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Zero entropy and stable rotation sets for monotone recurrence relations

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Abstract. In this paper, we show that each element in the convex hull of the rotation set of a compact invariant chain transitive set is realized by a Birkhoff solution, which is an improvement of the fundamental lemma of T. Zhou and W.-X. Qin [Pseudo solutions, rotation sets, and shadowing rotations for monotone recurrence relations. *Math. Z.* **297** (2021), 1673–1692] in the study of rotation sets for monotone recurrence relations. We then investigate the properties of rotation sets assuming the system has zero topological entropy. The rotation set for a Birkhoff recurrence class is a singleton and the forward and backward rotation numbers are identical for each solution in the same Birkhoff recurrence class. We also show the continuity of rotation numbers on the set of non-wandering points. If the rotation set is upper-stable, then we show that each boundary point is a rational number, and we also obtain a result of bounded deviation.

Key words: rotation set, topological entropy, chain transitivity, monotone recurrence relation
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1. *Introduction*

We continue the investigation of rotation sets for monotone recurrence relations in [29]. The solutions of a monotone recurrence relation correspond to orbits of a monotone twist map on the high-dimensional cylinder, a generalization of the classical monotone twist map on the annulus.

Let $k \geq 1$, $l \geq 1$ be integers, and $\Delta : \mathbb{R}^{k+l+1} \to \mathbb{R}$ be continuous. Consider solutions $\mathbf{x} = (x_n) \in \mathbb{R}^{\mathbb{Z}}$ of

$$\Delta(x_{n-k}, \ldots, x_n, \ldots, x_{n+l}) = 0 \quad \text{for all } n \in \mathbb{Z}. \tag{1.1}$$

We always assume in this paper that:

(1) $\Delta(x_{-k}, \ldots, x_0, \ldots, x_l)$ is a non-decreasing function of all the x_j except x_0. Moreover, it is strictly increasing in x_{-k} and x_l;

(2) $\Delta(x_{-k}+1, \ldots, x_l+1) = \Delta(x_{-k}, \ldots, x_l)$;

(3) $\lim_{x_{-k}\to\pm\infty} \Delta(x_{-k}, \ldots, x_l) = \pm\infty$ and $\lim_{x_l\to\pm\infty} \Delta(x_{-k}, \ldots, x_l) = \pm\infty$.

According to assumptions (1) and (3), we can solve equation (1.1) for x_{n+l} if $(x_{n-k}, \ldots, x_{n+l-1})$ is given. Thus we define a continuous map F_Δ from $\mathbb{R}^{k+l}$ to $\mathbb{R}^{k+l}$ by

$$F_\Delta(x_{n-k}, \ldots, x_{n+l-1}) = (x_{n-k+1}, \ldots, x_{n+l}).$$

The map F_Δ is a homeomorphism of $\mathbb{R}^{k+l}$ onto itself. Taking into account the periodicity assumption (2), we define on the high-dimensional cylinder $S^1 \times \mathbb{R}^{k+l-1}$ a homeomorphism φ_Δ which is a generalization of the class of monotone twist maps of the annulus or two-dimensional cylinder [3].

We say that a configuration $\mathbf{x} = (x_n) \in \mathbb{R}^\mathbb{Z}$ has bounded action if there is a constant $L > 0$ such that $|x_{n+1} - x_n| \le L$ for $n \in \mathbb{Z}$. Define the forward and backward rotation intervals of $\mathbf{x}$ to be

$$\rho(\mathbf{x}) = \left[\liminf_{n\to+\infty} \frac{x_n}{n}, \limsup_{n\to+\infty} \frac{x_n}{n}\right] \quad \text{and} \quad \rho^*(\mathbf{x}) = \left[\liminf_{n\to-\infty} \frac{x_n}{n}, \limsup_{n\to-\infty} \frac{x_n}{n}\right],$$

respectively. If $\rho(\mathbf{x})(\rho^*(\mathbf{x}))$ is a single point, that is, the limit $\lim_{n\to+\infty} x_n/n(\lim_{n\to-\infty} x_n/n)$ exists, we say that $\mathbf{x}$ has a forward (backward) rotation number. If $\rho(\mathbf{x}) = \rho^*(\mathbf{x})$ is a singleton, then we say $\mathbf{x}$ has a rotation number.

We define $\rho(\Delta)$, the union of $\rho(\mathbf{x})$, where $\mathbf{x} = (x_n)$ is a solution of equation (1.1) with bounded action, as the rotation set of equation (1.1).

We also consider the rotation sets of solutions with bounded action of

$$\Delta(x_{n-k}, \ldots, x_n, \ldots, x_{n+l}) = F \quad \text{for all } n \in \mathbb{Z}, \tag{1.2}$$

in which $F \in \mathbb{R}$ and Δ is the same as equation (1.1). In particular, if $k = l = 1$ and

$$\Delta(x_{-1}, x_0, x_1) = x_{-1} - 2x_0 + x_1 + a \sin 2\pi x_0, \quad a \in \mathbb{R},$$

then it is called the tilted Frenkel–Kontorova model [4, 10], in which the constant F represents the external driving force. We denote by $\rho(\Delta, F)$ the rotation set of solutions of equation (1.2) with bounded action.

Let

$$\tilde{S} = \{\mathbf{x} \in \mathbb{R}^\mathbb{Z} \mid \mathbf{x} \text{ is a solution of equation (1.1) with bounded action}\} \quad \text{and} \quad S = \tilde{S}/\langle \mathbf{1} \rangle,$$

where $\mathbf{1}$ denotes the configuration with all components being 1. For $L > 0$, let

$$S_L = \{\mathbf{x} = (x_n) \in S \mid |x_{n+1} - x_n| \le L, \quad \text{for all } n \in \mathbb{Z}\}.$$

Let $\tau_{m,n}$ denote the translation on $\mathbb{R}^\mathbb{Z}$ defined by $(\tau_{m,n}\mathbf{x})_i = x_{i-m} + n$ for $\mathbf{x} = (x_i) \in \mathbb{R}^\mathbb{Z}$ and $\sigma = \tau_{-1,0}/\langle \mathbf{1} \rangle$. Then the system generated by σ on S is equivalent to that by φ_Δ on the high-dimensional cylinder restricted to orbits with bounded action. Therefore, we would study the dynamical behavior and rotation set of σ on S rather than φ_Δ on the high-dimensional cylinder.

For each $\mathbf{y} \in S$, there is a lift $\mathbf{x} \in \tilde{S}$ such that $\mathbf{y} = \mathbf{x}/\langle \mathbf{1} \rangle$. Define

$$y_i - y_j = x_i - x_j \quad \text{for all } i, j \in \mathbb{Z},$$

which is independent of the lift $\mathbf{x}$. Define $\rho(\mathbf{y}) = \rho(\mathbf{x})$, which is also independent of $\mathbf{x}$, that is, $\rho(\mathbf{x}) = \rho(\mathbf{x}')$ if $\mathbf{x}/\langle \mathbf{1} \rangle = \mathbf{x}'/\langle \mathbf{1} \rangle$, where $\mathbf{x}, \mathbf{x}' \in \tilde{S}$. Let $\rho(K)$ denote the rotation set of K, that is, $\rho(K) = \bigcup_{\mathbf{x} \in K} \rho(\mathbf{x})$ for a set K of configurations with bounded action.

Some conclusions obtained in [29] are as follows. The rotation set $\rho(\Delta)$ is closed, each $\omega \in \rho(\Delta)$ is realized by a Birkhoff solution of equation (1.1), and if there exists a solution $\mathbf{x}$ of equation (1.1) with bounded action such that $\rho(\mathbf{x})$ is not a single point, then the topological entropy of φ_Δ is positive.

A fundamental lemma in [29] is that $\langle \rho(K) \rangle \subset \rho(\Delta)$, where $K = \omega(\mathbf{x})$ is the ω-limit set of the orbit $\{\sigma^n \mathbf{x} \mid n \in \mathbb{Z}\}$ and $\langle \cdot \rangle$ denotes the convex hull. In this paper, we shall improve this conclusion and show that $\langle \rho(K) \rangle \subset \rho(\Delta)$ provided K is a compact invariant chain transitive set for σ, which is an analogue, to some extent, of Franks's result on surface homeomorphisms [14], and then discuss its applications assuming the rotation set is upper-stable or σ has zero topological entropy on S.

If the monotone recurrence relation of equation (1.1) has a generating function, then zero topological entropy implies that Birkhoff minimizers with each rotation number form a continuous foliation [19]. Our first topic in this paper is to investigate, for the general monotone recurrence relations, the properties of rotation sets of equation (1.1) with zero topological entropy.

There are large amounts of research work on the relation between topological entropy and rotation sets of homeomorphisms on the torus and annulus, see [13, 23–25, 27] and references therein. In [24], Le Calvez and Tal investigated rotation sets of surface homeomorphisms with no topological horseshoe by developing a new criterion for the existence of topological horseshoes for surface homeomorphisms together with forcing theory [23]. For homeomorphisms on the two-dimensional cylinder isotopic to the identity, they showed that (among other things) each orbit with non-empty ω-limit set has a well-defined forward rotation number; the forward and backward rotation numbers for a non-wandering point are identical; the rotation number function is continuous on the set of non-wandering points; and each Birkhoff recurrence class has a unique rotation number.

The first part of this paper is devoted to the discussion of these questions for monotone recurrence relations with zero topological entropy. Applying the results in §3, we obtain that each Birkhoff recurrence class (see §2 for the definition) in S has a unique forward and a unique backward rotation number, which are actually identical, implying that $\rho(\mathbf{x}) = \rho^*(\mathbf{x})$ for $\mathbf{x}$ being non-wandering. Moreover, the rotation number function is continuous on the set of non-wandering points.

Let $\Omega \subset S$ denote the set of all non-wandering points of σ.

THEOREM A. *Assume σ has zero topological entropy on S. Let $L > 0$ and $K \subset S_L$ be a non-empty Birkhoff recurrence class. Then $\rho(K)$ is a single point and $\rho(\mathbf{y}) = \rho^*(\mathbf{y})$ for each $\mathbf{y} \in K$.*

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Remark. Let $L > 0$ and $\mathbf{x} \in S_L$ be a non-wandering point. Then (see §2) there is a Birkhoff cycle containing $\mathbf{x}$ and hence the Birkhoff recurrence class containing $\mathbf{x}$ is non-empty. It follows immediately from the above theorem that $\rho^*(\mathbf{x}) = \rho(\mathbf{x})$.

THEOREM B. *Assume σ has zero topological entropy on S. Then for each $L > 0$, $\rho : \Omega \cap S_L \to \mathbb{R}$ is continuous.*

For an endomorphism of the circle, the rotation set is a closed interval. Bamon, Malta, and Pacífico proved [5] that if the rotation interval is stable (persistent), then its endpoints must be rational numbers. For a homeomorphism on the torus homotopic to the identity, Addas-Zanata showed in [1] that its rotation set, which is a compact convex subset of the plane, is not upper-stable if it has an extremal point which is not a rational vector.

Inspired by these discussions, we shall investigate boundary points of rotation sets for monotone recurrence relations assuming the rotation sets are upper-stable.

We say that $\rho(\Delta)$ is upper-stable if there exists $\varepsilon_0 > 0$ such that $\rho(\Delta') \subset \rho(\Delta)$ for each Δ' continuous on $\mathbb{R}^{k+l+1}$ which satisfies assumptions (1)–(3) and $\sup_{u\in\Gamma} |\Delta(u) - \Delta'(u)| < \varepsilon_0$ for each compact set $\Gamma \subset \mathbb{R}^{k+l+1}$. We say that $\rho(\Delta)$ is upper-stable with respect to F if there exists $\varepsilon_0 > 0$ such that $\rho(\Delta, F) \subset \rho(\Delta)$ for $F \in (-\varepsilon_0, \varepsilon_0)$. It is obvious that if $\rho(\Delta)$ is upper-stable, then it is upper-stable with respect to F. It seems that the upper-stability with respect to F is a weaker assumption than the upper-stability. However, they are equivalent owing to Lemma 6.1.

THEOREM C. *Assume $\rho(\Delta)$ is upper-stable with respect to F. Then there exists a positive integer $q_0 \geq 1$ such that each boundary point of $\rho(\Delta)$ is rational and has the form p/q in lowest terms with $1 \leq q \leq q_0$.*

Remark 1. An immediate corollary of the above theorem is that if the rotation set is compact and upper-stable, then it has finite boundary points which are all rational numbers. We emphasize that a related result on rotation sets of homeomorphisms on the torus isotopic to the identity was obtained by Guihéneuf and Koropecki in [18].

