SUBORDINATE AND PSEUDO-SUBORDINATE SEMI-ALGEBRAS. II

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1. Introduction. This paper is a sequel to (**1**), to which the reader is referred for definitions and known results. As before, E is a compact Hausdorff space and $C^+(E)$ is the semi-algebra of all continuous non-negative functions defined on E. Recall that, for a uniformly closed subsemi-algebra A of $C^+(E)$, the semi-algebra A_u is the uniform closure of the set $\{f_1 \cup f_2 \cup \ldots \cup f_k : f_i \in A$, k a positive integer$\}$ where $\cup$ denotes the pointwise supremum operation; the semi-algebra A is *pseudo-subordinate* if and only if $A_u \neq C^+(E)$. It was conjectured in (**1**) that every proper closed subsemi-algebra of $C^+(E)$ is pseudo-subordinate. My aim in this note is to provide a counter-example for the conjecture. In addition, two other results are proved: one giving a peak point characterization of pseudo-subordinate semi-algebras, the second showing that for finitely generated closed semi-algebras the property of being pseudo-subordinate is equivalent to the property of being subordinate (i.e., contained in a maximal closed subsemi-algebra of $C^+(E)$). The latter result is a small step towards discovering whether every proper finitely generated closed subsemi-algebra is subordinate; cf. (**1**, Theorem 8).

2. A characterization of pseudo-subordinate semi-algebras. The proof of one of the implications in the following theorem is due essentially to Bishop and de Leeuw. Following (**2**, p. 49), we say that the semi-algebra $A \subseteq C^+(E)$ *satisfies Condition* II *at the point* $\xi \in E$ if and only if, given any G_δ-set S containing ξ, there exists a function $f \in A$ such that $f(\xi) = ||f||$ (the uniform norm) and f attains its maximum value only within S.

THEOREM 1. *Let A be a uniformly closed subsemi-algebra of $C^+(E)$. Then $A_u = C^+(E)$ if and only if A satisfies Condition* II *at each point ξ of E.*

Proof. If $A_u \neq C^+(E)$, then by (**1**, Theorem 5, Corollary), there exists a point ξ of E and a positive measure μ on E with no mass at ξ such that $f(\xi) \leqq \int f\, d\mu$ ($\forall f \in A$). Choose an open neighbourhood U of ξ such that $\mu(U) < \frac{1}{2}$. Then, for any function $g \in A$ with $1 = g(\xi) = ||g||$, $g^n \in A$ and

$$1 < \tfrac{1}{2} + \int_{\setminus U} g^n\, d\mu \qquad (n = 1, 2, \ldots),$$

whence one deduces that g must attain its maximum on $\setminus U$. Hence Condition II is sufficient for a semi-algebra to be non-pseudo-subordinate.

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Suppose that $A_u = C^+(E)$. For $0 < \epsilon < 1$, $\xi \in E$, and U an open neighbourhood of ξ, choose $g \in C^+(E)$ such that $||g|| = 1$, $g(\xi) = 1$, and $g(\backslash U) = \{0\}$. Since $g \in A_u$, there exist $f_i \in A$ with $||g - f_1 \cup f_2 \cup \ldots \cup f_k|| < \epsilon(2-\epsilon)^{-1}$. One of these functions, f_1 say, satisfies $f_1(\xi) > 1 - \epsilon(2-\epsilon)^{-1}$. This function must also satisfy $||f_1|| \leqq 2(2-\epsilon)^{-1}$ and $f_1(\backslash U) \subseteq [0, \epsilon]$. Taking

$$f = \tfrac{1}{2}(2-\epsilon)f_1,$$

we see that Condition I (**2**, p. 49) holds for A. Hence Condition II holds, since the proof given in (**2**, p. 51), involves only operations permissible in a semi-algebra.

COROLLARY. *If E is a metric space, then $A_u = C^+(E)$ if and only if every point of E is a peak point for A.*

3. The counter-example. Let $X_1X_2X_3X_4$ be a square of area 1 in the Euclidean plane, and $a \equiv A_1A_2$ a segment of length $\alpha \in [0, 1]$ contained in the side $X_1A_1A_2X_2$. The *pinnacle* of a is the unique point A_0 on X_1X_2 such that X_1A_0: A_0X_2: :A_1A_0: A_0A_2; let $\lambda = l(X_1A_0)$, the length of the segment X_1A_0. Each segment a is characterized by the pair $(\lambda, \alpha) \in [0, 1] \times [0, 1]$. A trapezoid $\tau(a)$ is associated with the segment a as follows: choose A_5 inside the square such that $A_0A_5 \perp X_1X_2$ and $l(A_0A_5) = 2\alpha(\alpha + \alpha^{\frac{1}{2}})^{-1}$; then choose A_3, A_4 such that

$$l(A_3A_5) = \lambda\alpha^{\frac{1}{2}}, \qquad A_3A_5||A_1A_2,$$
$$l(A_5A_4) = (1-\lambda)\alpha^{\frac{1}{2}}, \qquad A_4A_5||A_1A_2.$$

