AN UPPER BOUND FOR VOLUMES OF CONVEX BODIES

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1

Consider a non-degenerate convex body K in a Euclidean $(n+1)$-dimensional space of points $(x, z) = (x_1, \cdots, x_n, z)$ where $n \geqq 2$. Denote by μ the maximum length of segments in K which are parallel to the z-axis, and let A_j signify the area (two dimensional volume) of the orthogonal projection of K onto the linear subspace spanned by the z- and x_j-axes. We shall prove that the volume $V(K)$ of K satisfies

$$(1) \qquad (2^n \prod_{j=1}^{n} A_j)/(n+1)\mu^{n-1} - V(K) = \Delta(K) \geqq 0.$$

After this, some applications of (1) are discussed.

2

We first study the effect on $\Delta(K)$ of symmetrization of K in each one of the coordinate planes in succession. To symmetrize K in $z = 0$ for example, we translate each segment in K which is parallel to the z-axis along its containing line so that its midpoint falls in $z = 0$ and form the union K' of these translated segments. We say K' is obtained from K by symmetrization in $z = 0$. Analytically we may describe K as the set of (x, z) such that

$$f_1(x) \leqq z \leqq f_2(x), \qquad x \in k$$

where k is the orthogonal projection of K onto $z = 0$ and $x \in k$ means $(x, 0) \in k$. Then K' is the set of (x, z) such that

$$(2) \qquad -[f_2(x)-f_1(x)]/2 \leqq z \leqq [f_2(x)-f_1(x)]/2, \qquad x \in k.$$

The order in which a succession of symmetrizations is carried out affects the final figure in general, but this makes no difference for our discussion.

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It is essential to note that, when all the planes of symmetrization are mutually orthogonal, then from each step there results a convex body, with the same volume as K, which is symmetric with respect to all those planes of symmetrization used up through that step. For details see [1, pp. 69—70].

Clearly

$$\mu = \max_{k} [f_2(x)-f_1(x)]$$

is unchanged by symmetrization in $z = 0$. However, if the symmetrization is in some other coordinate plane, say $x_1 = 0$, and if μ' is the value of μ after symmetrization, then $\mu' \geqq \mu$. To see this, consider the orthogonal projection k' of K onto $x_1 = 0$. The maximal length of segments parallel to the z-axis in k' is μ'. On the other hand, a segment parallel to the z-axis in K of maximal length projects into a segment of length μ in k'.

After symmetrization in each one of the coordinate planes, we arrive at a convex body K^*; the greatest length μ^* of segments parallel to the z-axis in K^* is greater than μ or else equal to μ in case μ is the width of K in the direction of the z-axis.

As to the behaviour of the areas A_j under symmetrization, there are two situations: that in which the plane of symmetrization contains both the z- and x_j-axes and that in which the plane of symmetrization contains only one of these axes. We shall illustrate these two cases by examining the effect on A_1 of symmetrization in $x_2 = 0$ and in $z = 0$. The projection of K onto the two-dimensional subspace spanned by the x_1- and z-axes is the same as the projection of k'' onto that subspace, where k'' is the orthogonal projection of K onto $x_2 = 0$. Since k'' is not altered by symmetrization in $x_2 = 0$, neither is A_1.

If we write $A_1(K)$ and $A_1(K')$ for the values of A_1 before and after symmetrization in $z = 0$, then $A_1(K) \geqq A_1(K')$. This is shown as follows. Let

$$g_1(x_1) = \min_{*} f_1(x),\ g_2(x_1) = \max_{*} f_2(x),\ g(x_1) = \max_{*} [f_2(x)-f_1(x)]/2,$$

where the starred extrema are taken over those points x of k whose first coordinates have the fixed value x_1. The projection of K onto the subspace spanned by the z- and x_1-axes is the set of points $(x_1, 0, \cdots, 0, z)$ for which

$$g_1(x_1) \leqq z \leqq g_2(x_1), \quad a \leqq x_1 \leqq b,$$

where a and b are the least and greatest values of x_1 for x in k. For the projection of K' we have

$$-g(x_1) \leqq z \leqq g(x_1), \quad a \leqq x_1 \leqq b.$$

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Since

$$g(x_1) \leqq [g_2(x)-g_1(x)]/2,$$

and because

$$A_1(K) = \int_a^b [g_2(x_1)-g_1(x_1)]dx_1, \quad A_1(K') = 2\int_a^b g(x_1)dx_1,$$

we have $A_1(K) \geqq A_1(K')$.