Remark 2. If an irrational $\omega \in \rho(\Delta)$ which may not be upper-stable, then for each $\varepsilon > 0$, there exists $F \in (-\varepsilon, \varepsilon)$ such that $p/q \in \rho(\Delta, F)$ for some rational p/q close to ω. This is a straightforward consequence of Lemmas 6.3 and 6.4, the proof of which is the same as the last part of that of Theorem C. We remark that it is an analogue of a conclusion for twist maps on the two-dimensional cylinder obtained by Le Calvez, see [22, §1.5].

We say that a configuration $\mathbf{x} = (x_n)$ with forward rotation number ρ has bounded deviation if there exists $M > 0$ such that $|x_n - x_0 - n\rho| \leq M$ for all $n \in \mathbb{N}$. It is well known that if $\mathbf{x}$ is Birkhoff (see §2 for the definition), then $\mathbf{x}$ has a rotation number and bounded deviation. Generally, a solution of equation (1.1) which has forward rotation number does not necessarily have bounded deviation. Neither do the orbits of homeomorphisms on the annulus or torus [9, 20].

Recently, a great deal of attention has been gathered on the problem of bounded deviation for homeomorphisms isotopic to the identity on the torus, see [2, 12, 17, 21, 28] and references therein. For homeomorphisms isotopic to the identity on the closed

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annulus, Conejeros and Tal showed [11] that if f is a homeomorphism on a region of instability and the rotation numbers of the boundary components lie in the interior of the rotation set, then f has uniformly bounded deviations from its rotation set. We shall study a similar problem assuming the rotation set is upper-stable.

THEOREM D. *Assume $\rho(\Delta)$ is upper-stable with respect to F. Let $[a, b]$ be a connected component of $\rho(\Delta)$ and $L > 0$. Then for each compact and σ-invariant set $K \subset S_L$ with $\langle\rho(K)\rangle \subset [a, b]$, there exists $M > 0$, such that for each $\mathbf{x} = (x_n) \in K$,*

$$x_n - x_0 - nb \leq M \quad \text{and} \quad x_n - x_0 - na \geq -M \quad \text{for all } n \geq 1.$$

Remark. We should mention that Guihéneuf and Koropecki in [18] have also studied bounded deviations for torus homeomorphisms isotopic to the identity assuming upper-stability of rotation sets. Our approach is highly inspired by their methods.

Since the system corresponding to solutions of equation (1.1) is a monotone twist map on the high-dimensional cylinder, and we do not have powerful tools like forcing theory [23] for two-dimensional cases, we have to make full use of monotonicity condition of assumption (1). We define α-pseudo solutions (see also [29]) and introduce chain transitivity for α-pseudo solutions of equation (1.1), which are similar to, but not equivalent to, α-pseudo orbits and chain transitivity defined for general dynamical systems [14]. A fundamental result we proved in §3 is that each element in the convex hull of the rotation set of a compact invariant chain transitive set can be realized by a Birkhoff solution of equation (1.1), which forms the basis for the proofs of the main conclusions in this paper.

2. *Preliminaries*

We denote by X the configuration space $\mathbb{R}^{\mathbb{Z}}$ equipped with the product topology and $Y = X/\langle\mathbf{1}\rangle$, where $\mathbf{1}$ denotes the configuration with each component being 1. Let P denote the projection from X to Y defined by $P(\mathbf{x}) = \mathbf{y} = \mathbf{x}/\langle\mathbf{1}\rangle$ and call $\mathbf{x} \in X$ a lift of $\mathbf{y} \in Y$. Let $\sigma : Y \to Y$ be defined by $\sigma\mathbf{y} = P(\tau_{-1,0}\mathbf{x})$, where $\mathbf{x} \in X$ is a lift of $\mathbf{y} \in Y$, and the shift map $\tau_{m,n} : X \to X$ is defined for $m, n \in \mathbb{Z}$ by $(\tau_{m,n}\mathbf{x})_i = x_{i-m} + n$, for all $i \in \mathbb{Z}$.

For $L > 0$, let

$$\tilde{B}_L = \{\mathbf{x} = (x_i) \in X \mid |x_{i+1} - x_i| \leq L, \quad \text{for all } i \in \mathbb{Z}\}, \quad \tilde{B} = \bigcup_{L>0} \tilde{B}_L,$$

$$\tilde{S}_L = \{\mathbf{x} \in \tilde{B}_L \mid \mathbf{x} \text{ is a solution of equation (1.1)}\}, \quad \tilde{S} = \bigcup_{L>0} \tilde{S}_L.$$

Let $B_L = P(\tilde{B}_L)$, $B = P(\tilde{B})$, $S_L = P(\tilde{S}_L)$, and $S = P(\tilde{S})$. It then follows immediately from Tychonoff's theorem that B_L and S_L are compact.

Let $\mathbf{x}, \mathbf{y} \in S_L$. We say that there is a Birkhoff connection from $\mathbf{x}$ to $\mathbf{y}$ if for each neighborhood U of $\mathbf{x}$ and each neighborhood V of $\mathbf{y}$, there exists $n \geq 1$ such that $\sigma^n(U \cap S_L) \cap V \neq \emptyset$. A Birkhoff cycle is a finite sequence $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^p, \mathbf{x}^{p+1} = \mathbf{x}^1$ in S_L such that there is a Birkhoff connection from $\mathbf{x}^i$ to $\mathbf{x}^{i+1}$ for each $i \in \{1, 2, \ldots, p\}$. A solution $\mathbf{x} \in S_L$ is said to be Birkhoff recurrent for σ if there exists a Birkhoff cycle containing $\mathbf{x}$.

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A solution $\mathbf{x} \in S_L$ is said to be non-wandering for σ if for each neighborhood U of $\mathbf{x}$, there exists $n \geq 1$ such that $\sigma^n(U \cap S_L) \cap U \neq \emptyset$. Therefore, $\mathbf{x}$ is non-wandering if and only if there is a Birkhoff cycle containing $\mathbf{x}$ with the cycle length $p = 1$. We say that $\mathbf{x} \in S_L$ is Birkhoff equivalent to $\mathbf{y} \in S_L$ if there is a Birkhoff cycle containing both $\mathbf{x}$ and $\mathbf{y}$. The equivalence class will be called Birkhoff recurrence class, see [24] for the introduction of these concepts on surface homeomorphisms and related conclusions on rotation sets.

Let $\mathbf{w} = (w_n)$, $\mathbf{u} = (u_n) \in Y$, and $\tilde{\mathbf{w}} = (\tilde{w}_n) \in X$ be a lift of $\mathbf{w}$, that is, $P(\tilde{\mathbf{w}}) = \mathbf{w}$. Define

$$|w_0 - u_0| = \min\{|\tilde{w}_0 - \tilde{u}_0| \mid \tilde{\mathbf{u}} = (\tilde{u}_n) \in X,\ P(\tilde{\mathbf{u}}) = \mathbf{u}\}$$

which is independent of $\tilde{\mathbf{w}}$. Then there exists some $\hat{\mathbf{u}} = (\hat{u}_n) \in X$ with $P(\hat{\mathbf{u}}) = \mathbf{u}$ such that $|w_0 - u_0| = |\tilde{w}_0 - \hat{u}_0|$. If $|w_0 - u_0| = \frac{1}{2}$, take $\hat{\mathbf{u}}$ such that $\hat{u}_0 = \tilde{w}_0 + \frac{1}{2}$. Define

$$|w_n - u_n| = |\tilde{w}_n - \hat{u}_n| \quad \text{for all } n \in \mathbb{Z}. \tag{2.1}$$

For each $\mathbf{w} = (w_n) \in Y$ and $\delta > 0$, we define

$$U(\mathbf{w}, \delta) = \{\mathbf{u} = (u_n) \in Y \mid |u_n - w_n| < \delta,\ n = -k, \ldots, l-1\}. \tag{2.2}$$

Define the relations $\leq$, $<$, $\ll$ on the configuration space X as follows. We say that $\mathbf{x} = (x_i) \leq \mathbf{x}' = (x_i')$ if and only if $x_i \leq x_i'$ for $i \in \mathbb{Z}$, $\mathbf{x} < \mathbf{x}'$ if and only if $\mathbf{x} \leq \mathbf{x}'$ and $\mathbf{x} \neq \mathbf{x}'$, $\mathbf{x} \ll \mathbf{x}'$ if and only if $x_i < x_i'$ for $i \in \mathbb{Z}$. Similarly, we can define $\geq$, $>$, and $\gg$. We say that $\mathbf{x}$ and $\mathbf{x}'$ are ordered if $\mathbf{x} \leq \mathbf{x}'$ or $\mathbf{x} \geq \mathbf{x}'$, $\mathbf{x}$ and $\mathbf{x}'$ are strictly ordered if $\mathbf{x} \ll \mathbf{x}'$, or $\mathbf{x} \gg \mathbf{x}'$, or $\mathbf{x} = \mathbf{x}'$.

A configuration $\mathbf{x} \in X$ is said to be Birkhoff if for any $m, n \in \mathbb{Z}$, $\tau_{m,n}\mathbf{x}$ and $\mathbf{x}$ are ordered, that is, $\tau_{m,n}\mathbf{x} \leq \mathbf{x}$ or $\tau_{m,n}\mathbf{x} \geq \mathbf{x}$. We say that $\mathbf{y} \in Y$ is Birkhoff if a lift $\tilde{\mathbf{y}} \in X$ of $\mathbf{y}$ is Birkhoff.

Let $\tilde{\mathcal{B}} \subset X$ denote the set of Birkhoff configurations. It is easy to check that $\tilde{\mathcal{B}} \subset \tilde{B}$, $\tilde{\mathcal{B}}$ is closed in the product topology, and $\tau_{m,n}\tilde{\mathcal{B}} = \tilde{\mathcal{B}}$, for all $m, n \in \mathbb{Z}$. It follows that each Birkhoff configuration has a rotation number [6, 15].

LEMMA 2.1. *Let $\mathbf{x} \in X$ be a Birkhoff configuration. Then $\mathbf{x}$ has a rotation number $\rho(\mathbf{x}) = \rho^*(\mathbf{x}) = \rho$ and*

$$|x_j - x_i - (j-i)\rho| \leq 1 \quad \textit{for all } i, j \in \mathbb{Z}. \tag{2.3}$$

Moreover, the map $\mathbf{x} \mapsto \rho(\mathbf{x})$, $\tilde{\mathcal{B}} \to \mathbb{R}$ is continuous in the product topology [6, 15].

Definition 2.2. Given $\alpha > 0$, a configuration $\mathbf{x} = (x_i)$ is called an α-pseudo solution of equation (1.1) if

$$|\Delta(x_{i-k}, \ldots, x_{i+l})| \leq \alpha \quad \text{for all } i \in \mathbb{Z}.$$

Let $\rho_\alpha(\Delta)$ denote the rotation set of α-pseudo solutions of equation (1.1), that is,

$$\rho_\alpha(\Delta) = \bigcup \rho(\mathbf{x}),$$

where $\mathbf{x}$ is an α-pseudo solution of equation (1.1) with bounded action. It was proved [29] that the pseudo rotation set of equation (1.1) defined by $\rho_\psi(\Delta) = \bigcap_{\alpha>0} \rho_\alpha(\Delta)$ is identical to $\rho(\Delta)$. For the study of pseudo rotation sets, see [7] for circle endomorphisms and annulus homeomorphisms, and [18] for torus homeomorphisms.

The following conclusion was proved in [29], adapting Angenent's approach for the special case $\alpha = 0$ in [3].

LEMMA 2.3. *If* $\mathbf{x} = (x_i)$ *is an* α*-pseudo solution of equation (1.1) and there exists* $\omega \in \mathbb{R}$ *such that* $\sup_{i\in\mathbb{Z}} |x_i - x_0 - i\omega| < \infty$, *then equation (1.1) has a Birkhoff* α*-pseudo solution with rotation number* ω.

