Glasgow Math. J. **49** (2007) 121–126. © 2007 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089507003473. Printed in the United Kingdom

URSCM OR BI-URSCM FOR p-ADIC ANALYTIC OR MEROMORPHIC FUNCTIONS INSIDE A DISK

ABDELBAKI BOUTABAA and ALAIN ESCASSUT
Laboratoire de Mathématiques Pures, Université Blaise Pascal (Clermont-Ferrand), Les Cézeaux, 63177 Aubiere Cedex, France
e-mail: Abdelbaki.Boutabaa@math.univ-bpclermont.fr
Alain.Escassut@math.univ-bpclermont.fr

(Received 27 April, 2006; revised 18 September, 2006; accepted 29 September, 2006)

Abstract. Let K be an algebraically closed field of characteristic zero, complete with respect to an ultrametric absolute value. In a previous paper, we had found URSCM of 7 points for the whole set of unbounded analytic functions inside an open disk. Here we show the existence of URSCM of 5 points for the same set of functions. We notice a characterization of BI-URSCM of 4 points (and infinity) for meromorphic functions in K and can find BI-URSCM for unbounded meromorphic functions with 9 points (and infinity). The method is based on the p-Adic Nevanlinna Second Main Theorem on 3 Small Functions applied to unbounded analytic and meromorphic functions inside an open disk and we show a more general result based upon the hypothesis of a finite symmetric difference on sets of zeros, counting multiplicities.

2000 *Mathematics Subject Classification.* 12J25, 46S10.

Introduction and theorems.

Definitions and notation. The concept of unique range sets counting multiplicities for a family of meromorphic functions was first introduced by F. Gross and C. C. Yang in the eighties [**12**]. Many papers were published on this topic and on closely related topics involving uniqueness, on complex and p-adic meromorphic functions [**1**], [**3**], [**4**], [**5**], [**6**], [**7**], [**8**], [**10**], [**11**], [**13**], [**14**], [**16**], [**17**].

We denote by K an algebraically closed field of characteristic zero, complete with respect to an ultrametric absolute value. Let $\mathcal{A}(K)$ be the K-algebra of entire functions in K and let $\mathcal{M}(K)$ be the field of meromorphic functions in K, i.e. the field of fractions of $\mathcal{A}(K)$. Given $a \in K$ and $r > 0$, we denote by $d(a, r)$ the disk $\{x \in K | \, |x - a| \leq r\}$ and by $d(a, r^-)$ the disk $\{x \in K | \, |x - a| < r\}$. In the same way, we denote by $\mathcal{A}(d(a, r^-))$ the K-algebra of analytic functions in $d(a, r^-)$, i.e. the set of power series $\sum_{n=0}^{\infty} a_n(x - a)^n$ converging in $d(a, r^-)$ and by $\mathcal{M}(d(a, r^-))$ the field of meromorphic functions inside $d(a, r^-)$, i.e. the field of fractions of $\mathcal{A}(d(a, r^-))$.

We will denote by $\mathcal{A}_b(d(a, R^-))$ the K-subalgebra of $\mathcal{A}(d(a, R^-))$ consisting of the analytic functions $f \in \mathcal{A}(d(a, R^-))$ which are bounded in $d(a, R^-)$ and by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$. Next, we will denote by $\mathcal{A}_u(d(a, R^-))$ the set $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$ and, similarly, we set $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$. The Nevanlinna Theory applies to functions in $\mathcal{M}_u(d(a, R^-))$. This is why we may look for problems of uniqueness in this set of functions.

For a subset S of K and $f \in \mathcal{M}(d(a, R^-))$ we denote by $E(f, S)$ the set in $(d(a, R^-)) \times \mathbb{N}^*$: $\bigcup_{a \in S}\{(z, q) \in (d(a, R^-)) \times \mathbb{N}^* \mid z$ a zero of order q of $f(x) - a\}$.

