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# Mahler measure of polynomial iterates

Igor Pritsker

*Abstract.* Granville recently asked how the Mahler measure behaves in the context of polynomial dynamics. For a polynomial $f(z) = z^d + \cdots \in \mathbb{C}[z]$, $\deg(f) \geq 2$, we show that the Mahler measure of the iterates $f^n$ grows geometrically fast with the degree $d^n$, and find the exact base of that exponential growth. This base is expressed via an integral of $\log^+ |z|$ with respect to the invariant measure of the Julia set for the polynomial $f$. Moreover, we give sharp estimates for such an integral when the Julia set is connected.

## 1 Main results

For an arbitrary polynomial $P(z) = c_n \prod_{k=1}^n (z - z_k) \in \mathbb{C}[z]$ with $c_n \neq 0$, the Mahler measure is given by

$$M(P) := \exp\left(\frac{1}{2\pi}\int \log|P(e^{i\theta})|\, d\theta\right) = |c_n| \prod_{k=1}^n \max(1, |z_k|), \tag{1.1}$$

where the second equality is a well-known consequence of Jensen's formula (see [2, 7, 11] for background and applications).

Let $f(z) = z^d + \cdots \in \mathbb{C}[z]$, $\deg(f) \geq 2$, and consider the $n$-fold iterates for $f$ denoted by $f^n$, which are monic polynomials of degree $d^n$, $n \in \mathbb{N}$. At a recent conference [9], Granville asked interesting questions on the behavior of the Mahler measure under composition of polynomials. In particular, how the Mahler measure of the polynomial iterates $f^n$ behaves as $n \to \infty$. Our primary goal is to show that the Mahler measure of $f^n$ grows geometrically fast with the degree $d^n$. In order to state a precise result, we need to introduce the Julia set of $f$ denoted by $J$, which is a completely invariant compact set under iteration of $f$ (see, e.g., [4] for details). It is also known that there is a unique unit Borel measure $\mu_J$ supported on $J$ that is invariant under $f$. In fact, $\mu_J$ is the equilibrium measure of $J$ in the sense of logarithmic potential theory (see [4, 13]), and it expresses the steady-state distribution of charge if $J$ is viewed as conductor.

***Theorem 1.1*** *If* $f(z) = z^d + \cdots \in \mathbb{C}[z]$, $\deg(f) \geq 2$, *is different from the monomial* $z^d$, *then we have*

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$$\lim_{n\to\infty} d^{-n} \log M(f^n) = \int \log^+ |z| d\mu_J(z) > 0, \tag{1.2}$$

*where $\mu_J$ is the invariant (equilibrium) measure of the Julia set $J$ for $f$.*

***Remark 1.2*** If $f(z) = z^d$, then $f^n(z) = z^{d^n}$, $n \in \mathbb{N}$, and $M(f^n) = 1$, $n \in \mathbb{N}$, by (1.1). Also note that the smallest value of $\int \log^+ |z| d\mu_J(z)$ is 0 that is attained for $f(z) = z^d$ with $J = \mathbb{T} := \{|z| = 1\}$ and $d\mu_{\mathbb{T}}(e^{it}) = dt/(2\pi)$, $t \in [0, 2\pi)$.

In light of (1.2), we arrive at the question: How large can $\int \log^+ |z| d\mu_J(z)$ be? Since the location of the Julia set $J$ varies with $f$ in such a way that $J$ can be essentially anywhere in the complex plane, the value of this integral can be arbitrarily large with the values of $\log^+ |z|$. Indeed, if $J \subset \{z : |z| > R\}$, then $\int \log^+ |z| d\mu_J(z) \ge \log R$ because $\mu_J$ is the unit measure, where $R > 1$ can be arbitrarily large. However, if we make proper normalization assumptions, then we obtain some precise upper bounds stated below.

Let $K$ be the filled-in Julia set that consists of the Julia set $J$ and the union of the bounded components of its complement $\mathbb{C} \setminus J$ (see [4, p. 65]). It is clear that $J = \partial K$, so that $K$ is connected if and only if $J$ is connected, which is known to hold if and only if all the critical points of $f$ are contained in $K$ (see [4, p. 66]). Moreover, $J$ and $K$ share the same equilibrium measure $\mu_J = \mu_K$ (cf. [3, 13]).

***Theorem 1.3*** *If $f(z) = z^d + \cdots \in \mathbb{C}[z]$, $\deg(f) \ge 2$, $J$ is connected, and $0 \in K$, then*

$$\int \log^+ |z| d\mu_J(z) \le \int_1^4 \frac{\log t \, dt}{\pi\sqrt{t(4-t)}} \approx 0.6461318945. \tag{1.3}$$

*Equality holds above for $J = K = [0, 4]$ and $f(z) = 2\,T_d(z/2 - 1)$, where $T_d(z) = \cos(d \arccos z)$ is the classical Chebyshev polynomial.*

Symmetry assumptions also produce interesting results such as the one below.