Definition 2.4. Let $\delta > 0$, $i_0 \in \mathbb{Z}$, and $\mathbf{x} = (x_n)$, $\mathbf{x}' = (x'_n) \in X$ be two configurations with $|x_n - x'_n| < \delta$ for $n = i_0 - k, \ldots, i_0 + l - 1$. Then the configuration $\mathbf{z} = (z_n)$ is said to be a δ-gluing of $\mathbf{x}$ and $\mathbf{x}'$ (at site i_0) if

$$z_n = x_n \quad \text{for all } n < i_0 \quad \text{and} \quad z_n = x'_n \quad \text{for all } n \geq i_0.$$

Assume $\overline{\mathbf{x}} = (\overline{x}_n)$ and $\underline{\mathbf{x}} = (\underline{x}_n)$ are supersolution and subsolution of equation (1.1) respectively, that is,

$$\Delta(\overline{x}_{n-k}, \ldots, \overline{x}_{n+l}) \leq 0 \quad \text{and} \quad \Delta(\underline{x}_{n-k}, \ldots, \underline{x}_{n+l}) \geq 0 \quad \text{for all } n \in \mathbb{Z}.$$

It is said they exchange rotation numbers if

$$\liminf_{n\to\infty} \frac{\overline{x}_n}{n} \geq \omega_2, \quad \limsup_{n\to-\infty} \frac{\overline{x}_n}{n} \leq \omega_1, \quad \limsup_{n\to\infty} \frac{\underline{x}_n}{n} \leq \omega_1, \quad \liminf_{n\to-\infty} \frac{\underline{x}_n}{n} \geq \omega_2, \tag{2.4}$$

hold for some $\omega_1 < \omega_2$, see [3, §6].

A criterion presented by Angenent in [3] shows that if there exist a supersolution and a subsolution of equation (1.1) exchanging rotation numbers, then the homeomorphism φ_Δ defined by equation (1.1), or σ on S, has positive topological entropy, see [3, Theorem 7.1].

Remark. There are several objects called *Birkhoff* in this section. The notion of Birkhoff configuration, which is central in this paper, corresponds to that of Birkhoff orbit, which plays an important role in the Aubry–Mather theory, see, for example, [3, 6, 15]. Another terminology, Birkhoff recurrence class, was borrowed from [24]. It seems that the notion of Birkhoff recurrence class is related to that of Birkhoff region of instability for monotone twist maps on the annulus [26].

3. *Chain transitive sets*

In this section, we introduce chain transitivity for solutions or α-pseudo solutions of equation (1.1). Although it is not equivalent to that defined for general dynamical systems [14], it does help us to provide the proof of Theorems A and B since the ω-limit set $\omega(\mathbf{x})$ of $\mathbf{x} \in S_L$ and the Birkhoff recurrence class are chain transitive according to our definition, see Lemmas 3.6 and 4.3.

Definition 3.1. Assume $\alpha \geq 0$, and $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n) \in B_L$ are two α-pseudo solutions of equation (1.1). A β-pseudo solution $\mathbf{z} = (z_n) \in B_L$ is called a β-chain ($\beta > \alpha$) from $\mathbf{x}$ to $\mathbf{y}$ if

$$\tilde{z}_n = \tilde{x}_n \quad \text{and} \quad \tilde{z}_{j+n} = \tilde{y}_n, \quad n = -k, \ldots, l-1,$$

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for some $j > 0$, where $\tilde{\mathbf{z}} = (\tilde{z}_i)$, $\tilde{\mathbf{x}} = (\tilde{x}_i)$, and $\tilde{\mathbf{y}} = (\tilde{y}_i)$ are lifts of $\mathbf{z}$, $\mathbf{x}$, and $\mathbf{y}$, respectively. An α-pseudo solution $\mathbf{x} \in B_L$ is said to be chain recurrent if for each $\beta > \alpha$, there is a β-chain from $\mathbf{x}$ to itself. In particular, a solution $\mathbf{x} \in S_L$ is said to be chain recurrent if for each $\beta > 0$, there is a β-chain from $\mathbf{x}$ to itself.

Definition 3.2. Assume $\alpha \geq 0$ and $K \subset B_L$ is a set of α-pseudo solutions of equation (1.1). If for any $\mathbf{x}, \mathbf{y} \in K$ and any $\beta > \alpha$, there is a β-chain from $\mathbf{x}$ to $\mathbf{y}$, then we say that K is chain transitive.

LEMMA 3.3. *Assume $K \subset B_L$ is an invariant for σ and chain transitive set of α-pseudo solutions of equation (1.1). Then for any $\mathbf{x}, \mathbf{y} \in K$, any $\beta > \alpha$, and arbitrarily large integer $m' > k + l$, there exists a β-pseudo solution $\mathbf{z}$ and $m > m'$, such that*

$$\tilde{z}_n = \tilde{x}_n \quad \text{and} \quad \tilde{z}_{m+n} = \tilde{y}_n, \quad n = -k, \ldots, l-1,$$

where $\tilde{\mathbf{z}} = (\tilde{z}_i)$, $\tilde{\mathbf{x}} = (\tilde{x}_i)$, and $\tilde{\mathbf{y}} = (\tilde{y}_i)$ are lifts of $\mathbf{z}$, $\mathbf{x}$, and $\mathbf{y}$, respectively.

Proof. Let $\mathbf{y}' = \sigma^{-m'}\mathbf{y}$. Since $\mathbf{x}, \mathbf{y}' \in K$, which is chain transitive, then for each $\beta > \alpha$, there is a β-pseudo solution $\mathbf{z}' = (z'_n) \in B_L$ such that for some $j > 0$,

$$\tilde{z}'_n = \tilde{x}_n \quad \text{and} \quad \tilde{z}'_{j+n} = \tilde{y}'_n, \quad n = -k, \ldots, l-1,$$

where $\tilde{\mathbf{z}}' = (\tilde{z}'_i)$, $\tilde{\mathbf{x}} = (\tilde{x}_i)$, and $\tilde{\mathbf{y}}' = (\tilde{y}'_i)$ are lifts of $\mathbf{z}'$, $\mathbf{x}$, and $\mathbf{y}'$, respectively. Note that $\tilde{\mathbf{y}} = \tau_{-m',0}\tilde{\mathbf{y}}'$ is a lift of $\mathbf{y}$. Let $\tilde{\mathbf{z}} = (\tilde{z}_i)$ be constructed by

$$\tilde{z}_i = \begin{cases} \tilde{z}'_i, & i < j-k, \\ \tilde{z}'_i = \tilde{y}'_{i-j}, & j-k \leq i < j+l, \\ \tilde{y}'_{i-j} = \tilde{y}_{i-j-m'}, & i \geq j+l. \end{cases}$$

Then $\mathbf{z} = P(\tilde{\mathbf{z}})$ is the desired β-pseudo solution with $m = j + m'$. □

Let $K \subset B_L$ and denote $\rho(K) = \bigcup_{\mathbf{x}\in K} \rho(\mathbf{x})$ and its convex hull by $\langle\rho(K)\rangle$.

LEMMA 3.4. *Let $K \subset B_L$ be a compact invariant set for σ. Then for each $\rho \in \langle\rho(K)\rangle$, there exist $\mathbf{y}^1, \mathbf{y}^2 \in K$, such that*

$$y_n^1 - y_0^1 - n\rho \leq 1 \quad \text{and} \quad y_n^2 - y_0^2 - n\rho \geq -1 \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Proof. The proof is the same as that of [29, Lemma 2.4].

Let the ω-limit set of $\mathbf{x} \in B_L$ be denoted by $\omega(\mathbf{x})$, that is,

$$\omega(\mathbf{x}) = \{\mathbf{y} \in Y \mid \text{there exist } m_i \in \mathbb{N} \text{ such that } \sigma^{m_i}\mathbf{x} \to \mathbf{y} \text{ as } i \to \infty\}.$$ □

LEMMA 3.5. *Let $\mathbf{x} \in B_L$. Then $\rho(\mathbf{x}) \subset \langle\rho(\omega(\mathbf{x}))\rangle$.*

Proof. Let $\rho \in \rho(\mathbf{x})$. Then there exist $\mathbf{y}^1 = (y_n^1)$ and $\mathbf{y}^2 = (y_n^2) \in \omega(\mathbf{x})$ (the proof is the same as that of [29, Lemma 2.4] and hence omitted), such that equation (3.1) holds. It follows that $\liminf_{n\to+\infty}(y_n^1 - y_0^1)/n \leq \rho$ and $\limsup_{n\to+\infty}(y_n^2 - y_0^2)/n \geq \rho$, implying $\rho \in \langle\rho(\omega(\mathbf{x}))\rangle$. □

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LEMMA 3.6. *Let $\alpha \geq 0$ and $\mathbf{x}$ be an α-pseudo solution of equation (1.1) with bounded action. Then $\omega(\mathbf{x})$ is chain transitive.*

Proof. It is easy to check that each configuration in $\omega(\mathbf{x})$ is an α-pseudo solution of equation (1.1). Let $\mathbf{y}, \mathbf{z} \in \omega(\mathbf{x})$, $\beta > \alpha$, and $\delta > 0$. Then there exist $m_1, m_2 \in \mathbb{N}$ with $m_2 - m_1 > 2(k + l)$ such that

$$\sigma^{m_1}\mathbf{x} \in U(\mathbf{y}, \delta) \quad \text{and} \quad \sigma^{m_2}\mathbf{x} \in U(\mathbf{z}, \delta),$$

implying the existence of $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}}$, and $\tilde{\mathbf{z}}$, which are lifts of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, respectively, such that

$$|\tilde{x}_{m_1+n} - \tilde{y}_n| < \delta \quad \text{and} \quad |\tilde{x}_{m_2+n} - \tilde{z}_n| < \delta \quad \text{for } n = -k, \ldots, l - 1.$$

Let $\tilde{\mathbf{w}} = (\tilde{w}_i)$ be defined by

$$\tilde{w}_i = \begin{cases} \tilde{y}_{i-m_1}, & i \leq m_1 + l - 1, \\ \tilde{x}_i, & m_1 + l - 1 < i < m_2 - k, \\ \tilde{z}_{i-m_2}, & m_2 - k \leq i. \end{cases}$$

Then we have a β-chain from $\mathbf{y}$ to $\mathbf{z}$ provided δ is small enough. □

LEMMA 3.7. *Let $K \subset B_L$ be a compact invariant set for σ. Then there exist $\mathbf{x}, \mathbf{y} \in K$ such that $\rho(\mathbf{x}) = \rho^*(\mathbf{x}) = \sup \rho(K)$ and $\rho(\mathbf{y}) = \rho^*(\mathbf{y}) = \inf \rho(K)$.*

Proof. The proof is similar to that of [**29**, Lemma 5.2] and hence omitted here, see also [**7**, **8**]. □

LEMMA 3.8. *Let $\alpha \geq 0$ and $K \subset B_L$ be a set of α-pseudo solutions of equation (1.1) which is compact and invariant for σ. Assume K is chain transitive and $\rho \in \langle\rho(K)\rangle$. Then for each $\beta > \alpha$, there exists a β-pseudo solution $\mathbf{z} = (z_n)$ satisfying*

$$\sup_{n\in\mathbb{Z}} |z_n - z_0 - n\rho| < \infty.$$

Proof. The proof is postponed to Appendix A. □

THEOREM 3.9. *Let $\alpha \geq 0$ and $K \subset B_L$ be a set of α-pseudo solutions of equation (1.1) which is compact and invariant for σ. If K is chain transitive, then for each $\rho \in \langle\rho(K)\rangle$, there exists a Birkhoff α-pseudo solution $\mathbf{z}$ of equation (1.1) with $\rho(\mathbf{z}) = \rho$, and hence $\langle\rho(K)\rangle \subset \rho_\alpha(\Delta)$.*

Proof. For each $\rho \in \langle\rho(K)\rangle$, we deduce by Lemmas 3.8 and 2.3 that for each $\beta > \alpha$, there exists a Birkhoff β-pseudo solution $\mathbf{z}^\beta = (z_n^\beta) \in Y$ satisfying $|z_n^\beta - z_0^\beta - n\rho| \leq 1$, for all $n \in \mathbb{Z}$, due to Lemma 2.1. Applying Tychonoff's theorem, we obtain an accumulation point $\mathbf{z}$ of $\{\mathbf{z}^\beta\}$ as $\beta \to \alpha$, which is a Birkhoff α-pseudo solution of equation (1.1) with $\rho(\mathbf{z}) = \rho$. □

The following conclusions are actually generalizations of those in [**29**, Theorem A].

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THEOREM 3.10. *For each $\alpha \geq 0$, $\rho_\alpha(\Delta)$ is closed. Moreover, for each $\rho \in \rho_\alpha(\Delta)$, there exists a Birkhoff α-pseudo solution with rotation number ρ. If $\rho = p/q$ is rational in lowest terms, then there is a Birkhoff (p, q)-periodic α-pseudo solution.*

Proof. Let $\rho \in \rho_\alpha(\Delta)$. Then there is an α-pseudo solution $\mathbf{x}$ of equation (1.1) with bounded action such that $\rho \in \rho(\mathbf{x}) \subset \langle \rho(\omega(\mathbf{x}))\rangle$ by Lemma 3.5. Note that $\omega(\mathbf{x})$ is compact, invariant for σ, and chain transitive due to Lemma 3.6. We deduce by Theorem 3.9 the existence of a Birkhoff α-pseudo solution with rotation number ρ.

