) \neq 0$. Thus (n, kd, ka) is good. ∎

As the following example illustrates, (n, d, a) being good does not imply that any one of (kn, d, a), (n, kd, a), (kn, kd, a), (kn, d, ka), or (kn, kd, ka) is good.

https://doi.org/10.4153/CJM-2017-035-3 Published online by Cambridge University Press

Example 3.26 Consider $(n,d,a)=(4,15,1)$. Since $n=4\notin\Gamma(d)=\Gamma(15)=\langle 3,5\rangle$, we see that $(4,15,1)$ is good by Proposition 3.11.

Now consider $k=2$ and $(kn,d,a)=(8,15,1)$. Since $8\in\Gamma(15)$ and $1\notin\Gamma(15)$, we see that $(8,15,1)$ is bad by Proposition 3.14. The triple $(n,kd,a)=(4,30,1)$ is bad for similar reasons. Then $(kn,kd,a)=(8,30,1)$ is bad as well by Proposition 3.22 (ii).

We claim the triple $(kn,d,ka)=(8,15,2)$ is also bad. Using the fact that $(8,15,1)$ is bad, we deduce that there exists $Q\in\mathcal{V}_{15}$ such that $\mathcal{P}_1(Q)=0$. Since 2 and 15 are coprime, Remark 3.8 implies $\mathcal{P}_2(Q)=0$. Given that $\mathcal{P}_2$ and $\mathcal{P}_{(1,1)}=\mathcal{P}_1^2$ form a basis of the symmetric polynomials of degree 2, their simultaneous vanishing at Q implies the claim by Lemma 3.4. Finally, the claim just proved, together with Proposition 3.22 (i), imply that $(kn,kd,ka)=(8,30,2)$ is bad.

4 Regular Sequences of Type $S^{(2,2)}\oplus S^{(4)}\oplus S^{(4)}$

Throughout this section we fix $n=4$, so $R=\mathbb{C}[x_1,x_2,x_3,x_4]$.

As proved in [10, Proposition 2.5], there exist regular sequences g_1,g_2,f_1,f_2 in R such that g_1,g_2 form a basis of a graded representation isomorphic to $S^{(2,2)}$ and f_1,f_2 are symmetric polynomials. If $I\subset R$ is the ideal generated by g_1,g_2,f_1,f_2, then $I/\mathfrak{m}I$ is isomorphic to $S^{(2,2)}\oplus S^{(4)}\oplus S^{(4)}$. Setting $a:=\deg(g_1)=\deg(g_2)$, $c:=\deg(f_1)$, and $d:=\deg(f_2)$, we seek the possible tuples (a,a,c,d) corresponding to regular sequences g_1,g_2,f_1,f_2 of type $S^{(2,2)}\oplus S^{(4)}\oplus S^{(4)}$.

4.1 Sequences in Low Degree

We recall some facts of invariant theory; more details can be found in [2, Chapter 3,4]. There is an isomorphism $R\cong R^{\mathfrak{S}_4}\otimes_{\mathbb{C}}R/(e_1,e_2,e_3,e_4)$ of graded $\mathfrak{S}_4$-representations. The symmetric group acts trivially on $R^{\mathfrak{S}_4}$. On the other hand, the coinvariant algebra $R/(e_1,e_2,e_3,e_4)$ is isomorphic to the regular representation of $\mathfrak{S}_4$. We worked out the graded character of $R/(e_1,e_2,e_3,e_4)$ in [10, Example 3.1]. In particular, $R/(e_1,e_2,e_3,e_4)$ contains two copies of the irreducible representation $S^{(2,2)}$, one in degree 2 and one in degree 4.

Let us find an explicit description of these two representations. Specht's original construction shows that the polynomials

$$(x_1-x_2)(x_3-x_4),\quad (x_1-x_3)(x_2-x_4) \tag{4.1}$$

span a copy of $S^{(2,2)}$ inside the degree 2 component of R [9, §7.4, Example 17]. Now observe that the polynomials

$$(x_1^2-x_2^2)(x_3^2-x_4^2),\quad (x_1^2-x_3^2)(x_2^2-x_4^2) \tag{4.2}$$

behave in the same way under the action of $\mathfrak{S}_4$. Therefore, they span a copy of $S^{(2,2)}$ inside the degree 4 component of R. Note also that the polynomials in (4.1) and (4.2) do not belong to the ideal (e_1,e_2,e_3,e_4). Therefore, their residue classes span the desired copies of $S^{(2,2)}$ inside $R/(e_1,e_2,e_3,e_4)$.

