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LETTER TO THE EDITOR

Dear Editor,

Extension of Deltheil's study on random points in a convex quadrilateral

Deltheil's 1926 treatise [4] is sometimes cited in the context of Sylvester's famous four-point problem. The problem, finding the probability, p_4, that four uniformly distributed points within a planar convex body, K, have a triangular convex hull, has been solved – according to the research literature since 1964 – for only a few varieties of body, namely triangles, ellipses, parallelograms, and regular polygons. Whilst Delthiel's name is sometimes linked with these solutions and also with certain extremal issues, it has apparently been forgotten that his work gives a simple expression for p_4 when K is a general convex quadrilateral. In this note, we extend Deltheil's result and, as a pleasant ramification, draw attention to his forgotten study.

For n points, the probability p_n equals $\binom{n}{3}\,\mathrm{E}(A_3^{n-3})/|K|^{n-3}$, where A_n is the area of the convex hull formed by n points uniformly and independently distributed within a convex quadrilateral (see [5]). We report $\mathrm{E}(A_3^k)$ and $\mathrm{E}(A_n)$ for some small values of n and k, extending work of Deltheil. Hence, we focus on the affine-invariant moments $\mathrm{E}(A_n^k)/|K|^k$.

Let K be a convex quadrilateral $ABCD$, whose diagonal AC is cut by the other diagonal BD into two segments of ratio $a:1$, with BD in turn being divided in the ratio $b:1$. Using a straightforward analysis aided by symbolic calculations, we have derived the following formulae:

$$\mathrm{E}\left(\frac{A_3}{|K|}\right) = \frac{1}{12} - \frac{ab}{9(1+a)^2(1+b)^2}, \qquad \mathrm{E}\left(\frac{A_3^2}{|K|^2}\right) = \frac{1}{72} - \frac{ab}{18(1+a)^2(1+b)^2},$$

$$\mathrm{E}\left(\frac{A_3^3}{|K|^3}\right) = \frac{31}{9000} - \frac{ab(132ab + 74(a+b)(1+ab) + 41(1+a^2)(1+b^2))}{1500(1+a)^4(1+b)^4},$$

$$\mathrm{E}\left(\frac{A_3^4}{|K|^4}\right) = \frac{1}{900} - \frac{ab(28ab + 20(a+b)(1+ab) + 13(1+a^2)(1+b^2))}{900(1+a)^4(1+b)^4}.$$

Only $\mathrm{E}(A_3)/|K|$ was found by Deltheil (who expressed it using a different parametrization). Our analysis and further discussion can be found in [3].

Our (a,b)-quadrilateral collapses to a triangle when either a or b equals 0, and our leading terms agree with known results for triangles given by Reed [6]. Other special cases are $a=1$, yielding a (possibly skewed) kite; $a=b$, creating a trapezium; and $a=b=1$, creating a parallelogram. Our results do not agree with Reed's parallelogram formula. Instead, we agree with a formula of Trott, recently reported by Weisstein (see http://mathworld.wolfram.com/SquareTrianglePicking.html):

$$\mathrm{E}\left(\frac{A_3^k}{|K|^k}\right)_{\text{parallelogram}} = \frac{3(1+(k+2)\sum_{r=1}^{k+1} r^{-1})}{(1+k)(2+k)^3(3+k)^2 2^{k-3}}.$$

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We have also found some results for $\mathrm{E}(A_n)$ when $n > 3$. Buchta [1] showed that, for general K, $\mathrm{E}(A_4) = 2\,\mathrm{E}(A_3)$. We supplement this with

$$\mathrm{E}\left(\frac{A_5}{|K|}\right) = \frac{43}{180} - \frac{ab(108ab + 56(a+b)(1+ab) + 29(1+a^2)(1+b^2))}{90(1+a)^4(1+b)^4},$$
$$\mathrm{E}\left(\frac{A_6}{|K|}\right) = \frac{3}{10} - \frac{ab(124ab + 68(a+b)(1+ab) + 37(1+a^2)(1+b^2))}{90(1+a)^4(1+b)^4}.$$

Using Buchta's more recent theory [2] in combination with our results, we find that N_5, the number of sides of the convex hull when $n = 5$, takes the values 3, 4, and 5 with probabilities $\frac{5}{36} - \psi$, $\frac{5}{9}$, and $\frac{11}{36} + \psi$, respectively. Here,

$$\psi := \frac{5ab}{9(1+a)^2(1+b)^2}.$$

It is intriguing that the probability of this convex hull being four-sided does not depend on a or b.

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Yours sincerely,

Richard Cowan* and S. N. Chiu**

* School of Mathematics and Statistics,
University of Sydney,
NSW,
2006,
Australia.
Email address: rcowan@mail.usyd.edu.au

** Department of Mathematics,
Hong Kong Baptist University,
Kowloon,
Hong Kong.

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