Let $\rho = p/q$ in lowest terms be rational and $\rho(\mathbf{x}) = p/q$, where $\mathbf{x}$ is a Birkhoff α-pseudo solution. If $\mathbf{x}$ is (p, q)-periodic, then the proof is complete. If not, then we shall show that the limit point $\lim_{n\to\infty} \tau_{q,p}^n \mathbf{x}$ is a Birkhoff (p, q)-periodic α-pseudo solution of equation (1.1).

Indeed, if $\tau_{q,p}\mathbf{x} \neq \mathbf{x}$, then we assume $\tau_{q,p}\mathbf{x} \geq \mathbf{x}$ (the proof for the case $\tau_{q,p}\mathbf{x} \leq \mathbf{x}$ is the same) since $\mathbf{x}$ is Birkhoff. It then follows that

$$\mathbf{x} \leq \tau_{q,p}\mathbf{x} \leq \cdots \leq \tau_{q,p}^n\mathbf{x} \leq \cdots .$$

We claim that $\tau_{q,p}^n\mathbf{x} \leq \mathbf{x} + \mathbf{1} = \tau_{0,1}\mathbf{x}$, for all $n \geq 1$. Indeed, if this is not true, then we have a positive integer n_0 such that $\tau_{q,p}^{n_0}\mathbf{x} \geq \mathbf{x} + \mathbf{1}$ since $\tau_{q,p}^{n_0}\mathbf{x}$ and $\tau_{0,1}\mathbf{x}$ are ordered due to the fact that $\mathbf{x}$ is Birkhoff. Consequently, $\tau_{q,p}^{jn_0}\mathbf{x} \geq \mathbf{x} + j \cdot \mathbf{1}$, for all $j \geq 1$, implying $\rho(\mathbf{x}) \geq p/q + 1/(n_0 q)$, a contradiction. Therefore, we have $\tau_{q,p}^n\mathbf{x} \leq \mathbf{x} + \mathbf{1}$, for all $n \geq 1$, and hence for each $i \in \mathbb{Z}$, $\{(\tau_{q,p}^n\mathbf{x})_i\}_{n\geq 0}$ is a non-decreasing and bounded sequence, leading to the conclusion that $\{\tau_{q,p}^n\mathbf{x}\}_{n\geq 0}$ has a unique limit point, denoted by $\mathbf{z}$. Noting that

$$\tau_{q,p}\mathbf{z} = \tau_{q,p} \lim_{n\to\infty} \tau_{q,p}^n\mathbf{x} = \lim_{n\to\infty} \tau_{q,p}^{n+1}\mathbf{x} = \mathbf{z},$$

we obtain that $\mathbf{z}$ is a Birkhoff (p, q)-periodic α-pseudo solution of equation (1.1).

Let $\rho_n \in \rho_\alpha(\Delta)$ and $\rho_n \to \rho$ as $n \to \infty$. Then there are Birkhoff α-pseudo solutions $\mathbf{x}^n \in Y$ of equation (1.1) with $\rho(\mathbf{x}^n) = \rho_n$, for all $n \in \mathbb{N}$. The accumulation point $\mathbf{x}$ of $\{\mathbf{x}^n\}$ is an α-pseudo solution with $\rho(\mathbf{x}) = \rho$ by Lemma 2.1, implying that $\rho_\alpha(\Delta)$ is closed. □

4. *Proof of Theorem A*

LEMMA 4.1.

(i) *Let $\mathbf{x}, \mathbf{y} \in S_L$. If there is a Birkhoff connection from $\mathbf{x}$ to $\mathbf{y}$, then there is a Birkhoff connection from $\mathbf{x}$ to any point in $\{\sigma^n\mathbf{y} \mid n \in \mathbb{N}\} \cup \omega(\mathbf{y})$.*

(ii) *Let $\mathbf{x}, \mathbf{y} \in S_L$ and $\mathbf{y} \notin \{\sigma^n\mathbf{x} \mid n \in \mathbb{N}\}$. If there is a Birkhoff connection from $\mathbf{x}$ to $\mathbf{y}$, then there is a Birkhoff connection from any point in $\{\sigma^n\mathbf{x} \mid n \in \mathbb{N}\} \cup \omega(\mathbf{x})$ to $\mathbf{y}$.*

(iii) *Let $\mathbf{x}, \mathbf{y} \in S_L$ and $\mathbf{y} \notin \{\sigma^n\mathbf{x} \mid n \in \mathbb{N}\}$. If there is a Birkhoff connection from $\mathbf{x}$ to $\mathbf{y}$, then for each neighborhood V of $\mathbf{y}$, each neighborhood U of $\mathbf{x}$, and each $N \in \mathbb{N}$, there exists $n \geq N$ such that $\sigma^n(U \cap S_L) \cap V \neq \emptyset$.*

Proof. These facts are easy to check, see [24]. □

LEMMA 4.2. *Let $\mathbf{x}, \mathbf{y} \in S_L$ and $\mathbf{y} \notin \{\sigma^n\mathbf{x} \mid n \in \mathbb{N}\}$. If there is a Birkhoff connection from $\mathbf{x}$ to $\mathbf{y}$, then for each $\alpha > 0$, there is an α-chain from $\mathbf{x}$ to $\mathbf{y}$.*

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Proof. Let $\delta > 0$. Then $U(\mathbf{x}, \delta)$ and $U(\mathbf{y}, \delta)$ are neighborhoods of $\mathbf{x}$ and $\mathbf{y}$, respectively. From item (iii) of Lemma 4.1, it follows that there exists $\mathbf{u} \in U(\mathbf{x}, \delta) \cap S_L$ and $m > 2(k + l)$ such that $\sigma^m \mathbf{u} \in U(\mathbf{y}, \delta)$. Let $\tilde{\mathbf{u}}$, $\tilde{\mathbf{x}}$, and $\tilde{\mathbf{y}}$ be lifts of $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$, respectively, such that $|\tilde{u}_j - \tilde{x}_j| < \delta$ and $|\tilde{u}_{m+j} - \tilde{y}_j| < \delta$ for $-k \le j \le l - 1$. Let $\tilde{\mathbf{z}} = (\tilde{z}_i)$ be defined by

$$\tilde{z}_i = \begin{cases} \tilde{x}_i, & i \le l - 1, \\ \tilde{u}_i, & l \le i < m - k, \\ \tilde{y}_{i-m}, & m - k \le i. \end{cases}$$

Then $\mathbf{z} = P(\tilde{\mathbf{z}})$ is an α-pseudo solution of equation (1.1) if δ is small enough, and hence there is an α-chain from $\mathbf{x}$ to $\mathbf{y}$. □

LEMMA 4.3. *Assume $K \subset S_L$ is a Birkhoff recurrence class. Then the closure $\overline{K}$ of K is chain transitive.*

Proof. Let $\mathbf{x}', \mathbf{y}' \in \overline{K}$, and U and V be neighborhoods of $\mathbf{x}'$ and $\mathbf{y}'$, respectively. Then there exist $\mathbf{x} \in U \cap K$ and $\mathbf{y} \in V \cap K$. Since $\mathbf{x}, \mathbf{y} \in K$, which is a Birkhoff recurrence class, there are $\mathbf{x}^1, \ldots, \mathbf{x}^p \in S_L$ such that there exist Birkhoff connections from $\mathbf{x}$ to $\mathbf{x}^1$, $\mathbf{x}^1$ to $\mathbf{x}^2$, ..., and $\mathbf{x}^p$ to $\mathbf{y}$. From Lemma 4.2 it follows that for each $\alpha > 0$, there is an $\alpha/2$-chain from $\mathbf{x}$ to $\mathbf{x}^1$, ..., an $\alpha/2$-chain from $\mathbf{x}^p$ to $\mathbf{y}$. We then obtain an $\alpha/2$-chain from $\mathbf{x}$ to $\mathbf{y}$. Choosing U and V small enough, we have an α-chain from $\mathbf{x}'$ to $\mathbf{y}'$. □

LEMMA 4.4. *Let $\mathbf{y}^1, \mathbf{y}^2 \in S_L$ with $\rho^*(\mathbf{y}^1) = \rho(\mathbf{y}^1) = a < b = \rho^*(\mathbf{y}^2) = \rho(\mathbf{y}^2)$ and $\omega_1, \omega_2 \in \rho(\Delta)$ with $a < \omega_1 < \omega_2 < b$. Assume there is a Birkhoff connection from $\mathbf{y}^1$ to $\mathbf{y}^2$. Then there exists a supersolution $\overline{\mathbf{x}}$ of equation (1.1) with $\rho^*(\overline{\mathbf{x}}) = \omega_1$ and $\rho(\overline{\mathbf{x}}) = \omega_2$. Similarly, if there is a Birkhoff connection from $\mathbf{y}^2$ to $\mathbf{y}^1$, then there is a subsolution $\underline{\mathbf{x}}$ with $\rho(\underline{\mathbf{x}}) = \omega_1$ and $\rho^*(\underline{\mathbf{x}}) = \omega_2$.*

Proof. By Theorem 3.10, there are Birkhoff solutions in $X\mathbf{w}^1 = (w_n^1)$ and $\mathbf{w}^2 = (w_n^2)$ such that $\rho^*(\mathbf{w}^1) = \rho(\mathbf{w}^1) = \omega_1$ and $\rho^*(\mathbf{w}^2) = \rho(\mathbf{w}^2) = \omega_2$.

Let $\mathbf{z}^1 = (z_n^1)$ and $\mathbf{z}^2 = (z_n^2)$ be the lifts of $\mathbf{y}^1$ and $\mathbf{y}^2$, respectively. Then $\lim_{n\to\pm\infty}(z_n^1 - z_0^1)/n = a$ and $\lim_{n\to\pm\infty}(z_n^2 - z_0^2)/n = b$. Since $a < \omega_1 < \omega_2 < b$, there exists $N > 0$ such that

$$w_n^1 < z_n^1, \quad w_n^2 > z_n^2 \quad \text{for all } n \le -N \quad \text{and} \quad w_n^1 > z_n^1, \; w_n^2 < z_n^2 \quad \text{for all } n \ge N.$$

Next, we choose $\varepsilon_0 > 0$ such that

$$w_n^1 \le z_n^1 - \varepsilon_0, \quad w_n^2 \ge z_n^2 + \varepsilon_0 \quad \text{for } -N - k - l \le n \le -N,$$

and

$$w_n^1 \ge z_n^1 + \varepsilon_0, \quad w_n^2 \le z_n^2 - \varepsilon_0 \quad \text{for } N \le n \le N + k + l.$$

Since $\mathbf{y}^2 \notin \{\sigma^n \mathbf{y}^1 \mid n \in \mathbb{N}\}$ and there is a Birkhoff connection from $\mathbf{y}^1$ to $\mathbf{y}^2$, then by item (iii) of Lemma 4.1, for each neighborhood V of $\mathbf{y}^2$ and neighborhood U of $\mathbf{y}^1$, there exists $\mathbf{u} \in U \cap S_L$ such that $\sigma^m \mathbf{u} \in V$, where $m > 2(N + k + l)$.

For each $0 < \varepsilon \leq \varepsilon_0$, we choose U and V small enough such that

$$|x_n^1 - z_n^1| \leq \varepsilon \quad \text{and} \quad |x_n^2 - z_n^2| \leq \varepsilon \quad \text{for } -N-k-l \leq n \leq N+k+l,$$

where $\mathbf{x}^1 = (x_n^1)$ and $\mathbf{x}^2 = (x_n^2)$ are lifts of $\mathbf{u}$ and $\sigma^m\mathbf{u}$, respectively. Hence,

$$w_n^1 \leq x_n^1, \quad w_n^2 \geq x_n^2 \quad \text{for } -N-k-l \leq n \leq -N,$$

and

$$w_n^1 \geq x_n^1, \quad w_n^2 \leq x_n^2 \quad \text{for } N \leq n \leq N+k+l.$$

Note that $P(\mathbf{x}^1) = \mathbf{u}$ and $P(\mathbf{x}^2) = \sigma^m\mathbf{u}$. Then there exists $l_2 \in \mathbb{Z}$ such that

$$x_n^2 = x_{m+n}^1 + l_2 \quad \text{for all } n \in \mathbb{Z}.$$

We can replace $\mathbf{x}^2$, $\mathbf{z}^2$, and $\mathbf{w}^2$ by $\mathbf{x}^2 - l_2 \cdot \mathbf{1}$, $\mathbf{z}^2 - l_2 \cdot \mathbf{1}$, and $\mathbf{w}^2 - l_2 \cdot \mathbf{1}$, respectively. Therefore, we may assume $l_2 = 0$ without loss of generality. It then follows that $\mathbf{x}^2 = \tau_{-m,0}\mathbf{x}^1$.