Using the isomorphism $R \cong R^{\mathfrak{S}_4} \otimes_{\mathbb{C}} R/(e_1, e_2, e_3, e_4)$ together with our construction above, we can establish the following fundamental fact: any copy of $S^{(2,2)}$ contained inside the degree a component of R is spanned by

$$
\begin{aligned}
g_1 &= h(x_1 - x_2)(x_3 - x_4) + h'(x_1^2 - x_2^2)(x_3^2 - x_4^2),\\
g_2 &= h(x_1 - x_3)(x_2 - x_4) + h'(x_1^2 - x_3^2)(x_2^2 - x_4^2)
\end{aligned}
\tag{4.3}
$$

for some symmetric polynomials h of degree $a-2$ and h' of degree $a-4$.

Thus, when searching for degree tuples (a, a, c, d) corresponding to regular sequences g_1, g_2, f_1, f_2 of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$, we can assume that g_1, g_2 have the form given in equation (4.3).

We consider the cases where $a \leqslant 4$ first. Clearly, we must have $a \geqslant 2$.

Proposition 4.1 *Let $a = 2$ or 4. A regular sequence of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$ with degree tuple (a, a, c, d) exists if and only if $cd \geqslant 2$. If $a = 3$, then no such sequence exists.*

Proof Let $a = 2$. We form polynomials g_1, g_2 as in equation (4.3). By degree considerations, h is a unit and $h' = 0$. Therefore, we may take

$$g_1 = (x_1 - x_2)(x_3 - x_4), \quad g_2 = (x_1 - x_3)(x_2 - x_4).$$

Now we need symmetric polynomials f_1, f_2 such that g_1, g_2, f_1, f_2 is a regular sequence. Note that f_1, f_2 cannot both be linear, otherwise they would be scalar multiples of e_1. However, if we assume that $d = \deg(f_2) \geqslant 2$, then we can write $d = 2p + 3q$, where p, q are non-negative integers, and set $f_1 := e_1^c$, $f_2 := e_2^p e_3^q$. The sequence g_1, g_2, f_1, f_2 is regular with degree tuple $(2, 2, c, d)$.

Now let $a = 4$. We need h' to be a unit. In fact, we can take $h' = 1$ and $h = e_2$; this gives

$$
\begin{aligned}
g_1 &= e_2(x_1 - x_2)(x_3 - x_4) + (x_1^2 - x_2^2)(x_3^2 - x_4^2),\\
g_2 &= e_2(x_1 - x_3)(x_2 - x_4) + (x_1^2 - x_3^2)(x_2^2 - x_4^2).
\end{aligned}
$$

Again f_1, f_2 cannot both be linear. In fact, choosing the same f_1, f_2 as before gives a regular sequence g_1, g_2, f_1, f_2 with degree tuple $(4, 4, c, d)$ for $d \geqslant 2$.

Finally let $a = 3$. In this case, $h' = 0$ while h is a scalar multiple of e_1. Thus g_1, g_2 have a common factor and do not form a regular sequence. ∎

4.2 Sequences With $a \geqslant 5$

Here we obtain general results about regular sequences g_1, g_2, f_1, f_2 of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$ with degree tuple (a, a, c, d) and $a \geqslant 5$. We still refer to the form of g_1, g_2 given in equation (4.3).

Lemma 4.2 *Let*

$$h_1 := h + h'(x_1 + x_2)(x_3 + x_4), \quad h_2 := h + h'(x_1 + x_3)(x_2 + x_4),$$

so that $g_1 = (x_1 - x_2)(x_3 - x_4)h_1$ and $g_2 = (x_1 - x_3)(x_2 - x_4)h_2$. The sequence g_1, g_2, f_1, f_2 is regular if and only if the sequences

https://doi.org/10.4153/CJM-2017-035-3 Published online by Cambridge University Press

(i) h, h', f_1, f_2,
(ii) $(x_1 - x_2)(x_3 - x_4), (x_1 - x_3)(x_2 - x_4), f_1, f_2$,
(iii) $(x_1 - x_2)(x_3 - x_4), h_2, f_1, f_2$

are regular.