***Theorem 1.4*** *If $f(z) = z^d + \cdots \in \mathbb{C}[z]$, $\deg(f) \ge 2$, is either an odd or an even function, and $J$ is connected, then*

$$\int \log^+ |z| d\mu_J(z) \le 2 \int_1^2 \frac{\log t \, dt}{\pi\sqrt{1-t^2}} \approx 0.3230659472. \tag{1.4}$$

*Equality holds above for $J = [-2, 2]$ and $f(z) = 2\,T_d(z/2)$, where $T_d(z) = \cos(d \arccos z)$.*

A classical example that satisfies the assumptions of Theorem 1.4 is given by the family of quadratic polynomials $f(z) = z^2 + c$ with $c$ from the Mandelbrot set (see Chapter VIII of [4]).

We remark that the growth of the Mahler measure for the iterates exhibited here is essentially due to the intrinsic connection of the Mahler measure to the unit circle. A more suitable version of the Mahler measure for the dynamical setting is known (see the recent papers [5, 6], where the first one surveys many developments in the area). Another related notion is dynamical (or canonical) height (see [14] for a comprehensive exposition). There are many other connections of the Mahler measure and its generalizations with polynomial dynamics. Thus, the integral of (1.2) can be interpreted as the Arakelov–Zhang pairing of $f$ and $z^2$ that arises as a limit of average Weil heights in [12]. It is practically impossible to discuss all these interesting relations in detail in this short note.

For the proofs of Theorems 1.1, 1.3, and 1.4, we need the well-known result of Brolin [3, Theorem 16.1] on the equidistribution of preimages for the iterates $f^n$:

**Brolin's Theorem.** *Let $w \in \mathbb{C}$ be any point with one possible exception. Consider the preimages of w under $f^n$ denoted by $\{z_{k,n}\}_{k=1}^{d^n}$, i.e., all solutions of the equation $f^n(z) = w$ listed according to multiplicities. Define the normalized counting measures in those preimages by*

$$\tau_n := \frac{1}{d^n}\sum_{k=1}^{d^n}\delta_{z_{k,n}}, \tag{1.5}$$

*where $\delta_z$ denotes a unit point mass at $z$. Then we have the following weak* convergence:*

$$\tau_n \xrightarrow{*} \mu_J \quad \text{as } n \to \infty. \tag{1.6}$$

Brolin's result has the following implication, which is crucial for our purposes.

***Corollary 1.5.*** *If $f(z) = z^d + \cdots \in \mathbb{C}[z]$, $\deg(f) \geq 2$, is not the monomial $z^d$, then we have for the zeros of $f^n$ denoted by $\{z_{k,n}\}_{k=1}^{d^n}$ that*

$$\tau_n = \frac{1}{d^n}\sum_{k=1}^{d^n}\delta_{z_{k,n}} \xrightarrow{*} \mu_J \quad \text{as } n \to \infty. \tag{1.7}$$

**Proof** The exceptional points in Brolin's Theorem arise as values omitted by the family of iterates $\{f^n\}_{n=1}^{\infty}$ in a neighborhood of any point $\zeta \in J$. It follows that there are at most two such omitted values by Montel's theorem on normal families, for otherwise the family $\{f^n\}_{n=1}^{\infty}$ would be normal in that neighborhood, which contradicts the definition of the Julia set $J$ for $f$. Moreover, Lemma 2.2 of [3] states that the exceptional values are the same for all points $\zeta \in J$. Since $f$ is a polynomial in our settings, it certainly omits the value $\infty$ in every disk $\{z : |z - \zeta| < r\}$, where $r > 0$, $\zeta \in J$, so that at most one exceptional value can occur in this case. For example, if $f(z) = z^d$, then this exceptional value is 0 in every disk $\{z : |z - \zeta| < 1\}$, where $\zeta \in J = \mathbb{T}$ the unit circumference. However, 0 cannot be an exceptional value for any polynomial in Theorem 1.1. Indeed, since $\deg(f) \geq 2$ and $f$ is not the monomial $z^d$, there is a root $w_0 \neq 0$ of $f$. If we assume that 0 is an exceptional point for Brolin's Theorem, equivalently an omitted value for the family $\{f^n\}_{n=1}^{\infty}$ in a neighborhood $V$ of a point $\zeta \in J$, then the same must be true for $w_0$ because $f^n(z_0) = w_0$ for a point $z_0 \in V$ implies $f^{n+1}(z_0) = 0$. But two finite omitted values $0, w_0$ mean that the family $\{f^n\}_{n=1}^{\infty}$ must be normal in $V$, contradicting the definition of the Julia set $J$. Thus, 0 is not an exceptional point, and Corollary 1.5 is an immediate consequence of Brolin's Theorem. ∎

## 2 Proofs of the main results

We continue with the same notations as before.