We construct a supersolution $\overline{\mathbf{x}} = (\overline{x}_j)$ as follows. Let

$$\overline{x}_j = \begin{cases} w_j^1, & j \leq -N-l-k, \\ w_j^1 = \min\{w_j^1, x_j^1\}, & -N-l-k < j \leq -N, \\ \min\{w_j^1, x_j^1\}, & -N < j \leq N, \\ x_j^1 = \min\{w_j^1, x_j^1\}, & N < j \leq N+l+k, \\ x_j^1 = x_{j-m}^2, & N+l+k < j \leq m-N-l-k, \\ x_{j-m}^2 = \min\{x_{j-m}^2, w_{j-m}^2\}, & m-N-l-k < j \leq m-N, \\ \min\{x_{j-m}^2, w_{j-m}^2\}, & m-N < j \leq m+N, \\ w_{j-m}^2 = \min\{x_{j-m}^2, w_{j-m}^2\}, & m+N < j \leq m+N+l+k, \\ w_{j-m}^2, & m+N+l+k < j. \end{cases}$$

Note that $\mathbf{w}^1$, $\mathbf{w}^2$, $\mathbf{x}^1$, and $\mathbf{x}^2$ are solutions of equation (1.1). One can check by the construction of $\overline{x}_j$ and the monotonicity condition of assumption (1) that $\overline{\mathbf{x}} = (\overline{x}_j)$ is a supersolution of equation (1.1) which satisfies

$$\rho^*(\overline{\mathbf{x}}) = \rho^*(\mathbf{w}^1) = \omega_1 \quad \text{and} \quad \rho(\overline{\mathbf{x}}) = \rho(\mathbf{w}^2) = \omega_2.$$

Similarly, we can construct, due to a Birkhoff connection from $\mathbf{y}^2$ to $\mathbf{y}^1$, a subsolution $\underline{\mathbf{x}} = (\underline{x}_n)$ of equation (1.1) satisfying $\rho^*(\underline{\mathbf{x}}) = \omega_2$ and $\rho(\underline{\mathbf{x}}) = \omega_1$. □

Proof of Theorem A. Let $K \subset S_L$ be a non-empty Birkhoff recurrence class. Then K is invariant for σ due to Lemma 4.1, and the closure $\overline{K}$ of K is compact and chain transitive by Lemma 4.3. As a consequence of Theorem 3.9 by setting $\alpha = 0$, $\langle \rho(\overline{K}) \rangle \subset \rho(\Delta)$.

We shall show that $\rho(K)$ is a single point by contradiction. Assume $a, b \in \rho(K)$ with $a < b$. Then $[a, b] \subset \langle \rho(\overline{K}) \rangle \subset \rho(\Delta)$ and there exist $\mathbf{y}^1, \mathbf{y}^2 \in K$ such that $\rho(\mathbf{y}^1) = a$,

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$\rho(\mathbf{y}^2) = b$ since each solution of equation (1.1) with bounded action has a well-defined forward rotation number if σ has zero topological entropy on S, see [29, Theorem B].

Since $\mathbf{y}^1$ and $\mathbf{y}^2$ are in the same Birkhoff recurrence class, there is a Birkhoff cycle $\{\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^p, \mathbf{z}^{p+1} = \mathbf{z}^1\}(p \geq 2)$ such that $\mathbf{z}^1 = \mathbf{y}^1$ and $\mathbf{z}^n = \mathbf{y}^2$ for some $n \in \{2, \ldots, p\}$. We assume without loss of generality that $\mathbf{z}^{i+1} \notin \{\sigma^j \mathbf{z}^i \mid \text{for all } j \in \mathbb{N}\}$, and hence we may replace $\mathbf{z}^i$ by $\hat{\mathbf{z}}^i \in \omega(\mathbf{z}^i)$ by item (ii) of Lemma 4.1 if necessary. Note that the proof of [29, Theorem B] actually shows that $\langle \rho(\omega(\mathbf{z})) \rangle = \rho(\mathbf{z})$ is a single point for each $\mathbf{z} \in S$ if σ has zero topological entropy on S. Therefore, we may assume by Lemma 3.7 $\rho(\mathbf{z}^i) = \rho^*(\mathbf{z}^i)$ for all $i = 1, 2, \ldots, p$.

We assume $\rho(\mathbf{z}^2) = a_1 > a$, otherwise consider the Birkhoff connection from $\mathbf{z}^2$ to $\mathbf{z}^3$. We also assume $\rho(\mathbf{z}^p) = a_2 > a$, otherwise consider the Birkhoff connection from $\mathbf{z}^{p-1}$ to $\mathbf{z}^p$. By the first part of Lemma 4.4, we can construct a supersolution $\bar{\mathbf{x}}$ with $a < \rho^*(\bar{\mathbf{x}}) = \omega_1 < \rho(\bar{\mathbf{x}}) = \omega_2 < \min\{a_1, a_2, b\}$, and by the second part of Lemma 4.4, we have a subsolution $\underline{\mathbf{x}}$ with $\rho(\underline{\mathbf{x}}) = \omega_1 < \omega_2 = \rho^*(\underline{\mathbf{x}})$; hence a supersolution and a subsolution exchanging rotation numbers, which implies that σ on S has positive topological entropy by [3, Theorem 7.1], a contradiction. Consequently, $\rho(K)$ is a single point.

Finally, we show that $\rho^*(\mathbf{y}) = \rho(\mathbf{y})$ for each $\mathbf{y} \in K$. Indeed, we can show as above that $\rho^*(K)$ is a singleton since σ^{-1} also has zero topological entropy. Note that by Lemma 4.1, we have $\omega(\mathbf{y}) \subset K$, and hence there exists $\mathbf{z} \in \omega(\mathbf{y}) \subset K$ such that $\rho^*(\mathbf{z}) = \rho(\mathbf{z})$ by Lemma 3.7, implying $\rho^*(K) = \rho(K)$. □

5. *Proof of Theorem B*

We denote by $\mathbf{y} \sim \mathbf{x}$ if there is a Birkhoff connection from $\mathbf{y}$ to $\mathbf{x}$ and a Birkhoff connection from $\mathbf{x}$ to $\mathbf{y}$. Let $\mathbf{x} \in S_L$ and

$$B(\mathbf{x}) = \{\mathbf{y} \in S_L \mid \mathbf{y} \sim \mathbf{x}\}.$$

LEMMA 5.1. *Let $\mathbf{x} \in \Omega \cap S_L$. Then $B(\mathbf{x})$ is non-empty, invariant, and closed.*

Proof. Since $\mathbf{x} \in S_L$ for some $L > 0$ is a non-wandering point, that is, for each neighborhood U of $\mathbf{x}$, there exists $m \in \mathbb{N}$ such that $\sigma^m(U \cap S_L) \cap U \neq \emptyset$, then there is a Birkhoff connection from $\mathbf{x}$ to itself, and hence $\mathbf{x} \in B(\mathbf{x})$.

For each $\mathbf{y} \in B(\mathbf{x})$, if $\sigma\mathbf{y} \neq \mathbf{x}$, then from Lemma 4.1, it follows that there is a Birkhoff connection from $\mathbf{x}$ to $\sigma\mathbf{y}$ and a Birkhoff connection from $\sigma\mathbf{y}$ to $\mathbf{x}$, implying $\sigma\mathbf{y} \in B(\mathbf{x})$. If $\sigma\mathbf{y} = \mathbf{x}$, then naturally $\sigma\mathbf{y} \in B(\mathbf{x})$. Therefore, $\sigma(B(\mathbf{x})) \subset B(\mathbf{x})$. Similarly, we have $\sigma^{-1}(B(\mathbf{x})) \subset B(\mathbf{x})$, and hence $B(\mathbf{x})$ is invariant for σ.

Let $\{\mathbf{y}^n\} \subset B(\mathbf{x})$ and $\mathbf{y}^n \to \mathbf{y}$ in the product topology as $n \to \infty$. For each neighborhood U of $\mathbf{x}$ and each neighborhood V of $\mathbf{y}$, there exists $N \in \mathbb{N}$ such that $\mathbf{y}^N \in B(\mathbf{x}) \cap V$. We deduce the existence of $n_1, n_2 \in \mathbb{N}$ such that $\sigma^{n_1}(U \cap S_L) \cap V \neq \emptyset$ and $\sigma^{n_2}(V \cap S_L) \cap U \neq \emptyset$, implying $\mathbf{y} \in B(\mathbf{x})$ and hence $B(\mathbf{x}) \subset S_L$ is closed. □

LEMMA 5.2. *Assume σ has zero topological entropy on S and $\mathbf{x} \in \Omega$. Then $\rho(\mathbf{y}) = \rho(\mathbf{x})$ for each $\mathbf{y} \in B(\mathbf{x})$.*

Proof. Since each $\mathbf{y} \in B(\mathbf{x})$ and $\mathbf{x}$ are in the same Birkhoff recurrence class containing $\mathbf{x}$ which is non-empty, we derive the conclusion by Theorem A. □

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For $\delta > 0$, let

$$\mathscr{U}(B(\mathbf{x}), \delta) = \bigcup_{\mathbf{z} \in B(\mathbf{x})} U(\mathbf{z}, \delta),$$

where $U(\mathbf{z}, \delta)$ is defined as in equation (2.2).

LEMMA 5.3. *Let $L > 0$, $\mathbf{x}^n \in \Omega \cap S_L$, and $\mathbf{x}^n \to \mathbf{x}$ as $n \to \infty$ in the product topology. Then for each $\delta > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $B(\mathbf{x}^n) \subset \mathscr{U}(B(\mathbf{x}), \delta)$.*

Proof. We prove by contradiction. Assume the conclusion is not true. Then there exists $\delta_0 > 0$ and a sequence $\mathbf{y}^{n_i} \in B(\mathbf{x}^{n_i}) \subset S_L$ such that $\mathbf{y}^{n_i} \notin \mathscr{U}(B(\mathbf{x}), \delta_0)$, for all $i \geq 1$. There is a convergent subsequence of $\{\mathbf{y}^{n_i}\}$, not relabeled, such that $\lim_{i \to \infty} \mathbf{y}^{n_i} = \mathbf{y} \notin \mathscr{U}(B(\mathbf{x}), \delta_0)$. For each neighborhood U of $\mathbf{x}$ and each neighborhood V of $\mathbf{y}$, there is a sufficiently large n_j such that $\mathbf{x}^{n_j} \in U$ and $\mathbf{y}^{n_j} \in V$. Since $\mathbf{x}^{n_j} \sim \mathbf{y}^{n_j}$, then there exist $m_1 \geq 1$ and $m_2 \geq 1$ such that $\sigma^{m_1}(U \cap S_L) \cap V \neq \emptyset$ and $\sigma^{m_2}(V \cap S_L) \cap U \neq \emptyset$, and hence $\mathbf{x} \sim \mathbf{y}$, which is a contradiction to $\mathbf{y} \notin \mathscr{U}(B(\mathbf{x}), \delta_0)$. □

For $\delta > 0$, let $K \subset S_L$,

$$\mathscr{U}(K, \delta) = \bigcup_{\mathbf{x} \in K} U(\mathbf{x}, \delta) \quad \text{and} \quad \mathscr{O}(K, \delta) = \{\mathbf{y} \in S_L \mid \sigma^n \mathbf{y} \in \mathscr{U}(K, \delta), \quad \text{for all } n \geq 0\}.$$

LEMMA 5.4. *Let $K \subset S_L$ be a compact and invariant set for σ and denote $\langle \rho(K) \rangle = [a, b]$. Then,*

$$\lim_{\delta \to 0} \inf_{\mathbf{y} \in \mathscr{O}(K,\delta)} \rho(\mathbf{y}) = a \quad \textit{and} \quad \lim_{\delta \to 0} \sup_{\mathbf{y} \in \mathscr{O}(K,\delta)} \rho(\mathbf{y}) = b.$$

Proof. Since $\sup \rho(K) = b$, then for each $\varepsilon > 0$ and each $\mathbf{x} = (x_n) \in K$, there exists $n \geq 1$ such that $x_n - x_0 < n(b + \varepsilon)$. It follows from the continuity of σ on S that there exists a neighborhood $U(\mathbf{x}, \gamma)$ with $\gamma > 0$ small enough, such that for each $\mathbf{y} \in U(\mathbf{x}, \gamma) \cap S$, we have $y_n - y_0 < n(b + \varepsilon)$. The compactness of K implies the existence of $N \geq 1$ such that

$$K \subset \bigcup_{j=1}^{N} U(\mathbf{x}^j, \gamma_j),$$

and $y_{n_j} - y_0 < n_j(b + \varepsilon)$ if $\mathbf{y} = (y_n) \in U(\mathbf{x}^j, \gamma_j) \cap S$.