Let $\mathcal{F}$ be a non-empty subset of $\mathcal{M}(d(a, R^-))$. A subset S of K is called a *unique range set counting multiplicities* (an *URSCM* in brief) *for* $\mathcal{F}$ if for any non-constant $f, g \in \mathcal{F}$ such that $E(f, S) = E(g, S)$, we have $f = g$.

It is known that the algebra of complex entire functions admits URSCM of 7 points and that the field of complex meromorphic functions admits URSCM of 11 points [**10**].

For the field K, it is known that the USRCM for $\mathcal{A}(K)$ are the URSCM for polynomials which actually are the sets which are preserved by no affine mapping but the identity [**3**], [**4**]. So, there exist URSCM for $\mathcal{A}(K)$ having just 3 points.

In [**5**] we proved the existence of URSCM and URSIM for functions in $\mathcal{A}_u(d(a, R^-))$ and in $\mathcal{M}_u(d(a, R^-))$: there exist URSCM of 7 points for $\mathcal{A}_u(d(a, R^-))$. We also found smaller URSCM for subsets of $\mathcal{A}_u(d(a, R^-))$ consisting of functions with "a small derivative" by using a method due to Frank and Reinders, also developed by H. Fujimoto [**11**]. Here we shall use a more simple method based upon the p-adic Second Main Theorem on Three Small Functions [**15**], [**17**] in order to show the existence of URSCM of 5 points for $\mathcal{A}_u(d(a, R^-))$, without assuming any additional hypotheses on the functions.

By the same method, we will also show the existence of BI-URSCM for $\mathcal{M}_u(d(a, r^-))$ of the form $(\{a_1, \dots, a_9\}, \{\infty\})$. A set of the form $(S, \{\infty\})$ with $S \subset K$ (or $(S, \{b\})$ with $b \in K$) is called *a BI-URSCM* for a subset $\mathcal{F}$ of $\mathcal{M}(d(a, R^-))$ if, given $f, g \in \mathcal{F}$ such that $E(f, S) = E(g, S)$ and $E(f, \{\infty\}) = E(g, \{\infty\})$ (or $E(f, \{b\}) = E(g, \{b\})$), we have $f = g$. Currently, when S is finite, the cardinal of S is called the number of points of the BI-URSCM. As a consequence of [**8**, Theorem 2], BI-URSCM are easily seen to have at least 4 points. In [**4**] we showed the existence of BI-URSM of 5 points for $\mathcal{M}(K)$. In [**13**] T.T.H. An and H.H. Khoai showed the existence of BI-URSCM for $\mathcal{M}(K)$ having only 4 points and showed the role of Condition (2) in Theorem 1 below. As a corollary of [**9**, Theorem 3.7], BI-URSCM of 4 points for $\mathcal{M}(K)$ of the form $(S, \{\infty\})$ may be characterized in the following way (which was not mentioned in [**9**]).

PROPOSITION. *Let $S = \{a_1, a_2, a_3, a_4\} \subset K$ with $a_i \neq a_j\ \forall i \neq j$ and let $T(x) = \prod_{j=1}^{4}(x - a_j)$. Then $(S, \{\infty\})$ is a BI-URSCM for $\mathcal{M}(K)$ if and only if T' admits 3 distinct zeros $c_1,\ c_2,\ c_3$ satisfying the two following conditions:*

(i) $T(c_i) \neq T(c_j)\ \forall i \neq j$;

(ii) *the equality* $\dfrac{T(c_1)}{T(c_2)} = \dfrac{T(c_2)}{T(c_3)} = \dfrac{T(c_3)}{T(c_1)}$ *is not true.*

REMARK. If (ii) is violated in the Proposition, then $\frac{T(c_1)}{T(c_2)}$ is a number λ such that $\lambda^2 + \lambda + 1 = 0$.

Here we shall show the existence of BI-URSCM for $\mathcal{M}_u(d(a, R^-))$ having 9 points.