Proof By Lemma 2.9, g_1, g_2, f_1, f_2 is regular if and only if

(1) h_1, h_2, f_1, f_2,
(2) $(x_1 - x_2)(x_3 - x_4), (x_1 - x_3)(x_2 - x_4), f_1, f_2$,
(3) $(x_1 - x_2)(x_3 - x_4), h_2, f_1, f_2$,
(4) $h_1, (x_1 - x_3)(x_2 - x_4), f_1, f_2$

are regular. Note that (2) and (3) are the same as (ii) and iii) above. Moreover, the transposition $(2\,3) \in \mathfrak{S}_4$ permutes (3) and (4), therefore it is enough to assume (3) is regular. Thus the statement of the lemma will follow if we can prove that (i), (ii), and (iii) are regular if and only if (1), (2), and (3) are regular.

Let us show that if (1) is regular, then (i) is regular. Since we have an equality of ideals $(h_1, h_2, f_1, f_2) = (h_1, h_2 - h_1, f_1, f_2)$ and (1) is regular, $h_1, h_2 - h_1, f_1, f_2$ is also regular. Notice that

$$h_2 - h_1 = h'(x_1 - x_4)(x_2 - x_3). \tag{4.4}$$

This implies that h_1, h', f_1, f_2 is regular. We deduce that (i) is regular, because of the equality $(h_1, h', f_1, f_2) = (h, h', f_1, f_2)$.

Now assume that (i) and (iii) are regular and let us prove that (1) is regular. Since (i) is regular, the equality $(h, h', f_1, f_2) = (h_1, h', f_1, f_2)$ implies that h_1, h', f_1, f_2 is regular. As previously observed, (iii) being regular implies (3) and (4) are regular. Note that $(3\,4)h_1 = h_1$. Therefore, applying $(3\,4)$ to (4), we obtain the regular sequence $h_1, (x_1 - x_4)(x_2 - x_3), f_1, f_2$. Since both h_1, h', f_1, f_2 and $h_1, (x_1 - x_4)(x_2 - x_3), f_1, f_2$ are regular, we can multiply their second elements to obtain a new regular sequence. By equation (4.4), this sequence is simply $h_1, h_2 - h_1, f_1, f_2$. Finally the ideal equality $(h_1, h_2 - h_1, f_1, f_2) = (h_1, h_2, f_1, f_2)$ allows us to conclude that (1) is regular. ∎

By Lemma 4.2, a necessary condition for g_1, g_2, f_1, f_2 to be a regular sequence of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$ and degrees (a, a, c, d) is that $(a - 2, a - 4, c, d)$ is a regular degree sequence. In fact, we will show this condition is also sufficient when $a \geqslant 5$.

Proposition 4.3 *Let $a \geqslant 5$. Suppose that $(a-2, a-4, c, d)$ is a regular degree sequence for $n = 4$. Then there exists a regular sequence of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$ and degrees (a, a, c, d).*

Proof First we suppose that a is even. Then both $a - 2$ and $a - 4$ are even, and exactly one of them is divisible by 4. Also (2.2) implies that $3 \mid (a-2)(a-4)cd$. Since $c = d = 1$ is impossible by condition (2.2), we can assume without loss of generality that $c \geqslant 2$; in particular, we can write $c = 2p + 3q$ for some non-negative integers p, q.

We claim that the sequence $(a - 2, a - 4, c, d)$ corresponds to (at least) one of the rows of Table 2. If $3 \mid cd$, then without loss of generality, $3 \mid c$ and $(a - 2, a - 4, c, d)$ corresponds to either row 1 or row 2 according to whether $4 \mid a - 4$ or $4 \mid a - 2$.