$\tau(a)$ is the trapezoid $A_1A_2A_4A_3$. (Note that if $\alpha = 0$, all the A_i are taken to be coincident.) Observe that: (1) the area of $\tau(a)$ is equal to the length of a; (2) $l(A_0A_5)$ is a strictly increasing function of α mapping $[0, 1]$ onto $[0, 1]$; (3) cotangent angle $A_1A_3A_5 = \frac{1}{2}\lambda(1 - \alpha)$; (4) if a and b are two segments contained in X_1X_2, then $b \subseteq a \Rightarrow \tau(b) \subseteq \tau(a)$, and the intersection of the boundaries of $\tau(a)$ and $\tau(b)$ is $a \cap b$ unless a and b have an endpoint in common. The proof of (4) is as follows. Let $a \sim (\lambda, \alpha)$, $b \sim (\rho, \beta)$, $\tau(a) = A_1A_2A_4A_3$, $\tau(b) = B_1B_2B_4B_3$, $b \subseteq a$. The point C_1 such that

$$C_1B_1 \perp X_1X_2, \qquad l(B_1C_1) = 2\beta(\beta + \beta^{\frac{1}{2}})^{-1}$$

is collinear with B_3B_4 and lies inside $\tau(a)$. If A_1A_3 and B_4B_3 intersect in D_1, then D_1 and B_3 lie on the same side of C_1B_1 and

$$\begin{aligned} l(D_1C_1) - l(B_3C_1) &= l(A_1B_1) \\ &\qquad + l(C_1B_1)(\text{cotangent angle } A_1A_3A_4) - l(B_3C_1) \\ &= l(A_1B_1) + \lambda(1-\alpha)\beta(\beta + \beta^{\frac{1}{2}})^{-1} - \rho(\beta^{\frac{1}{2}} - \beta) \\ &= l(A_1B_1) + \beta(\beta + \beta^{\frac{1}{2}})^{-1}[\lambda(1-\alpha) - \rho(1-\beta)] \\ &= l(A_1B_1)[1 - \beta(\beta + \beta^{\frac{1}{2}})^{-1}] \geqq 0. \end{aligned}$$

Similarly, if $C_2B_2 \perp X_1X_2$, $l(C_2B_2) = 2\beta(\beta + \beta^{\frac{1}{2}})$, and C_2B_4 intersects A_2A_4 in D_2, then $l(C_2D_2) - l(C_2B_4) \geqq 0$. Hence, the segment B_3B_4 is contained in the segment D_1D_2. We are now in a position to state the following fact.

(5) *For s segments $a_1, a_2, \ldots, a_s$ in X_1X_2 with non-void intersection,*

$$\text{Area}[\tau(a_1) \cap \tau(a_2) \cap \ldots \cap \tau(a_s)] \geqq \text{Area}[\tau(a_1 \cap a_2 \cap \ldots \cap a_s)]$$
$$= l(a_1 \cap a_2 \cap \ldots \cap a_s).$$

THEOREM 2. *There exists a compact Hausdorff space E such that $C^+(E)$ contains a proper closed non-pseudo-subordinate subsemi-algebra.*

Construction. Let E be the square $X_1X_2X_3X_4$ of area 1, Y_1 the set of functions in $C^+(E)$ which vanish on side X_1X_2, and Y_2 the set of functions f in $C^+(E)$ such that for each $\gamma \in [0, ||f||]$,

$$\{\eta: f(\eta) \geqq \gamma\} = \tau(a)$$

for some segment a contained in X_1X_2. Let Z be the closed semi-algebra generated by the set $Y_1 \cup Y_2$; this is the required semi-algebra.

Proof that $Z_u = C^+(E)$. It will be shown that the set $Y_1 \cup Y_2 \subseteq Z$ contains a function which peaks exactly at any prescribed point of E, so that the corollary of Theorem 1 can be applied. If $\xi \in E\backslash$side X_1X_2, then, clearly, Y_1 contains a function whose maximum value is attained only at ξ. Now let $\xi \in$ side X_1X_2. If $\xi \neq X_1$ or X_2, then Y_2 contains a function which, when restricted to X_1X_2, vanishes at X_1, increases linearly to the value 1 at ξ, and decreases linearly to the value 0 at X_2; if ξ is either X_1 or X_2, then Y_2 contains a function which is linear on X_1X_2, takes the value 1 at ξ, and vanishes at the other endpoint.

