A THEOREM ON HENSELIAN RINGS

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It is known that if K is a field, then the ring of formal power series in one or more variables, with coefficients in K, is Henselian at its maximal ideal. In this note we show that if R is a ring (commutative and with identity element) which is Henselian at the maximal ideals $M_1, M_2, \ldots$ then $R[[x]]$ - the ring of formal power series in x with coefficients from R - is also Henselian at the maximal ideals $M_1 \cdot R[[x]] + x \cdot R[[x]]$, etc.

DEFINITION. A ring R is said to be Henselian at a maximal ideal M if for each monic polynomial $f(y) \varepsilon R[y]$ whose image $\bar{f}(y)$ in $\frac{R}{M}[y] = k[y]$ factors into a product $\bar{g}(y) \cdot \bar{h}(y)$ where $\bar{g}(y)$ and $\bar{h}(y)$ are both monic and relatively prime, there exist monic polynomials g(y) and h(y) in R[y] which are mutually prime, and satisfy

(i) $f(y) = g(y) . h(y)$

(ii) $g(y) \equiv \bar{g}(y) \pmod{M}$ and $h(y) \equiv \bar{h}(y) \pmod{M}$.

LEMMA 1. *Let f(y), g(y) be two polynomials in* R[y] *where at least one of them is monic. Then the following statements are equivalent.*

(i) g(y) *and* h(y) *are relatively prime in* R[y].

(ii) $\bar{g}(y)$ *and* $\bar{h}(y)$ *are relatively prime in* $\frac{R}{M}[y]$ *where* M *is a maximal ideal of* R.

For the proof of this lemma, we refer the reader to Lafon [1].

THEOREM 1. *If the ring* R *is Henselian at a maximal ideal* M (*say*) *then* R[[x]] *is Henselian at the maximal ideal* $M.R[[x]] + x.R[[x]] = m$.

Proof. Let f(y) in R[[x]][y] be monic and of degree n such that $\bar{f}(y) = \bar{g}(y).\bar{h}(y)$ where $\bar{f}(y) \equiv f(y) \pmod{m}$, and $\bar{g}(y), \bar{h}(y)$ be relatively prime and monic.

We now rewrite $f(y) = f_o(y) + f_1(y).x + f_2(y).x^2 + \ldots$ where $f_o(y)$ is a monic polynomial of degree n over R and $f_i(y)$

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are polynomials over R of degree at most $n-1$. Now to prove the theorem observe that it is enough to establish the following: if $f_o(y) = g_o(y).h_o(y)$ where $g_o(y)$ and $h_o(y)$ are both monic in $R[y]$ and of degrees s and t respectively such that $s+t = n$ and $g_o(y)$ and $h_o(y)$ are relatively prime in $R[y]$ then there exist polynomials $\{g_i(y)\}$ and $\{h_j(y)\}$ of degrees at most $s-1$ and $t-1$ respectively, such that

$$(1) \qquad \sum_{s=0}^{\infty} f_s(y).x^s = \sum_{s=0}^{\infty} \sum_{i+j=s} g_i(y).h_j(y).x^s .$$

Now from the Henselian nature of R and from the assumption on $\bar{g}(y)$, $\bar{h}(y)$ we can get $g_o(y)$ and $h_o(y)$ in $R[y]$ such that these are monic. These are also relatively prime by lemma 1. Consequently it is possible to find $g_1(y)$ and $h_1(y)$ such that

$$f_1(y) = g_o(y).h_1(y) + h_o(y).g_1(y)$$

with degree of $g_1(y) \leq (s-1)$ and the degree of $h_1(y) \leq (t-1)$. Continuing the above procedure we can inductively construct $\{g_i(y)\}$ and $\{h_j(y)\}$ satisfying (1).

$$\text{Now set } g(y) = \sum_{i=0}^{\infty} g_i(y).x^i \text{ and } h(y) = \sum_{j=0}^{\infty} h_j(y).x^j .$$

Then it is clear from our construction that

$$g(y) \equiv \bar{g}(y) \pmod{m},$$

and

$$h(y) \equiv \bar{h}(y) \pmod{m}.$$

We remark that the above theorem is no longer valid if we take the polynomial ring instead of the power series ring. For example, take a field K. Trivially it is Henselian while the polynomial ring over K is not.

COROLLARY 1. If the ring R is Henselian at all its maximal ideals then $R[[x]]$ is also Henselian at all its maximal ideals.

For the proof observe that there is a one to one correspondence between the maximal ideals of R and the maximal ideals of $R[[x]]$. This observation in conjunction with theorem 1 yields the corollary.

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COROLLARY 2. If R is Henselian at M, then $R[[x_1, \ldots, x_n]]$ is also Henselian at the maximal ideal lying over M.

Proof of this follows by induction on the number of variables.

However, if we confine ourselves to restricted power series rings (see [2] for definition) the above result (theorem 1) no longer holds. For example let K be a field, endowed with the discrete topology. Then K is Henselian. The restricted power series ring over K in the variable x (or more generally $x_1, x_2, \ldots, x_n$) is $K[x]$ (or $K[x_1, \ldots, x_n]$) and this is not Henselian at the maximal ideal (x) $((x_1 \ldots x_n))$ that lies over the zero ideal.

REFERENCES

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2. P. Salmon, Sur les sẽries formelles restreintes Bull. Soc. Math. France, 92 (1964) 385-410.

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