NOTATION. Throughout the paper, we shall denote by P a polynomial of the form $P(x) = x^n - \alpha x^m + 1$ with m, n relatively prime such that $2 \leq m \leq n - 1$ and such that $\alpha^n \neq \frac{n^n}{m^m(n-m)^{n-m}}$. We shall denote by $S(n, m, \alpha)$ its set of zeros.

We denote by Δ the symmetric difference on subsets of a set.

REMARK. Since $\alpha^n \neq \frac{n^n}{m^m(n-m)^{n-m}}$, P has n distinct zeros.

THEOREM 1. *Let* $f,\ g \in \mathcal{A}_u(d(a, R^-))$ *be two different non-constant functions satisfying* $\#(E(f, S(n, m, \alpha))\Delta E(g, S(n, m, \alpha))) < \infty$. *Then* $2m - n \leq 2$.

COROLLARY 1.1. *Suppose that* $2m > n + 2$. *Then* $S(n, m, \alpha)$ *is an URSCM for* $\mathcal{A}_u(d(a, R^-))$.

REMARK. In particular, Corollary 1.1 holds with $n \geq 5$ and $m = n - 1$.

THEOREM 2. *Let* $f,\ g \in \mathcal{A}_u(d(a, R^-))$ *be two different non-constant functions satisfying* $\#(E(f, S(n, m, \alpha))\Delta E(g, S(n, m, \alpha))) < \infty$ *and* $\#(E(f, \{\infty\})\Delta E(g, \{\infty\})) < \infty$. *Then* $2m - n \leq 3$.

COROLLARY 2.1. *Suppose* $m \leq n - 2$ *and* $2m > n + 3$. *Then* $S(n, m, \alpha)$ *is a BI-URSCM for* $\mathcal{M}_u(d(a, R^-))$.

The proofs. Let log be the real logarithm function of base $p > 1$. Let $R \in]0, +\infty[$ and let $f \in \mathcal{M}(d(0, R^-))$ such that 0 is neither a zero nor a pole of f. Let $r \in]\rho, R[$.

We denote by $Z(r, f)$ and $\overline{Z}(r, f)$ the counting functions of zeros of f in $d(0, R) \setminus \{0\}$, (counting multiplicities or not) i.e. if (a_n) is the finite or infinite sequence of zeros of f in $d(0, R^-) \setminus \{0\}$, with respective multiplicity order s_n, we put

$$Z(r, f) = \sum_{|a_n| \leq r} s_n(\log r - \log|a_n|) \quad \text{and} \quad \overline{Z}(r, f) = \sum_{|a_n| \leq r} (\log r - \log|a_n|).$$

In the same way, we denote by $N(r, f)$ and by $\overline{N}(r, f)$ the counting functions of poles of f: considering the sequence (b_n) of poles of f in $d(0, r) \setminus \{0\}$, with respective multiplicity order t_n, we put

$$N(r, f) = \sum_{|b_n| \leq r} t_n(\log r - \log|b_n|) \quad \text{and} \quad \overline{N}(r, f) = \sum_{|b_n| \leq r} (\log r - \log|b_n|).$$

For a function f having no zero and no pole at 0, the Nevanlinna function $T(r, f)$ is defined by $T(r, f) = \max(Z(r, f) + \log|f(0)|, N(r, f))$.

In order to prove the Theorems, we must recall the Nevanlinna Second Main Theorem on 3 small functions showed in $\mathcal{M}(K)$ in **[15]** which actually also holds in $\mathcal{M}(d(0, R^-))$ **[17]**.

THEOREM A. *Let* $f,\ u_1,\ u_2,\ u_3 \in \mathcal{M}(d(0, R^-))$ *have no zero and no pole at* 0 *and let* $S(r) = \max_{j=1,2,3}(T(r, u_j))$. *Then* $T(r, f) \leq \sum_{j=1}^{3} \overline{Z}(r, f - u_j) + S(r),\ r \in]\rho, \mathrm{R}[$.