Taking $\delta > 0$ small enough such that

$$\mathscr{U}(K, \delta) \subset \bigcup_{j=1}^{N} U(\mathbf{x}^j, \gamma_j),$$

we deduce that for each $\mathbf{z} = (z_n) \in \mathscr{O}(K, \delta)$ and each $s \geq 1$,

$$\begin{aligned}
&z_{n_{j_1}} - z_0 < n_{j_1}(b + \varepsilon) && \text{for some } j_1 \in \{1, \ldots, N\},\\
&(\sigma^{n_{j_1}} \mathbf{z})_{n_{j_2}} - (\sigma^{n_{j_1}} \mathbf{z})_0 < n_{j_2}(b + \varepsilon) && \text{for some } j_2 \in \{1, \ldots, N\},\\
&\cdots\cdots\\
&(\sigma^{n_{j_1} + \cdots + n_{j_{s-1}}} \mathbf{z})_{n_{j_s}} - (\sigma^{n_{j_1} + \cdots + n_{j_{s-1}}} \mathbf{z})_0 < n_{j_s}(b + \varepsilon) && \text{for some } j_s \in \{1, \ldots, N\}.
\end{aligned}$$

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Let $\mathbf{x} = (x_n)$ be a lift of $\mathbf{z}$. Then $x_i - x_j = z_i - z_j$ for $i, j \in \mathbb{Z}$, and hence

$$x_{n_{j_1}} - x_0 < n_{j_1}(b + \varepsilon), \ldots, x_{n_{j_1}+\cdots+n_{j_s}} - x_{n_{j_1}+\cdots+n_{j_{s-1}}} < n_{j_s}(b + \varepsilon),$$

implying

$$x_{k_s} - x_0 \le k_s(b + \varepsilon),$$

where $k_s = n_{j_1} + \cdots + n_{j_s} > 0$, $s \ge 1$, and $k_0 = 0$. Therefore, it follows that

$$\limsup_{s\to\infty}(x_{k_s} - x_0)/k_s \le b + \varepsilon.$$

Let $m = \max\{n_j \mid j = 1, \ldots, N\}$. For each $n \in \mathbb{N}$, there exists $s \ge 0$ such that $k_s \le n < k_{s+1}$ and $n - k_s \le m$. Note that

$$\frac{x_n - x_0}{n} = \frac{x_n - x_{k_s}}{n} + \frac{x_{k_s} - x_0}{k_s} \cdot \frac{k_s}{n}.$$

Consequently, we obtain that $\limsup_{n\to\infty}(x_n - x_0)/n \le b + 2\varepsilon$ due to the facts

$$|x_n - x_{k_s}| \le mL \quad \text{and} \quad \lim_{n\to\infty} k_s/n = 1,$$

and hence $\sup_{\mathbf{z}\in\mathscr{O}(K,\delta)} \rho(\mathbf{z}) \le b + 2\varepsilon$, leading to the second equality by the fact $K \subset \mathscr{O}(K, \delta)$. The proof for the other equality is similar. □

Proof of Theorem B. Note that each solution of equation (1.1) with bounded action has a well-defined forward rotation number if σ has zero topological entropy on S, see [**29**, Theorem B]. Let $\mathbf{x}^n, \mathbf{x} \in \Omega \cap S_L$, $\mathbf{x}^n \to \mathbf{x}$ as $n \to \infty$, and $\rho(\mathbf{x}) = a$. Note that $B(\mathbf{x}) \subset S_L$ is compact and invariant by Lemma 5.1. Then from Lemma 5.2, it follows that $\rho(\mathbf{y}) = a$ for each $\mathbf{y} \in B(\mathbf{x})$, implying by Lemma 5.4 that for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(\mathbf{y}) \in [a - \varepsilon, a + \varepsilon]$ for each $\mathbf{y} \in \mathscr{O}(B(\mathbf{x}), \delta)$. For this $\delta > 0$, there exists $N \in \mathbb{N}$ due to Lemma 5.3 such that for $n \ge N$, $B(\mathbf{x}^n) \subset \mathscr{U}(B(\mathbf{x}), \delta)$. According to Lemma 5.1, we have $\mathbf{x}^n \in \mathscr{O}(B(\mathbf{x}), \delta)$, and hence $\rho(\mathbf{x}^n) \in [a - \varepsilon, a + \varepsilon]$ for $n \ge N$. □

6. *Proof of Theorem C*

LEMMA 6.1. *If there exists a Birkhoff α-pseudo solution $\mathbf{x}$ of equation (1.1) with $\rho(\mathbf{x}) = \omega$, then there exists $F \in [-\alpha, \alpha]$, such that $\omega \in \rho(\Delta, F)$.*

Proof. Let $\mathbf{x} = (x_n)$ be a Birkhoff α-pseudo solution $\mathbf{x}$ of equation (1.1) with $\rho(\mathbf{x}) = \omega$. Note that for $\omega \in \mathbb{R}$, it follows from [**3**, Theorem 9.1] that there are a Birkhoff configuration $\mathbf{y} = (y_n)$ with $\rho(\mathbf{y}) = \omega$ and some $\lambda \in \mathbb{R}$ satisfying

$$\Delta(y_{n-k}, \ldots, y_{n+l}) = \lambda \quad \text{for all } n \in \mathbb{Z}.$$

If $\lambda \in [-\alpha, \alpha]$, then the proof is complete. If $\lambda > \alpha$, then

$$\Delta(y_{n-k}, \ldots, y_{n+l}) - \alpha > 0 \quad \text{for all } n \in \mathbb{Z}.$$

Note that by Lemma 2.1, we have

$$y_n \le y_0 + n\omega + 1 \quad \text{and} \quad x_0 + n\omega - 1 \le x_n \quad \text{for all } n \in \mathbb{Z}.$$

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Take an integer $m \geq y_0 - x_0 + 2$. Then we have

$$y_n \leq y_0 + n\omega + 1 \leq x_0 - 2 + m + n\omega + 1 \leq x_0 + n\omega - 1 + m \leq x_n + m \quad \text{for all } n \in \mathbb{Z},$$

and hence $\mathbf{y} \leq \mathbf{z} = \mathbf{x} + m \cdot \mathbf{1}$. Combining the assumption

$$\Delta(z_{n-k}, \ldots, z_{n+l}) - \alpha \leq 0 \quad \text{for all } n \in \mathbb{Z},$$

we obtain by [3, Theorem 4.2] a configuration $\mathbf{w} = (w_n)$ satisfying

$$\mathbf{y} \leq \mathbf{w} \leq \mathbf{z} \quad \text{and} \quad \Delta(w_{n-k}, \ldots, w_{n+l}) = \alpha \quad \text{for all } n \in \mathbb{Z},$$

implying $\omega = \rho(\mathbf{w}) \in \rho(\Delta, \alpha)$. The case $\lambda < -\alpha$ is proved similarly. □

LEMMA 6.2. *Assume that $\rho(\Delta)$ is upper-stable with respect to F. Then there exists $\varepsilon_0 > 0$ such that $\rho_\alpha(\Delta) = \rho(\Delta)$ for $0 \leq \alpha < \varepsilon_0$.*

Proof. Let $\varepsilon_0 > 0$ such that $\rho(\Delta, F) \subset \rho(\Delta)$ for $F \in (-\varepsilon_0, \varepsilon_0)$. Let $0 \leq \alpha < \varepsilon_0$ and $\omega \in \rho_\alpha(\Delta)$. We deduce by Theorem 3.10 the existence of a Birkhoff α-pseudo solution $\mathbf{x}$ of equation (1.1) with $\rho(\mathbf{x}) = \omega$ and hence by Lemma 6.1 that $\omega \in \rho(\Delta, F) \subset \rho(\Delta)$ for some $F \in [-\alpha, \alpha] \subset (-\varepsilon_0, \varepsilon_0)$. We conclude that $\rho_\alpha(\Delta) \subset \rho(\Delta)$ and hence $\rho_\alpha(\Delta) = \rho(\Delta)$. □

LEMMA 6.3. *Let $0 \leq \alpha < \alpha'$. Then there exists $\delta' > 0$ such that the following conclusion holds true for $0 < \delta < \delta'$. Assume $\mathbf{x} = (x_n) \in Y$ is an α-pseudo solution of equation (1.1) with bounded action, and there is an integer $q \geq 1$ such that $\sigma^q \mathbf{x} \in U(\mathbf{x}, \delta)$. Then there exist $p \in \mathbb{Z}$ such that $|x_q - x_0 - p| < \delta$, a (p, q)-periodic α'-pseudo solution $\mathbf{y} \in \mathscr{O}(K', \tilde{\delta})$, where $K' = \{\sigma^n \mathbf{x} \mid \text{for all } n \in \mathbb{Z}\}$, $\tilde{\delta} = (k + l + 1)\delta$, and an $F \in [-\alpha', \alpha']$ with $p/q \in \rho(\Delta, F)$.*

Proof. The assumption $\sigma^q \mathbf{x} \in U(\mathbf{x}, \delta)$ implies the existence of a lift $\tilde{\mathbf{x}}$ of $\mathbf{x}$ and $p \in \mathbb{Z}$ such that

$$|\tilde{x}_{i+q} - \tilde{x}_i - p| < \delta, \quad i = -k, \ldots, 0, \ldots, l - 1. \tag{6.1}$$

Let

$$\tilde{y}_j = \tilde{x}_i + mp \quad \text{where } j = i + mq,\ i \in \{0, 1, \ldots, q - 1\},\ m \in \mathbb{Z}. \tag{6.2}$$

Then $\tilde{\mathbf{y}} = (\tilde{y}_j)$ is a (p, q)-periodic configuration. We shall show that $\tilde{\mathbf{y}}$ is an α'-pseudo solution if δ' is small enough.

Let $j = i + mq$ as above and

$$z_{j-k} = \tilde{x}_{i-k} + mp, \ldots, z_j = \tilde{y}_j = \tilde{x}_i + mp, \ldots, z_{j+l} = \tilde{x}_{i+l} + mp. \tag{6.3}$$

The assumption $\mathbf{x}$ is an α-pseudo solution implies that

$$|\Delta(z_{j-k}, \ldots, z_j, \ldots, z_{j+l})| \leq \alpha \quad \text{for all } j \in \mathbb{Z}.$$

In what follows, we shall estimate $|\tilde{y}_{j+n} - z_{j+n}|$ for $n \in \{-k, \ldots, l\}$. First we consider $|\tilde{y}_{j+l} - z_{j+l}|$. Note that

$$j = i + mq,\ 0 \leq i \leq q - 1 \quad \text{and} \quad j + l = i' + m'q \quad \text{where } 0 \leq i' \leq q - 1,\ m' \geq m.$$

If $m' = m$, then $i < i' = i + l \leq q - 1$ and hence

$$\tilde{y}_{j+l} = \tilde{x}_{i'} + mp = z_{j+l}.$$

If $m' > m$, then we choose integers $i_1, i_2, \ldots, i'$ such that

$$i_1 = i + l - q,\ i_2 = i + l - 2q, \ldots, i' = i + l - (m' - m)q,$$

and hence

$$j + l = i_1 + (m+1)q = i_2 + (m+2)q = \cdots = i' + m'q.$$

Note that $-k < 0 \leq i' < \cdots < i_2 < i_1 = i + l - q \leq l - 1$. It follows from equation (6.1) that

$$|z_{j+l} - (\tilde{x}_{i_1} + (m+1)p)| = |\tilde{x}_{i_1+q} - \tilde{x}_{i_1} - p| < \delta,$$
$$|\tilde{x}_{i_1} + (m+1)p - (\tilde{x}_{i_2} + (m+2)p)| = |\tilde{x}_{i_2+q} - \tilde{x}_{i_2} - p| < \delta, \ \ldots, \quad \text{and}$$
$$|\tilde{x}_{i'+q} + (m'-1)p - (\tilde{x}_{i'} + m'p)| = |\tilde{x}_{i'+q} - \tilde{x}_{i'} - p| < \delta.$$

Consequently, we deduce that

$$|z_{j+l} - \tilde{y}_{j+l}| = |z_{j+l} - (\tilde{x}_{i'} + m'p)| < (m' - m)\delta.$$