Row	Degrees				Symmetric Polynomials			
	$a-2$	$a-4$	c	d	h	h'	f_1	f_2
1	$4\alpha+2$	4α	3γ	d	$e_2^{2\alpha+1}$	e_4^{α}	e_3^{γ}	e_1^d
2	4α	$4\alpha-2$	3γ	d	e_4^{α}	$e_2^{2\alpha-1}$	e_3^{γ}	e_1^d
3	$12\alpha+2$	12α	$2p+3q$	d	$(e_2^3+e_3^2)e_4^{3\alpha-1}$	$(e_2^6+e_3^4+e_4^3)^{\alpha}$	$e_2^p e_3^q$	e_1^d
4	$6\alpha+2,\ (2\nmid\alpha)$	6α	$2p+3q$	d	$e_4^{(3\alpha+1)/2}$	$(e_2^3+e_3^2)^{\alpha}$	$e_2^p e_3^q$	e_1^d
5	$6\alpha,\ (2\nmid\alpha)$	$6\alpha-2$	$2p+3q$	d	$(e_2^3+e_3^2)^{\alpha}$	$e_4^{(3\alpha-1)/2}$	$e_2^p e_3^q$	e_1^d
6	12α	$12\alpha-2$	$2p+3q$	d	$(e_2^6+e_3^4+e_4^3)^{\alpha}$	$(e_2^3+e_3^2)e_4^{3\alpha-2}$	$e_2^p e_3^q$	e_1^d

Table 2: a even

If $3 \mid a-4$, then $(a-2, a-4, c, d)$ corresponds to either row 3 or row 4 according to whether $4 \mid a-4$ or $4 \mid a-2$.

If $3 \mid a-2$, then $(a-2, a-4, c, d)$ corresponds to either row 5 or row 6 according to whether $4 \mid a-4$ or $4 \mid a-2$.

In each case, Table 2 contains possible choices of polynomials h, h', f_1, f_2. One can easily verify that in each case the given choices make h, h', f_1, f_2 a regular sequence (see Remark 4.4). The polynomials g_1, g_2 are obtained using equation (4.3). Using Lemma 4.2, we conclude that in each case, g_1, g_2, f_1, f_2 is a regular sequence of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$.

Next suppose that a is odd. Then $a-2$ and $a-4$ are also odd. By (2.2), without loss of generality, we may assume that $2 \mid c$ and $4 \mid d$. Also (2.2) implies that $3 \mid (a-2)(a-4)cd$. We claim that that the sequence $(a-2, a-4, c, d)$ corresponds to (at least) one of the rows of Table 3. If $3 \mid a-2$, then $(a-2, a-4, c, d)$ corresponds to row 1. If $3 \mid a-4$, then $(a-2, a-4, c, d)$ corresponds to row 2. If $3 \mid c$, then

Row	Degrees				Symmetric Polynomials			
	$a-2$	$a-4$	c	d	h	h'	f_1	f_2
1	3α	$3\alpha-2$	2γ	4δ	e_3^{α}	$e_2^{(3\alpha-3)/2}e_1$	$(e_1^2+e_2)^{\gamma}$	e_4^{δ}
2	$3\alpha+2$	3α	2γ	4δ	$e_2^{(3\alpha-1)/2}e_1$	e_3^{α}	$(e_1^2+e_2)^{\gamma}$	e_4^{δ}
3	$4\alpha+1$	$4\alpha-1$	6γ	4δ	$e_4^{\alpha}e_1$	$e_2^{2\alpha-2}e_3$	$(e_2^3+e_3^2)^{\gamma}$	$(e_1^4+e_4)^{\delta}$
4	$4\alpha-1$	$4\alpha-3$	6γ	4δ	$e_2^{2\alpha-2}e_3$	$e_4^{\alpha-1}e_1$	$(e_2^3+e_3^2)^{\gamma}$	$(e_1^4+e_4)^{\delta}$
5	$4\alpha+1$	$4\alpha-1$	2γ	12δ	$e_2^{2\alpha}e_1$	$e_4^{\alpha-1}e_3$	$(e_1^2+e_2)^{\gamma}$	$(e_3^4+e_4^3)^{\delta}$
6	$4\alpha-1$	$4\alpha-3$	2γ	12δ	$e_4^{\alpha-1}e_3$	$e_2^{2\alpha-2}e_1$	$(e_1^2+e_2)^{\gamma}$	$(e_3^4+e_4^3)^{\delta}$

Table 3: a odd

$(a-2, a-4, c, d)$ corresponds to either row 3 or row 4 according to whether $a \equiv 3 \pmod 4$ or $a \equiv 1 \pmod 4$. If $3 \mid d$, then $(a-2, a-4, c, d)$ corresponds to either row 5 or row 6 according to whether $a \equiv 3 \pmod 4$ or $a \equiv 1 \pmod 4$.