By Replacing f by $\frac{1}{f}$ and taking $u_3 = 0$, we obtain Corollary A1 **[17]**:

COROLLARY A.1. *Let* $f \in \mathcal{M}(d(0, R^-))$ *and* $u_1,\ u_2, \in \mathcal{M}_b(d(0, R^-))$ *have no zero and no pole at* 0 *and let* $S(r) = \max_{j=1,2}(T(r, u_j))$.

Then $T(r, f) \leq \sum_{j=1}^{2} \overline{Z}(r, f - u_j) + \overline{N}(r, f) + O(1),\ r \in]\rho, R[$.

We shall also use the following Lemma B which is classical **[2]**, **[3]**.

https://doi.org/10.1017/S0017089507003473 Published online by Cambridge University Press

LEMMA B. *Let* $f, g \in \mathcal{A}(d(0, R^-))$.
(1) *Then* $T(r, fg) = T(r, f) + T(r, g)$.
(2) *Let* $P \in K[x]$. *Then* $T(r, P \circ f) = \deg(P)T(r, f) + O(1)$.

Proof of Theorems 1 and 2. Without loss of generality we may obviously assume that $a = 0$. By hypothesies, in both Theorems 1 and 2 $\#(E(f, S(n, m, \alpha))\,\Delta E(g, S(n, m, \alpha)))$ and $\#(E(f, \{\infty\})\Delta E(g, \{\infty\}))$ are finite (whereas $E(f, \{\infty\}) = E(g, \{\infty\}) = \emptyset$ in Theorem 1). Since all zeros of P are of order 1, we see that $P \circ f$ and $P \circ g$ have the same zeros and the same poles, counting multiplicities, except maybe finitely many. Consequently, the function $u(x) = \frac{P \circ f}{P \circ g}$ which obviously lies in $\mathcal{M}(d(0, R^-))$, has finitely many zeros and finitely many poles in $d(0, R^-)$. Hence, $u \in \mathcal{M}_b(d(0, R^-))$.

Without loss of generality we may obviously assume that 0 is neither a zero nor a pole for all functions we have to consider in Theorems 1 and 2.

On the other hand, we notice that

$$T(r, P \circ f) = nT(r, f) + O(1),$$
$$T(r, P \circ g) = nT(r, g) + O(1)$$

But since u belongs to $\mathcal{M}_b(d(0, R^-))$, $T(r, u)$ is bounded, hence $T(r, P \circ f) = T(r, P \circ g) + O(1)$ and therefore

$$T(r, f) = T(r, g) + O(1). \tag{1}$$

Now, let $F(x) = f^n - \alpha f^m$, let $G(x) = u(x) - (g^n - \alpha g^m)$ and let $w(x) = 1 - u(x)$. Thus, we have $F(x) = u(x)(g^n - \alpha g^m) + u(x) - 1$.

Suppose that u is not identically 1. By Corollary A.1 we have

$$T(r, F) \leq \overline{Z}(r, F) + \overline{Z}(r, F - w) + \overline{N}(r, f) + O(1). \tag{2}$$

But

$$\begin{aligned}\overline{Z}(r, F) = \overline{Z}(r, f^m(f^{n-m} - \alpha)) &= \overline{Z}(r, f) + \overline{Z}(r, f^{n-m} - \alpha)\\ &\leq (n - m + 1)T(r, f) + O(1).\end{aligned} \tag{3}$$