The estimates for $|z_{j+n} - \tilde{y}_{j+n}|$ for $n \in \{-k, \ldots, l-1\}$ can be obtained similarly. Therefore, we arrive at the conclusion that for $j \in \mathbb{Z}$, $n \in \{-k, \ldots, l\}$,

$$|z_{j+n} - \tilde{y}_{j+n}| < ((k+l)/q + 1)\delta \leq (k + l + 1)\delta = \tilde{\delta}. \tag{6.4}$$

Let $\mathbf{y} = P(\tilde{\mathbf{y}})$. Then $\mathbf{y} \in \mathscr{O}(K', \tilde{\delta})$ is a (p, q)-periodic α'-pseudo solution provided $0 < \delta < \delta'$ and δ' is taken to be small enough. We deduce by Lemma 2.3 the existence of a Birkhoff α'-pseudo solution with rotation number p/q, and hence by Lemma 6.1 that $p/q \in \rho(\Delta, F)$ for some $F \in [-\alpha', \alpha']$. □

LEMMA 6.4. *Let $\mathbf{x} = (x_j) \in Y$ be a Birkhoff configuration with $\rho(\mathbf{x}) = \omega$, $t \in \mathbb{Z}$, and $s \in \mathbb{N}$. Then for each $\varepsilon > 0$, there exists $\mathbf{z} \in \{\sigma^n \mathbf{x} \mid n \in \mathbb{Z}\}$ such that*

$$|z_{i+s} - z_i - t| \leq (k + l)|s\omega - t| + \varepsilon, \quad i = -k, \ldots, l - 1.$$

Proof. Let $\tilde{\mathbf{x}}$ be a lift of $\mathbf{x}$ and $n \in \mathbb{N}$. Since $\tilde{\mathbf{x}}$ is Birkhoff, it follows from Lemma 2.1 that

$$|\tilde{x}_{i+sn} - \tilde{x}_i - sn\omega| \leq 1 \quad \text{and} \quad |\tilde{x}_{i+sn} - nt - \tilde{x}_i| \leq n|s\omega - t| + 1 \quad \text{for all } i \in \mathbb{Z}.$$

Assume $\tau_{-s,-t}\tilde{\mathbf{x}} \geq \tilde{\mathbf{x}}$. The proof for the case $\tau_{-s,-t}\tilde{\mathbf{x}} \leq \tilde{\mathbf{x}}$ is similar. Note that

$$\sum_{i=-k}^{l-1} (\tau_{-s,-t}^n \tilde{\mathbf{x}})_i - \tilde{x}_i = \sum_{i=-k}^{l-1} |\tilde{x}_{i+sn} - nt - \tilde{x}_i| \leq (k + l)(n|s\omega - t| + 1), \tag{6.5}$$

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and

$$0 \leq \sum_{i=-k}^{l-1} (\tau^n_{-s,-t}\tilde{\mathbf{x}})_i - \tilde{x}_i = \sum_{i=-k}^{l-1} \sum_{j=0}^{n-1} (\tau^{j+1}_{-s,-t}\tilde{\mathbf{x}})_i - \tau^j_{-s,-t}\tilde{\mathbf{x}}_i$$
$$= \sum_{j=0}^{n-1} \sum_{i=-k}^{l-1} (\tau^{j+1}_{-s,-t}\tilde{\mathbf{x}})_i - (\tau^j_{-s,-t}\tilde{\mathbf{x}})_i.$$

We deduce by the drawer principle and equation (6.5) the existence of $j_0 \in \{0, 1, \ldots, n-1\}$ such that

$$0 \leq \sum_{i=-k}^{l-1} (\tau^{j_0+1}_{-s,-t}\tilde{\mathbf{x}})_i - (\tau^{j_0}_{-s,-t}\tilde{\mathbf{x}})_i \leq (k+l)(|s\omega - t| + 1/n).$$

Taking n large enough and denoting $\tilde{\mathbf{z}} = \tau^{j_0}_{-s,-t}\tilde{\mathbf{x}}$ and $\mathbf{z} = P(\tilde{\mathbf{z}})$, we obtain

$$\sum_{i=-k}^{l-1} |z_{i+s} - z_i - t| \leq (k+l)|s\omega - t| + \varepsilon. \qquad \square$$

LEMMA 6.5. *Let $0 \leq \alpha < \alpha'$. Then there exists an integer $q_0 > 0$ such that if $p/q \in \rho_\alpha(\Delta)$ in lowest terms with $q > q_0$, then for each t/s with $t, s \in \mathbb{Z}$, $s > 0$, and $qt - ps = \pm 1$, there exists $F \in [-\alpha', \alpha']$ satisfying $t/s \in \rho(\Delta, F)$.*

Proof. Let $\delta' > 0$ be determined by Lemma 6.3 and q_0 be an integer with $q_0 \geq (k + l + 1)/\delta'$. Then for $p/q \in \rho_\alpha(\Delta)$ in lowest terms with $q > q_0$, there exists by Theorem 3.10 a (p, q)-periodic Birkhoff α-pseudo solution $\mathbf{x}$. We deduce by Lemma 6.4 the existence of $\mathbf{z} \in \{\sigma^n\mathbf{x} \mid n \in \mathbb{Z}\}$ for $\varepsilon = 1/q$ such that

$$|z_{i+s} - z_i - t| \leq (k+l)|sp/q - t| + 1/q \leq (k+l+1)/q = \delta < \delta', \quad i = -k, \ldots, l-1,$$

implying $\sigma^s\mathbf{z} \in U(\mathbf{z}, \delta)$ and hence by Lemma 6.3 that $t/s \in \rho(\Delta, F)$ for some $F \in [-\alpha', \alpha']$. $\square$

We say that $[p'/q', p/q]$ is a Farey interval if $q'p - qp' = 1$, where $p'/q', p/q \in \mathbb{Q}$ in lowest terms.

Proof of Theorem C. Since $\rho(\Delta)$ is upper-stable with respect to F, there exists $\varepsilon_0 > 0$ such that $\rho(\Delta, F) \subset \rho(\Delta)$ for $F \in (-\varepsilon_0, \varepsilon_0)$. Let $0 = \alpha < \alpha' < \varepsilon_0$, and $\delta' > 0$ and q_0 be determined by Lemmas 6.3 and 6.5, respectively.

Assume $p/q \in \rho(\Delta)$ in lowest terms with $q > q_0$. Then it follows from Lemma 6.5 that $t'/s', t/s \in \rho(\Delta)$ where $t, s, t', s' \in \mathbb{Z}$ and $s > 0$, $s' > 0$ satisfying $qt - ps = 1$ and $qt' - ps' = -1$.

Note that both $[t'/s', p/q]$ and $[p/q, t/s]$ are Farey intervals. If we denote by $p'/q' = (p+t)/(q+s)$ the mediant of p/q and t/s, then we deduce by Lemma 6.5 that $p'/q' \in \rho(\Delta)$.

Applying Lemma 6.5 again, we obtain that both the mediant of p'/q' and t/s and the mediant of p/q and p'/q' lie in $\rho(\Delta)$ since $q' > q_0$. Therefore, by induction all rational numbers (see [16]) in $[p/q, t/s]$ are in $\rho(\Delta)$ and hence $[p/q, t/s] \subset \rho(\Delta)$ since $\rho(\Delta)$ is

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closed (see [29, Theorem A]). Similarly, we have $[t'/s', p/q] \subset \rho(\Delta)$ and hence p/q is in the interior of $\rho(\Delta)$.

Let $\omega \in \rho(\Delta)$ be irrational and $\mathbf{x} \in S$ be the corresponding Birkhoff solution with $\rho(\mathbf{x}) = \omega$ by Theorem 3.10. Take two consecutive convergents of ω, p'/q' and p/q, such that $p'/q' < \omega < p/q, q > q_0$, and $|q\omega - p| < 1/q$. Then it follows from Lemma 6.4 that for $\varepsilon = 1/q$, there exists $\mathbf{z} \in \{\sigma^n \mathbf{x} \mid n \in \mathbb{Z}\}$ such that

$$|z_{i+q} - z_i - p| \le (k+l)|q\omega - p| + 1/q \le (k+l+1)/q < \delta', \quad i = -k, \ldots, l-1,$$

and hence $p/q \in \rho(\Delta, F) \subset \rho(\Delta)$ for some $F \in [-\alpha', \alpha']$ by Lemma 6.3. We then arrive at the conclusion that $p'/q' \in \rho(\Delta, F) \subset \rho(\Delta)$ for some $F \in [-\alpha', \alpha']$ by Lemma 6.5, and hence the Farey interval $[p'/q', p/q] \subset \rho(\Delta)$ by repeating the previous argument, implying ω is in the interior of $\rho(\Delta)$. This completes the proof. □

7. *Proof of Theorem D*

The conclusion of Theorem D is a straightforward consequence of the following theorem by setting $\alpha = 0$.

THEOREM 7.1. *Assume $\rho(\Delta)$ is upper-stable with respect to F and $[a, b]$ is a connected component of $\rho(\Delta)$. Then there exists $\varepsilon_0 > 0$, such that for $0 \le \alpha < \varepsilon_0$, $L > 0$, it follows that for each compact and σ-invariant set $K \subset B_L$ with $\langle \rho(K) \rangle \subset [a, b]$, there exists $M > 0$, such that for each $\mathbf{x} = (x_n) \in K$ which is an α-pseudo solution of equation (1.1),*

$$x_n - x_0 - nb \le M \quad \textit{and} \quad x_n - x_0 - na \ge -M \quad \textit{for all } n \ge 1.$$

Proof. From Theorem C, we know that the boundary points of $\rho(\Delta)$ are isolated, implying the connected component $[a, b]$ is isolated. By Lemma 6.2, we deduce the existence of $\varepsilon_0 > 0$ such that $\rho_\alpha(\Delta) = \rho(\Delta)$ for $0 \le \alpha < \varepsilon_0$.

Let $0 \le \alpha < \alpha' < \varepsilon_0$, $\delta' > 0$ be defined by Lemma 6.3, and $0 < \delta < \delta'$. Since K is compact, there exist $k_0 \in \mathbb{N}$, $\mathbf{z}^1, \ldots, \mathbf{z}^{k_0} \in K$ such that $K \subset \bigcup_{i=1}^{k_0} U(\mathbf{z}^i, \delta/2)$.

Let $\mathbf{x} = (x_j) \in K$ be an α-pseudo solution with $\rho(\mathbf{x}) \subset \langle \rho(K) \rangle \subset [a, b]$ and $n \in \mathbb{N}$. Define a sequence of integers $0 = q_0 < q_1 < \cdots < q_m = n$ recursively as follows. Let q_1 be the smallest number of $\{1, \ldots, n\}$ such that $\sigma^j \mathbf{x} \notin U(\mathbf{x}, \delta)$ for $q_1 \le j \le n$. If $\sigma^n \mathbf{x} \in U(\mathbf{x}, \delta)$, set $q_1 = n$. Assume q_i has been defined and $q_i < n$. Define q_{i+1} as the smallest element of $\{q_i + 1, \ldots, n\}$ such that

$$\sigma^j \mathbf{x} \notin U(\sigma^{q_i} \mathbf{x}, \delta) \quad \text{for } q_{i+1} \le j \le n.$$

If $\sigma^n \mathbf{x} \in U(\sigma^{q_i} \mathbf{x}, \delta)$, set $q_{i+1} = n$ and then $m = i + 1$.

The sequence $\{q_0, q_1, \ldots, q_m\}$ has the property that

$$\sigma^{q_j} \mathbf{x} \notin U(\sigma^{q_i} \mathbf{x}, \delta) \quad \text{for } 0 \le i < j \le m-1. \tag{7.1}$$

We claim that $m \le k_0$. Indeed, since $\mathbf{x} \in K \subset \bigcup_{i=1}^{k_0} U(\mathbf{z}^i, \delta/2)$, we may assume without of loss of generality that $\mathbf{x} \in U(\mathbf{z}^1, \delta/2)$. Then $\sigma^{q_i} \mathbf{x} \notin U(\mathbf{z}^1, \delta/2)$ for $i = 1, \ldots, m-1$. Otherwise, due to $\mathbf{x} \in U(\mathbf{z}^1, \delta/2)$, we shall have $\sigma^{q_i} \mathbf{x} \in U(\mathbf{x}, \delta)$, a contradiction to equation (7.1).

We assume without loss of generality again that $\sigma^{q_1} \mathbf{x} \in U(\mathbf{z}^2, \delta/2)$. Then we deduce that $\sigma^{q_i} \mathbf{x} \notin U(\mathbf{z}^2, \delta/2)$ for $2 \le i \le m-1$ with the same reason as above, leading to the conclusion that $\sigma^{q_2}\mathbf{x} \notin U(\mathbf{z}^1, \delta/2) \cup U(\mathbf{z}^2, \delta/2)$. Inductively, we conclude that

$$\sigma^{q_{m-1}}\mathbf{x} \notin \bigcup_{i=1}^{k_0} U(\mathbf{z}^i, \delta/2) \supset K$$

if we assume $m \geq k_0 + 1$, which is a contradiction to that K is invariant for σ. Therefore, $m \leq k_0$.