In each case, Table 3 lists a possible choice for the polynomials h, h', f_1, f_2. As above one can easily verify that the given choices make h, h', f_1, f_2 a regular sequence. Again the polynomials g_1, g_2 are obtained using equation (4.3). Using Lemma 4.2, we conclude that, in each case, g_1, g_2, f_1, f_2 is a regular sequence of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$. ∎

Remark 4.4 For each line in Table 2 and Table 3, one can prove that the polynomials h, h', f_1, f_2 form a regular sequence using Lemma 2.9 and [8, Corollary 17.8 a]. As an example, we show that the polynomials in row 3 of Table 2, specifically $(e_2^3 + e_3^2)e_4^{3\alpha-1}$, $(e_2^6 + e_3^4 + e_4^3)^\alpha$, $e_2^p e_3^q$, and e_1^d, form a regular sequence.

By Lemma 2.9 and [8, Corollary 17.8 a], it is enough to show that the sequences

$$
\begin{aligned}
&e_2^3 + e_3^2, e_2^6 + e_3^4 + e_4^3, e_2, e_1, && e_2^3 + e_3^2, e_2^6 + e_3^4 + e_4^3, e_3, e_1,\\
&e_4, e_2^6 + e_3^4 + e_4^3, e_2, e_1, && e_4, e_2^6 + e_3^4 + e_4^3, e_3, e_1,
\end{aligned}
$$

are regular.

Let us show the first sequence is regular. The ideal it generates is equal to e_3^2, $e_3^4 + e_4^3, e_2, e_1$, therefore it suffices to show that these generators form a regular sequence. Using [8, Corollary 17.8 a] again, it is enough to prove that $e_3, e_3^4 + e_4^3, e_2, e_1$ is regular. Because of the ideal equality $(e_3, e_3^4 + e_4^3, e_2, e_1) = (e_3, e_4^3, e_2, e_1)$, we only need to prove that e_3, e_4^3, e_2, e_1 is regular. This follows immediately from [8, Corollary 17.8 a] and the fact that the elementary symmetric polynomials form a regular sequence.

The other sequences are handled similarly.

In summary, we have the following result.

Theorem 4.5 *There exists a regular sequence of type $S^{(2,2)} \oplus S^{(4)} \oplus S^{(4)}$ and degrees (a, a, c, d) if and only if*

(i) *$a = 2$ or 4 and $(c, d) \neq (1, 1)$, or*
(ii) *$a \geqslant 5$ and $(a-2, a-4, c, d)$ is a regular degree sequence.*

A Macaulay2 code

We present here the Macaulay2 code used to produce the example in Remark 3.3.

```
needsPackage "Depth"
R=QQ[x_1..x_5]
e=apply(5,i->sum(apply(subsets(gens R,i+1),product)))
l=apply(4,i->x_(i+1)^6-x_5^6)
g=apply(4,i->sum(apply(4,j->e_(j+1)*(x_(i+1)^(4-j)-x_5^(4-j)))))
isRegularSequence(l|{e_0})
isRegularSequence(g|{e_0})
```

Acknowledgements As we were working on the research for this article, A. V. (Tony) Geramita became ill and was confined to hospital in Vancouver, BC in late December 2015. During this period we continued working, with Tony contributing comments and suggestions by email as his health permitted. The proofs of the results were mostly completed by March 2016 and Tony sent his last comments on April 26. Unfortunately, Tony passed away in Kingston on June 22, 2016. He was a wonderful mathematician, colleague, and friend whom we miss greatly.