Similarly:

$$\begin{aligned}\overline{Z}(r, F - w) = \overline{Z}(r, u(x)(g^n - \alpha g^m)) &= \overline{Z}(r, g)\\ &+ \overline{Z}(r, g^{n-m} - \alpha) + \overline{Z}(r, u) =\leq (n - m + 1)T(r, g) + O(1),\end{aligned}$$

hence by (1), we have

$$\overline{Z}(r, F - w) \leq (n - m + 1)T(r, f) + O(1). \tag{4}$$

On the other hand, obviously

$$\overline{N}(r, F) = \overline{N}(r, f) \leq T(r, f). \tag{5}$$

Now, by Lemma B we have $T(r, F) = nT(r, f) + O(1)$ hence by (1), (2), (3), (4) we obtain

$$nT(r, f) \leq 2(n - m + 1)T(r, f) + \overline{N}(r, f) + O(1). \tag{6}$$

Thus, in the hypotheses of Theorem 1, we have $nT(r,f) \leq 2(n-m+1)T(r,f) + O(1)$. And since $T(r,f)$ is unbounded when r tends to R, we see that $2m - n \leq 2$. Now, in the hypotheses of Theorem 2, by (5) and (6) we obtain $2m - n \leq 3$.

We can now assume that u is identically 1, hence $f^n - \alpha f^m = g^n - \alpha g^m$. Putting $h = \frac{f}{g}$, we obtain $g^{n-m}(h^n - 1) = \alpha(h^m - 1)$. Since $m, \ n$ are relatively prime, we notice that $(h^n - 1)$ and $(h^m - 1)$ may not be both identically zero, hence we have

$$g^{n-m} = \alpha \frac{h^m - 1}{h^n - 1}. \tag{7}$$

Let ξ_k, $1 \leq k \leq n$ be the n-th roots of 1 with $\xi_1 = 1$ and let ζ_j, $1 \leq j \leq m$ be the m-th roots of 1 with $\zeta_1 = 1$. Since $m < n$ there exists $k \in [2, n]$ such that $\xi_k \neq \zeta_j \ \forall j = 1, \ldots, m$ and therefore, each zero of $h - \xi_k$ is a pole of g^{n-m}, a contradiction to the hypothesis of Theorem 1. Thus, in the hypothesis of Theorem 1, u is not identically 1 which completes the proof.

Assume now the hypothesis of Theorem 2. Since $\mathcal{M}_b(d(0, r^-))$ is a field, by (7) h does not belong to $\mathcal{M}_b(d(0, r^-))$ because if it belonged to $\mathcal{M}_b(d(0, r^-))$ then g should also lie in $\mathcal{M}_b(d(0, r^-))$. Thus, since $n - m \geq 2$, for every $j = 2, \ldots, m$ we have $\overline{Z}(r, h - \xi_j) \leq \frac{1}{2} Z(r, h - \xi_j)$ and for every $k = 2, \ldots, n$ we have $\overline{Z}(r, h - \zeta_k) \leq \frac{1}{2} Z(r, h - \xi_j)$.

Since $m, \ n$ are relatively prime, we notice that $\xi_k \neq \zeta_j \ \forall k = 2, \ldots, n \ j = 2, \ldots, m$. Consequently, each zero of $h - \xi_k$ is a pole of g^{n-m} (and hence is a zero of order at least $n - m$ of $h - \xi_k$). And similarly, each zero of $h - \zeta_j$ is zero of g^{n-m} (and hence is a zero of order at least $n - m$ of $h - \zeta_j$). Consequently,

$$\overline{Z}(r, h - \xi_k) \leq \frac{1}{n-m} Z(r, h - \xi_k), \quad \forall k = 2, \ldots, n \tag{8}$$

and

$$\overline{Z}(r, h - \zeta_j) \leq \frac{1}{n-m} Z(r, h - \zeta_j), \quad \forall j = 2, \ldots, m. \tag{9}$$