**Proof of Theorem 1.1** It is clear from (1.1) that

$$d^{-n}\log M(f^n) = \frac{1}{d^n}\sum_{k=1}^{d^n}\log^+|z_{k,n}| = \int \log^+|z|\,d\tau_n(z).$$

Since $\log^+|z|$ is a continuous function in $\mathbb{C}$, the limit relation in (1.2) follows from the weak* convergence of (1.7). One only needs to observe here that the sets $\{z_{k,n}\}_{k=1}^{d^n}$ are uniformly bounded for all $n\in\mathbb{N}$, say belong to a fixed disk $D_R=\{z:|z|\le R\}$, so that $\log^+|z|$ can be extended from $D_R$ to $\mathbb{C}\setminus D_R$ as a continuous function with compact support in $\mathbb{C}$.

The inequality in (1.2) follows from the work of Fernández [8], who showed that the Julia set $J$ of $f$ different from $z^d$ must have points in the domain $\Delta=\{z:|z|>1\}$. It is well known that supp $\mu_J = J$ (see [3, Lemma 15.2] and [13, pp. 195–197]). Thus,

$$\int \log^+|z|d\mu_J(z) = \int_\Delta \log|z|d\mu_J(z) > 0. \qquad \blacksquare$$

**Proof of Theorem 1.3** Recall that the logarithmic capacity of the Julia set for a monic polynomial is equal to 1 (see Lemma 15.1 of [3] and Theorem 6.5.1 of [13] for a detailed proof). The book [13] contains a complete account on logarithmic potential theory, and on capacity in particular. Since $J=\partial K$, the equilibrium measure of $K$ is $\mu_K=\mu_J$, and the capacity of $K$ is 1 (cf. [13]). Clearly, $K$ is a connected set because $J$ is so. The conditions that the capacity of $K$ is 1, $0\in K$ and $K$ is connected introduce restrictions on the size of $K$ and, consequently, on the size of the integral $\int\log^+|z|d\mu_J(z)$ in (1.2). Theorem 6.2 of [1] (see also Corollary 6 of [10]) gives that the largest value of this integral is attained when $K=[0,4]=J$, in which case it is well known [13] that

$$d\mu_K(x) = d\mu_J(x) = \frac{dx}{\pi\sqrt{x(4-x)}}, \qquad x\in(0,4).$$

To apply Theorem 6.2 of [1], we also need to note that $\log^+|z| = \max(0,\log|z|)$ is clearly a convex function of $\log|z|$. Thus, we have the upper bound (1.3)

$$\int \log^+|z|d\mu_J(z) \le \int_1^4 \frac{\log t\,dt}{\pi\sqrt{t(4-t)}} \approx 0.6461318945.$$

The case of equality for $J=[0,4]$ is attained by the polynomial $f(z) = 2\,T_d(z/2-1)$, where $T_d(z)=\cos(d\arccos z)$ is the classical Chebyshev polynomial of the first kind (see Sections 1.6.2 and 6.2 of [14] for details). ■

**Proof of Theorem 1.4** We proceed with a proof similar to the previous one, but use Corollary 6.3 of [1] instead of Theorem 6.2 of [1]. We have that capacity of $J$ is 1 by Theorem 6.5.1 of [13], and $J$ is connected by our assumption. Corollary 6.3 of [1] is applied to the filled-in Julia set $K$, so that $J=\partial K$, where the equilibrium measure of $K$ is $\mu_K=\mu_J$, and the capacity of $K$ is 1. Again, $K$ is connected because $J$ is so. Moreover,

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both $J$ and $K$ are symmetric with respect to the origin because $f$ is even or odd. If $f$ is odd, then 0 is a fixed point of $f$, implying that $0 \in K$. If $f$ is even, then 0 is a critical point of $f$; hence, $0 \in K$ because we assume that $J$ is connected (cf. [4, p. 66]). Thus, $0 \in K$ under our assumptions, and we obtain from Corollary 6.3 of [1] that the largest value of the integral in (1.4) is attained for $J = K = [-2, 2]$ :

$$\int \log^+ |z| d\mu_J(z) = \int \log^+ |z| d\mu_K(z) \le 2 \int_1^2 \frac{\log t\, dt}{\pi\sqrt{1-t^2}} \approx 0.3230659472,$$

where we used that the equilibrium measure for $J = K = [-2, 2]$ is the Chebyshev distribution [13]

$$d\mu_K(x) = d\mu_J(x) = \frac{dx}{\pi\sqrt{4-x^2}}, \quad x \in (-2, 2).$$

It is well known that $J = [-2, 2]$ for $f(z) = 2\,T_d(z/2)$, where $T_d(z) = \cos(d \arccos z)$ (see Sections 1.6.2 and 6.2 of [14]). ■

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*Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA*
*e-mail*: igor@math.okstate.edu

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