For K^*, the final result of all the symmetrizations, we have by the foregoing discussion

$$(3) \qquad \Delta(K) \geqq \Delta(K^*).$$

3

For each choice of j from $1, 2, \cdots, n$, the intersection of K^* with

$$x_1 = x_2 = \cdots = x_{j-1} = x_{j+1} = \cdots = x_n = 0$$

is a two dimensional convex body whose points $(0, \cdots, 0, x_j, 0, \cdots, 0, z)$ satisfy

$$(4) \qquad -\phi_j(z) \leqq x_j \leqq \phi_j(z), \quad -\zeta \leqq z \leqq \zeta,$$

where $\zeta = \mu^*/2$ is the greatest value of z in K^*. If C_j is the set of all points (x, z) which satisfy (4), then the symmetry of K^* with respect to each coordinate plane shows that

$$K^* \subseteqq \bigcap_{j=1}^{n} C_j = C.$$

Observe that the convexity of the cylinders C_j is reflected in the non-negativity and concavity of each of the functions $\phi_1, \phi_2, \cdots, \phi_n$ over their common domain. Clearly C is a convex body for which the quantities A_j and μ have the same value as they do for K^*; further $V(C) \geqq V(K^*)$. Therefore

$$\Delta(K^*) \geqq \Delta(C).$$

There is equality if and only if each section of K^* by a plane $z = t$, $-\zeta \leqq t \leqq \zeta$, is an n-dimensional rectangular parallelopiped.

Since

$$(5) \qquad \Delta(C) = \left(2^{n+1}\zeta/(n+1)\right)\left(E_1(C)-E_2(C)\right),$$

where

$$E_1(C) = \prod_{j=1}^{n} \left(2\int_0^\zeta \phi_j(t)dt/\zeta\right),$$

$$E_2(C) = (n+1)\int_0^\zeta \prod_{j=1}^{n} \phi_j(t)dt/\zeta,$$

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to prove (1), we must show $E_1(C) \geqq E_2(C)$.

In the first orthant of our $(n+1)$-dimensional space, the edge curve Γ of C has the equations

$$x_j = \phi_j(z), \quad 0 \leqq z \leqq \zeta, \qquad j = 1, 2, \cdots, n.$$

If Γ is the line segment L joining $(\phi_1(0), \phi_2(0), \cdots, \phi_n(0), 0)$ to $(0, \cdots, 0, \zeta)$, then

$$\phi_j(z) = \phi_j(0)(1-z/\zeta)$$

and, by direct computation $E_1(C) = E_2(C)$.

If Γ and L do not coincide, then the orthogonal projection of Γ onto the two dimensional subspace spanned by the z-axis and some one of the x_j-axes is not a line segment. Suppose this happens for $j = 1$; we write the points of this subspace as (x_1, z). The projection Γ' of Γ onto this subspace has the equation

$$x_1 = \phi_1 (z).$$

From the point $P:(0, \zeta)$ we draw a line segment M to a point $R:(\xi_1, 0)$, where $\xi_1 > \phi_1(0)$. Then, in addition to the point P, M intersects Γ' in a second point $Q : (\xi_1', \zeta')$, at least for ξ_1 near $\phi_1(0)$, because of the concavity of ϕ_1. Let r be the region bounded by the segment PQ and that arc of Γ' which joins P and Q; let r' be the region bounded by the segment QR, the rest of Γ' and part of the x_1-axis. We choose ξ_1 so that the areas of r and r' are equal, which is to say so that

$$\int_0^\zeta \phi_1(t)dt = \int_0^\zeta \psi(t)dt,$$

where

$$\psi(t) = \xi_1(1-t/\zeta).$$

Thus $E_1(C)$ is unaltered when we replace ϕ_1 by ψ.

Such a replacement increases $E_2(C)$. Each function ϕ_j is concave with respect to z and non-increasing at $z = 0$. Consequently

$$\prod_{j=2}^n \phi_j(z) \gtreqless \prod_{j=2}^n \phi_j(\zeta') \quad \text{according as} \quad z \lesseqgtr \zeta'. \tag{6}$$

Also, by our choice of M,

$$\psi(z) \gtreqless \phi_1(z) \quad \text{according as} \quad z \lesseqgtr \zeta'.$$

Hence, because inequalities (6) are strict for some z,

$$\int_0^\zeta (\psi(t)-\phi_1(t)) \prod_{j=2}^n \phi_j(t)dt$$
$$> \prod_{j=2}^n \phi_j(\zeta') \left\{ \int_0^{\zeta'} [\psi(t)-\phi_1(t)]dt - \int_{\zeta'}^{\zeta} [\phi_1(t)-\psi(t)]dt \right\}.$$

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Since the areas of r and r' are equal, the expression in curly brackets vanishes.