References

[1] Aart Blokhuis, Andries E. Brouwer, and Tamás Szőnyi, *Proof of a conjecture by Đoković on the Poincaré series of the invariants of a binary form*. Indag. Math. (N.S.) **24**(2013), no. 4, 766–773. http://dx.doi.org/10.1016/j.indag.2012.12.004
[2] Michel Broué, *Introduction to complex reflection groups and their braid groups*. Lecture Notes in Mathematics, 1988. Springer-Verlag, Berlin, 2010. http://dx.doi.org/10.1007/978-3-642-11175-4
[3] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*. Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993.
[4] Ri-Xiang Chen, *On two classes of regular sequences*. J. Commut. Algebra **8**(2016), no. 1, 29–42. http://dx.doi.org/10.1216/JCA-2016-8-1-29
[5] Aldo Conca, Christian Krattenthaler, and Junzo Watanabe, *Regular sequences of symmetric polynomials*. Rend. Semin. Mat. Univ. Padova **121**(2009), 179–199. http://dx.doi.org/10.4171/RSMUP/121-11
[6] Jacques Dixmier, *Quelques résultats et conjectures concernant les séries de Poincaré des invariants des formes binaires*. In: Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin, 36ème année (Paris, 1983–1984). Lecture Notes in Mathematics, 1146. Springer, Berlin, 1985, pp. 127–160. http://dx.doi.org/10.1007/BFb0074537
[7] Dragomir Ž. Đoković, *A heuristic algorithm for computing the Poincaré series of the invariants of binary forms*. Int. J. Contemp. Math. Sci. **1**(2006), 557–568. http://dx.doi.org/10.12988/ijcms.2006.06059
[8] David Eisenbud, *Commutative algebra*. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995. http://dx.doi.org/10.1007/978-1-4612-5350-1
[9] William Fulton, *Young tableaux*. London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.
[10] Federico Galetto, Anthony V. Geramita, and David L. Wehlau, *Symmetric complete intersections*. Comm. Algebra **46**(2018), 2194–2204. http://dx.doi.org/10.1080/00927872.2017.1372453
[11] Roe Goodman and Nolan R. Wallach, *Symmetry, representations, and invariants*. Graduate Texts in Mathematics, 255. Springer, Dordrecht, 2009. http://dx.doi.org/10.1007/978-0-387-79852-3
[12] Andrew Granville, *Arithmetic properties of binomial coefficients. I. Binomial coefficients modulo prime powers*. In: Organic mathematics. CMS Conf. Proc., 20. Amer. Math. Soc., Providence, RI, 1997, pp. 253–276.
[13] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*. http://www.math.uiuc.edu/Macaulay2/.
[14] Neeraj Kumar and Ivan Martino, *Regular sequences of power sums and complete symmetric polynomials*. Matematiche (Catania), **67**(2012), no. 1, 103–117.
[15] Tsit Yuen Lam and Ka Hin Leung, *On vanishing sums of roots of unity*. J. Algebra **224**(2000), no. 1, 91–109. http://dx.doi.org/10.1006/jabr.1999.8089
[16] Serge Lang, *Algebra*. Third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002. http://dx.doi.org/10.1007/978-1-4613-0041-0
[17] Peter Littelmann and Claudio Procesi, *On the Poincaré series of the invariants of binary forms*. J. Algebra **133**(1990), no. 2, 490–499. http://dx.doi.org/10.1016/0021-8693(90)90284-U
[18] Jorge Luis Ramírez Alfonsín, *The Diophantine Frobenius problem*. Oxford Lecture Series in Mathematics and its Applications, 30. Oxford University Press, Oxford, 2005. http://dx.doi.org/10.1093/acprof:oso/9780198568209.001.0001
[19] Bruce E. Sagan, *The symmetric group*. Second edition. Graduate Texts in Mathematics, 203. Springer-Verlag, New York, 2001. http://dx.doi.org/10.1007/978-1-4757-6804-6
[20] Richard P. Stanley, *Enumerative combinatorics*. Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, Cambridge, 1999. http://dx.doi.org/10.1017/CBO9780511609589

Department of Mathematics and Statistics, McMaster University, Hamilton ON L8S 4K1
e-mail: galettof@math.mcmaster.ca

Department of Mathematics, Queen's University, Kingston ON K7L 3N6
e-mail: tony@mast.queensu.ca

Department of Mathematics and Computer Science, Royal Military College, Kingston ON K7K 7B4
e-mail: wehlau@rmc.ca

https://doi.org/10.4153/CJM-2017-035-3 Published online by Cambridge University Press