Proof that Z is proper. Let μ_1 be the two-dimensional Lebesgue measure on the square E and μ_2 the linear Lebesgue measure on the side X_1X_2; let $\mu \equiv \mu_1 - \mu_2$. Then $\mu \notin M^+(E)$. It will be shown that if g is a finite product of elements in $Y_1 \cup Y_2$, then $\int g\, d\mu \geqq 0$, so that μ belongs to the dual cone of the closed convex cone generated by such products, i.e., the semi-algebra Z. Suppose then that $g = f_1 f_2 \ldots f_s$. If any of the f_i belong to Y_1, then clearly $\int g\, d\mu \geqq 0$. Assume now that each of the f_i is a member of Y_2. For integers i, m, and n with $1 \leqq i \leqq s$, $1 \leqq n$, $1 \leqq m \leqq 2^n - 1$, define segments $a_{i,m,n}$ such that

$$\tau(a_{i,m,n}) \equiv \{\eta: f_i(\eta) \geqq m\, 2^{-n}||f_i||\}$$

and let

$$f_i^{(n)} = 2^{-n} \sum_{m=1}^{2^n-1} k(a_{i,m,n}),$$

where $k(a)$ denotes the characteristic function of the trapezoid $\tau(a)$. Since $\prod k(a_i)$ is the characteristic function of $\bigcap \tau(a_i)$ (a set whose intersection with the side X_1X_2 is $\bigcap a_i$), it is a consequence of the Beppo Levi theorem and the fact stated in (5) above that

$$\int f_1 f_2 \ldots f_s \, d\mu = \lim_{n\to\infty} \int f_1^{(n)} f_2^{(n)} \ldots f_s^{(n)} \, d\mu$$
$$= \lim_{n\to\infty} 2^{-sn} \sum_{(m_i)} \int \prod_{i=1}^{s} k(a_{i,m_i,n}) \, d\mu \geqq 0.$$

The proof of Theorem 2 is now complete. The reason for making the conjecture originally was to permit us to state that each finitely generated closed semi-algebra is subordinate if it is proper; this result may indeed still hold. It will be indicated in Theorem 3 that it suffices to prove this with the property 'subordinate' replaced by the weaker property 'pseudo-subordinate'.

THEOREM 3. *Let A be a closed subsemi-algebra of $C^+(E)$ generated by a finite set. Then A is subordinate if and only if A is pseudo-subordinate.*

Proof. Let A be the closed semi-algebra generated by $f_1, f_2, \ldots, f_n$, and suppose that A is pseudo-subordinate. A_1 is defined to be the least closed semi-algebra containing A and all positive powers of the function f_1; for $i = 2, 3, \ldots, n$, A_i is defined to be the least closed semi-algebra containing A_{i-1} and all positive powers of the function f_i. The semi-algebra A_n is the closed semi-algebra generated by all positive real powers of the functions $f_1, f_2, \ldots, f_n$, so that A_n is generated by a power-closed set.

Since A is pseudo-subordinate, there exists a point ξ in E, and a positive measure μ on E with $\mu(\{\xi\}) = 0$ and $f(\xi) \leqq \int f \, d\mu$ $(\forall f \in A)$. If $f_1(\xi) = 0$, then for $\lambda_i > 0$, $g_i \in A$, and k a positive integer, we have that

$$\left(g_0 + \sum_{i=1}^{k} g_i f_1^{\lambda_i}\right)(\xi) = g_0(\xi) \leqq \int g_0 \, d\mu \leqq \int \left(g_0 + \sum_{i=1}^{k} g_i f_1^{\lambda_i}\right) d\mu$$

so that $\mu - \delta_\xi \in A_1'$. On the other hand, if $f_1(\xi) \neq 0$, then, as in the proof of Proposition 5 in (**1**), we have that

$$(f_1(\xi))^{-1}\mu - \delta_\xi = f_1(\xi)^{-1}(\mu - f_1 \cdot \delta_\xi) \in A_1'.$$

In either case, by (**1**, Theorem 5, Corollary), A_1 is pseudo-subordinate. Continuing in the same manner, one proves inductively that $A_2, \ldots, A_n$ are all pseudo-subordinate. But A_n is generated by a power-closed set, and hence, by (**1**, Theorem 7), is subordinate. This implies that A, being contained in A_n, is subordinate. Since the reverse implication is trivial, the theorem is proved.

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