The same argument applies to those of the remaining functions ϕ_j not of the form ψ. In this way, we arrive at a replacement for the final factor on the right of (5) which vanishes. Hence $E_1(C) > E_2(C)$. In summary: we have $E_1(C) \geqq E_2(C)$ with equality if and only if $L = \Gamma$. This proves (1).

4

It remains to describe the cases of equality.

To begin with, there is equality in (1) if and only if $\Delta(K) = \Delta(K^*)$ and K^* has the following properties: its sections by planes $z = t$ are rectangular parallelopipeds and $\Gamma = L$. That is to say, K^* is the convex closure of the union of a rectangular parallelopiped Π in $z = 0$ with the segment τ from $(0, \cdots, 0, -\zeta)$ to $(0, \cdots, 0, \zeta)$. Π has its centre at the origin and its edges parallel to the coordinate axes.

We shall show that, for equality in (1), it is necessary and sufficient that K is, to within a translation, the convex closure of the union of τ with a translate of Π which intersects τ. For convenience, we call such a figure a dipyramid. Note that this includes the case in which τ intersects the translate of Π in an end point of τ. The sufficiency is trivial and so we need only show that

$$\Delta(K) = \Delta(K^*) = 0 \tag{7}$$

requires K to be a dipyramid.

We noted earlier that the length of the longest segment or segments in K parallel to the z-axis, equals the width of K in the direction of the z-axis. Thus K contains a translate τ' of τ. We suppose K translated so that τ and τ' coincide. This will be true also for all those convex bodies which result from K by symmetrization in one or more coordinate planes.

Let K' be the result of symmetrizing K in each of the coordinate planes excepting $z = 0$. When we symmetrize K' in $z = 0$, we obtain K^*. We shall first show that K' is a dipyramid. The difference $f_1 - f_2$ of a concave function f_1 and a convex function f_2 is linear if and only if f_1 and f_2 are linear. From this and the analytic description of symmetrization, it follows that K' is a polyhedron, symmetric with respect to each of the planes $x_j = 0, j = 1, \cdots, n$. The number of vertices in a convex polyhedron cannot decrease under symmetrization and so K' has at most 2^n+2 vertices. Consider a vertex of K' which is not in any one of the coordinate planes $x_j = 0$. There must be such vertices since the projection of K' onto $z = 0$ is identical with Π. By reflecting this vertex in each one of the coordinate planes $x_j = 0$,

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we obtain the 2^n vertices of a translate Π' of Π. Π' lies in some plane $z = t$ and is centred on the z-axis. In addition, K' has one or two more vertices at the ends of τ according as Π' intersects τ in an endpoint or an interior point of τ, and can have no further vertices because such vertices would have to be off the z-axis, and symmetry considerations show K' would have to have more than 2^n+2 vertices which is impossible. Since K' is the convex closure of its vertices, K' is a dipyramid as asserted.

The intersection $\Pi'(a)$ of K' by a plane $z = a$, $-\zeta \leqq a \leqq \zeta$, is homothetic to $\Pi' = \Pi'(t)$. $\Pi'(a)$ is obtained from the intersection $\Pi(a)$ of K and $z = a$ by symmetrizing $\Pi(a)$ with respect to all the coordinate planes $x_j = 0$. Moreover, $\Pi'(a)$ is independent of the order in which these symmetrizations are performed.

We next prove that $\Pi(a)$ must be a translate of $\Pi'(a)$. Suppose $x_j = 0$ is the final plane of symmetrization. The pair of $(n-1)$-dimensional faces of $\Pi'(a)$ which are parallel to $x_j = 0$ necessarily come from a pair of parallel $(n-1)$-dimensional faces F_j, G_j of $\Pi(a)$, because any line in $z = a$, perpendicular to $x_j = 0$, must intersect $\Pi(a)$ and $\Pi'(a)$ in segments of the same length. Since j can be any one of the numbers $1, \cdots, n$, $\Pi(a)$ has n pairs of parallel faces. The number of faces of a convex polyhedron cannot decrease under symmetrization. Therefore the pairs $F_1, G_1, \cdots, F_n, G_n$ make up the totality of faces of $\Pi(a)$, and $\Pi(a)$ is a parallelopiped. If F_j, G_j were not parallel to $x_j = 0$, then they would fail to be perpendicular to some one of the planes $x_i = 0$, $i \neq j$. Symmetrization of $\Pi(a)$ with respect to $x_i = 0$ would cause $\Pi'(a)$ to have more than $2n$ faces. Thus $\Pi(a)$ is a rectangular parallelopiped which, it is easy to see, must be a translate of $\Pi'(a)$.