Now, since $h \in \mathcal{M}_u(d(0, r^-))$, we may apply to h the classical p-adic Second Main Theorem in $\mathcal{M}_u(d(0, r^-))$. We have $(n + m - 3)T(r, h) \leq \sum_{j=2}^{n} \overline{Z}(r, h - \xi_j) + \sum_{k=2}^{m} \overline{Z}(r, h - \zeta_k) + \overline{N}(r, h) + O(1)$ and therefore, by (8) and (9), we obtain $(n + m - 3)T(r, h) \leq \frac{1}{2} \left(\sum_{j=2}^{n} Z(r, h - \xi_j) + \sum_{k=2}^{m} Z(r, h - \zeta_k)\right) + N(r, h) + O(1) \leq \left(\frac{m-1+n-1}{2} + 1\right) T(r, h) + O(1)$. Thus we check that $m + n \leq 6$. In fact, we can easily see that $m + n \leq 6$ is incompatible with $2m - n \geq 4$, consequently, the hypotheses of Theorem 2 led to $2m - n \leq 3$ in all cases. This completes the proof of Theorem 2.

REMARK. In [**4**], we neglected the fact that when m, n are not relatively prime, $h^m - 1$ and $h^n - 1$ may have common zeros different from 1. This is why Theorem 4 in [**4**] is not correct: when $P(x) = x^6 - \alpha x^4 + 1$, any function f satisfy $P \circ f = P \circ (-f)$.

ACKNOWLEDGMENT. We are grateful to the referee for pointing out to us misprints and abstraction mistakes in a first version of this paper.

REFERENCES

1. W. W. Adams and E. G. Straus, Non archimedian analytic functions taking the same values at the same points, *Illinois J. Math.* **15** (1971) 418–424.

2. A. Boutabaa, Théorie de Nevanlinna p-adique, *Manuscripta Math.* **67** (1990), 251–269.

3. A. Boutabaa, A. Escassut and L. Haddad, On uniqueness of p-adic entire functions, *Indag. Math.* **8** (1997), 145–155.

4. A. Boutabaa and A. Escassut, On uniqueness of p-adic meromorphic functions, *Proc. Amer. Math. Soc.* **126** (1998), 2557–2568.

5. A. Boutabaa and A. Escassut, Urs and ursim for p-adic meromorphic functions inside a p-adic disk, *Proc. Edinburgh Math. Soc.* **44** (2001), 485–504.

6. W. Cherry and C. C. Yang, Uniqueness of non-archimedean entire functions sharing sets of values counting multiplicities, *Proc. Amer. Math. Soc.* **127** (1998), 967–971.

7. A. Boutabaa and A. Escassut, URS' for Weierstrass products without exponential factors, *Complex Var. Theory Appl.* **47** (2002), 409–415.

8. A. Escassut, L. Haddad and R. Vidal, Urs, ursim and non-urs for p-adic functions and polynomials, *J. Number Theory* **75** (1999), 133–144.

9. A. Escassut and C. C. Yang, The functional equation $P(f) = Q(g)$ in a p-adic field, *J. Number Theory* **105** (2004), 344–360.

10. G. Frank and M. Reinders, A unique range set for meromorphic functions with 11 elements, *Complex Variable Theory Appl.* **37** (1998), 185–193.

11. H. Fujimoto, On uniqueness of meromorphic functions sharing finite sets, *Amer. J. Math.* **122** (2000), 1175–1203.

12. F. Gross and C. C. Yang, On preimage and range sets of meromorphic functions, *Proc. Japan Acad.* **58** (1982), 17–20.

13. Ha Huy Khoai and Ta Thi Hoai An, On uniqueness polynomials and bi-URs for p-adic meromorphic functions, *J. Number Theory* **87** (2001), 211–221.

14. P. C. Hu and C. C. Yang, A unique range set of p-adic functions meromorphic functions with 10 elements, *Acta Math. Vietnam.* **24** (1999), 95–108.

15. P. C. Hu and C. C. Yang, *Meromorphic functions over non archimedean fields*, Mathematics and its applications, vol. 522 (Kluwer, 2000).

16. P. Li and C. C. Yang, On the unique range set of meromorphic functions, *Proc. Amer. Math. Soc.* **124** (1996), 177–185.

17. J. Ojeda, Applications of the p-adic Nevanlinna theory to problems of uniqueness, preprint.