For $0 \leq i \leq m-2$, if $q_{i+1} - q_i \geq 2$, since $\sigma^{q_{i+1}-1-q_i}(\sigma^{q_i}\mathbf{x}) \in U(\sigma^{q_i}\mathbf{x}, \delta)$, applying Lemma 6.3, we obtain $p_i \in \mathbb{Z}$ such that

$$|x_{q_{i+1}-1} - x_{q_i} - p_i| < \delta \quad \text{and} \quad r_i = \frac{p_i}{q_{i+1} - 1 - q_i} \in \rho(\Delta). \tag{7.2}$$

Furthermore, we deduce that $r_i \in [a, b]$. Indeed, from Lemma 6.3, we know that the corresponding $(p_i, q_{i+1} - 1 - q_i)$-periodic configuration $\mathbf{y}$ constructed by Lemma 6.3 lies in $\mathscr{O}(K', \tilde{\delta})$, where $\tilde{\delta} = (k + l + 1)\delta$, and $K' = \{\sigma^n\mathbf{x} \mid \text{for all } n \in \mathbb{Z}\}$, implying $\mathbf{y} \in \mathscr{O}(K, \tilde{\delta})$. Then Lemma 5.4 and the assumption $\langle\rho(K)\rangle \subset [a, b]$ imply that

$$r_i \in [a - \varepsilon, b + \varepsilon] \cap \rho(\Delta)$$

for arbitrarily small ε if we choose δ small enough. Since $[a, b]$ is an isolated component of $\rho(\Delta)$, we choose a smaller δ which is independent of $\mathbf{x}$ if necessary such that $r_i \in [a, b]$.

As a consequence of equation (7.2), we have $-\delta < x_{q_{i+1}-1} - x_{q_i} - p_i < \delta$, and hence

$$-\delta + a(q_{i+1} - q_i - 1) < x_{q_{i+1}-1} - x_{q_i} < \delta + b(q_{i+1} - q_i - 1). \tag{7.3}$$

Combining $-L \leq x_{q_{i+1}} - x_{q_{i+1}-1} \leq L$, we derive for $0 \leq i \leq m-2$,

$$-L - \delta + a(q_{i+1} - q_i - 1) \leq x_{q_{i+1}} - x_{q_i} \leq L + \delta + b(q_{i+1} - q_i - 1). \tag{7.4}$$

Note that equation (7.4) also holds for the case $q_{i+1} - q_i = 1$.

For $i = m-1$, there are two cases. One is that $\sigma^{q_{i+1}}\mathbf{x} \in U(\sigma^{q_i}\mathbf{x}, \delta)$, the other is that $\sigma^{q_{i+1}}\mathbf{x} \notin U(\sigma^{q_i}\mathbf{x}, \delta)$. For the former case, we have by the same discussion as above

$$-\delta + a(q_{i+1} - q_i) < x_{q_{i+1}} - x_{q_i} < \delta + b(q_{i+1} - q_i).$$

For the latter case, we must have $\sigma^{q_{i+1}-1}\mathbf{x} \in U(\sigma^{q_i}\mathbf{x}, \delta)$ according to the definition of q_{i+1}, and again we have equations (7.3) and (7.4).

Consequently, we have for $n \in \mathbb{N}$,

$$x_n - x_0 - nb \leq m(L + \delta - b) \quad \text{or} \quad x_n - x_0 - nb \leq m(L + \delta - b) + b,$$

and

$$x_n - x_0 - na \geq -m(L + \delta + a) \quad \text{or} \quad x_n - x_0 - na \geq -m(L + \delta + a) + a.$$

Taking $M = \max\{k_0(L + 1 - b) + |b|, k_0(L + 1 + a) + |a|\}$, we complete the proof. □

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A. *Appendix. Proof of Lemma 3.8*

The proof is a slight modification of that of [29, Lemma 4.1]. Lemma 4.2 in [29] says that if we already have a β-pseudo solution $\mathbf{z} = (z_i)$ with $\sup_{i \geq 0} |z_i - z_0 - i\rho| < \infty$, then there exists a β-pseudo solution $\mathbf{u} = (u_n)$ satisfying $\sup_{n \in \mathbb{Z}} |u_n - u_0 - n\rho| < \infty$.

By Lemma 3.4, there exist $\mathbf{y}^1, \mathbf{y}^2 \in K$, such that

$$y_i^1 - y_0^1 - i\rho \leq 1 \quad \text{and} \quad y_i^2 - y_0^2 - i\rho \geq -1 \quad \text{for all } i \in \mathbb{N}.$$

If there exists $C > 0$ such that for all $i \in \mathbb{N}$,

$$-C \leq y_i^1 - y_0^1 - i\rho \leq 1 \quad \text{or} \quad -1 \leq y_i^2 - y_0^2 - i\rho \leq C,$$

then we can construct an α-pseudo solution $\mathbf{z} = (z_n)$ with $\sup_{n \in \mathbb{Z}} |z_n - z_0 - n\rho| < \infty$ by [29, Lemma 4.2]. Otherwise, for each $C > 0$, there exist $i, j \in \mathbb{N}$ such that

$$y_i^1 - y_0^1 - i\rho < -C \quad \text{and} \quad y_j^2 - y_0^2 - j\rho > C. \tag{A.1}$$

We shall construct a β-pseudo solution $\mathbf{z} = (z_n)$ with $\sup_{n \geq 0} |z_n - z_0 - n\rho| < \infty$ for each $\beta > \alpha$.

Given $\beta > \beta' > \alpha \geq 0$, there exists $0 < \delta < \min\{1, L\}$ such that for arbitrary two β'-pseudo solutions in $\tilde{B}_L$, their δ-gluing is a β-pseudo solution. This is a straightforward consequence of the uniform continuity of Δ on B_L. In fact, we can glue more than two β'-pseudo solutions to obtain a β-pseudo solution, as we do in what follows.

For each $\mathbf{w} = (w_n) \in K$, let $U(\mathbf{w}, \delta/2)$ be defined as in equation (2.2). Then $\{U(\mathbf{w}, \delta/2)\}_{\mathbf{w} \in K}$ is an open cover of K and hence it has a finite subcover, say, $\{U_i \mid i = 3, \ldots, q\}$ since $K \subseteq Y$ is compact. Denote $U_1 = U(\mathbf{y}^1, \delta/2)$ and $U_2 = U(\mathbf{y}^2, \delta/2)$.

Since K is chain transitive, we can construct by Lemma 3.3 a β'-pseudo solution $\mathbf{y} = (y_n) \in B_L$ with the following properties.

(i) $\mathbf{y} \in U_q$. Denote $\mathbf{y}$ by $\mathbf{u}^q$.

(ii) $\sigma^{m_i}(\mathbf{u}^i) = \mathbf{u}^{i-1} \in U_{i-1}$ with $m_i > k + l$, $i = 2, 3, \ldots, q$, that is, $\mathbf{y} = \mathbf{u}^q \to \cdots \to \mathbf{u}^3 \to \mathbf{u}^2 \to \mathbf{u}^1 \in U_1$.

We remark that there is an integer $N > 0$ such that

$$k + l < m_2 + \cdots + m_q \leq N - 1 \quad \text{and} \quad N \geq 2 + 2/L. \tag{A.2}$$

Take a lift of $\mathbf{y}^1 \in K$, $\mathbf{z}^1 = (z_n^1) \in X$ with $z_0^1 \in [0, 1]$. Note that $z_i^1 - z_j^1 = y_i^1 - y_j^1$ for all $i, j \in \mathbb{Z}$. Let

$$z_n = z_n^1 \quad \text{for all } n \leq 0. \tag{A.3}$$

Choose $C_1 = 2NL$. Note that

$$z_j^1 - z_0^1 - j\rho \geq -jL - j|\rho| \geq -(k+l)L - (k+l)L \geq -2NL \quad \text{for } j = 1, \ldots, k+l.$$

Then by equation (A.1), there exists $j_1 > k + l$ such that

$$z_{j_1}^1 - z_0^1 - j_1\rho < -C_1 = -2NL,$$

and

$$-2NL = -C_1 \leq z_j^1 - z_0^1 - j\rho \leq 1 \quad \text{for } j = 1, \ldots, j_1 - 1. \tag{A.4}$$

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Since $|z^1_{j_1} - z^1_{j_1-1}| \le L$ and $|\rho| \le L$, then by equation (A.4)

$$z^1_{j_1} - z^1_0 - j_1\rho = (z^1_{j_1} - z^1_{j_1-1}) + (z^1_{j_1-1} - z^1_0 - (j_1-1)\rho) - \rho \ge -L - C_1 - L \ge -3NL.$$

As a consequence, we have

$$-3NL \le z^1_{j_1} - z^1_0 - j_1\rho < -2NL. \tag{A.5}$$

Let

$$z_n = z^1_n \quad \text{for } n = 1, \ldots, j_1 - 1. \tag{A.6}$$

Since $\sigma^{j_1}\mathbf{y}^1 \in K$, there exists some $i \in \{3, \ldots, q\}$ such that $\sigma^{j_1}\mathbf{y}^1 \in U_i$. We may assume $i = 3$. The construction is the same for $i = 4, \ldots, q$.

Step 1: Note that $P(\tau_{-j_1,0}\mathbf{z}^1) = \sigma^{j_1}\mathbf{y}^1 \in U_3$ and $\mathbf{u}^3 \in U_3$. We take a lift $\mathbf{x}^3 = (x^3_n) \in X$ of $\mathbf{u}^3$, that is, $P(\mathbf{x}^3) = \mathbf{u}^3$, such that

$$|(\tau_{-j_1,0}\mathbf{z}^1)_n - x^3_n| < \delta \quad \text{for } n = -k, \ldots, l-1, \tag{A.7}$$

and then we obtain a δ-gluing of $\tau_{-j_1,0}\mathbf{z}^1$ and $\mathbf{x}^3$ as follows. Note that due to equation (A.6),

$$z_{j_1+n} = (\tau_{-j_1,0}\mathbf{z}^1)_n \quad \text{for } n = -j_1+1, \ldots, -1.$$

Take $n_1 = m_3 > k + l$ such that $\sigma^{n_1}(\mathbf{u}^3) = \mathbf{u}^2$. Let

$$z_{j_1+n} = x^3_n \quad \text{for } n = 0, 1, \ldots, n_1 - 1. \tag{A.8}$$

Then we obtain z_n for $n = 1, \ldots, j_1, \ldots, j_1 + n_1 - 1$ by gluing $\tau_{-j_1,0}\mathbf{z}^1$ and $\mathbf{x}^3$ at site j_1. Since $\mathbf{x}^3 \in \tilde{B}_L$, then by equations (A.6) and (A.7), with $n = 0$, we have for $1 \le n \le n_1 - 1 \le N - 1$,

$$-(n_1-1)L \le z_{j_1+n} - z_{j_1} = x^3_n - x^3_0 \le (n_1-1)L.$$

Combining equation (A.5), we derive

$$-\delta - 3NL \le z_{j_1} - z_0 - j_1\rho = x^3_0 - z^1_{j_1} + z^1_{j_1} - z^1_0 - j_1\rho \le \delta - 2NL,$$

and hence

$$\begin{aligned}-(n_1-1)L - \delta - 3NL - n|\rho| &\le z_{j_1+n} - z_0 - (j_1+n)\rho \\ &= (z_{j_1+n} - z_{j_1}) + (z_{j_1} - z_0 - j_1\rho) - n\rho \\ &\le (n_1-1)L + \delta - 2NL + n|\rho|,\end{aligned}$$

for $0 \le n \le n_1 - 1 \le N - 1$. Note that $|\rho| \le L$. It follows that

$$-5NL \le z_{j_1+n} - z_0 - (j_1+n)\rho \le 0 \quad \text{for } n = 0, 1, \ldots, n_1 - 1. \tag{A.9}$$

Steps 2–4 are similar to those in the proof of [29, Lemma 4.1] and hence omitted here. We know that these four steps can be repeated. Therefore, $\mathbf{z} = (z_n)$ is a β-pseudo solution satisfying $\sup_{n\ge 0} |z_n - z_0 - n\rho| < M$, where $M = 6NL$ (see the proof of [29, Lemma 4.1]). We use [29, Lemma 4.2] to complete the proof. □

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