From its convexity, K contains the convex closure $\bar{K}$ of the union of $\Pi(t)$ with τ, where we recall that $\Pi(t)$ is the largest of the parallelopipeds $\Pi(a)$. If v is the n-dimensional volume of $\Pi(t)$ and of Π, then the volumes of $\bar{K}$ and K^* equal $v\zeta$. But $v\zeta$ must also be the volume of K. Hence K is the dipyramid $\bar{K}$ as originally asserted.

5

The inequality $E_1(C) \geqq E_2(C)$ may have independent analytic interest. It can be written in the slightly more general form

$$\prod_{j=1}^{n} \left[2 \int_a^b \phi_j(t)dt/(b-a) \right] \geqq (n+1) \int_a^b \prod_{j=1}^{n} \phi_j(t)dt/(b-a) \tag{8}$$

for non-negative, concave functions $\phi_1, \phi_2, \cdots, \phi_n$ over $a \leqq t \leqq b$. To see this, choose for K the set of points (x, z) which satisfy the inequalities

$$-\phi_j(z) \leqq x_j \leqq \phi_j(z), \quad a \leqq z \leqq b,$$

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and apply (1). The cases of equality in (1) show that there is equality in (8) if and only if

$$\phi_j(z) = \phi_j(a)(b-z)/(b-a), \qquad j = 1, 2, \cdots, n,$$

or

$$\phi_j(z) = \phi_j(b)(z-a)/(b-a), \qquad j = 1, 2, \cdots, n.$$

In particular, if we set $\phi_1(z) = \phi_2(z) = \cdots = \phi_n(z) = \phi(z)$ in (8), then for integers $n \geqq 2$ and non-negative, concave functions ϕ we have

$$\text{(9)} \qquad [(1+n)^{1/n}/2] \sqrt[n]{\left[\int_a^b (\phi(t))^n dt/(b-a)\right]} \leqq \int_a^b \phi(t)dt/(b-a).$$

We contrast this with

$$\text{(10)} \qquad \int_a^b \phi(t)dt/(b-a) \leqq \sqrt[n]{\left[\int_a^b (\phi(t))^n dt/(b-a)\right]},$$

which holds for non-negative, integrable functions ϕ, cf. [3].

From (9) and (10) we can get upper and lower volume bounds for convex bodies of revolution in $(n+1)$-dimensional space. Take the axis of such a body K as the z-axis; the boundary of K is made up of points (x, z) which satisfy $\rho = \phi(z)$ where $\rho^2 = x_1^2+x_2^2+\cdots+x_n^2$. The function ϕ is non-negative and concave. If μ is the length of the axis of K, we may assume ϕ to be defined over $0 \leqq z \leqq \mu$. A meridian section of K is a two dimensional body obtained from cutting K with a two dimensional linear subspace which contains the axis of K. Denote its area by A. Then (9) yields for the volume V of K:

$$V \leqq \kappa_n A^n/(n+1)\mu^{n-1},$$

where κ_n is the n-dimensional volume of the unit ball in n-dimensional space.

On the other hand, (10) gives

$$V \geqq \kappa_n A^n/2^n \mu^{n-1}.$$

In the upper bound for V, there is equality if and only if K is a cone or double cone of revolution; in the lower bound there is equality if and only if K is a cylinder.

Inequality (1) can also be used directly to estimate other geometrical quantities associated with a general convex body K in $(n+1)$-dimensional space. As an example, if D is the diameter of K and σ is the least brightness of K, which we assume to be positive, then

$$\text{(11)} \qquad (D/2)^n < \left(\prod_{j=1}^{n} A_j\right)/\sigma.$$

The brightness of K in any direction is the n-dimensional volume of its orthogonal projection onto a plane normal to that direction; the least

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brightness is the attained minimum of the brightnesses over all directions.

To prove (11), choose the z-axis in the direction of maximal width of K so that we have $D = \mu$. In [2] it was shown that, for non-degenerate convex bodies,

$$D\sigma/(n+1) < V(K)$$

and this, together with (1) yields (11). Although (11) is a strict inequality, it cannot be improved. This can be seen by computing the quotient of the two sides of (11) for the following family of convex bodies $K(\zeta)$ and then letting ζ tend to infinity. $K(\zeta)$ is the dipyramid with vertices

$$(\pm 1, \pm 1, \cdots, \pm 1, 0), \quad (0, \cdots, 0, \pm \zeta)$$

formed by allowing all possible sign combinations. the least brightness of $K(\zeta)$ occurs in a direction which tends to that of z-axis as ζ tends to infinity.

References

[1] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, (Berlin, 1934).

[2] W. Firey, 'Lower bounds for volumes of convex bodies', *Archiv der Math.* 16 (1965), 69—74.

[3] G. Hardy, J. Littlewood and G. Pólya, *Inequalities*, (Cambridge, 1934).

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