Diffraction by a Half-Plane

By Ll. G. Chambers

1. Introduction.

The diffraction of a simple harmonic wave train by a straight-edged semi-infinite screen was originally discussed by Sommerfeld in 1895. The analysis is of a recondite character, involving the use of multivalued functions and Riemann surfaces (1). An alternative formulation of the problem is as an inhomogeneous Wiener-Hop integral equation, the solution of which also involves considerable difficulties (2). It is the purpose of this note to show that following Friedlander (3) it is possible by the use of parabolic co-ordinates to solve the problem by elementary methods. The method can be applied either to the case of sound or that of electromagnetism, the results being formally identical.

2. Preliminary Analysis.

We assume that the plane is defined by $x = 0, y > 0, -\infty < z < \infty$, the problem being two dimensional.

Let us now take parabolic co-ordinates.

$$x = l\,(\xi^2 - \eta^2), \qquad y = 2l\xi\eta,$$

where l is any convenient quantity of the dimensions of a length.

The ranges over which ξ, η run are $-\infty < \xi < \infty$, $0 \leqq \eta < \infty$.

By restricting η to positive values, it is possible to avoid the troubles of a two-valued function. The range of negative values of η corresponds to the second sheet of the Riemann surface corresponding to the plane cut along the positive real axis.

We write
$$\Box_0^2\, g \equiv \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 g}{\partial t^2}.$$

It can be shown that

$$\chi = \chi\,(\xi,\ \eta) = \int_0^P F\,(ct + l\overline{\xi^2 - \eta^2}\cos\alpha + 2l\xi\eta\sin\alpha - l\zeta^2)\,d\zeta,$$

where
$$P = \{\xi^2 + \eta^2 + \overline{\xi^2 - \eta^2}\cos\alpha + 2l\xi\eta\sin\alpha\}^{\frac{1}{2}}$$

is a solution of $\square_0^2\, X = 0$, as is $\chi\, (\xi, -\eta)$ also. This may be done by direct differentiation or by using the transformation

$$x' = -x \sin a + y \cos a, \qquad y' = x \cos a + y \sin a,$$

when χ takes the form

$$\chi = \int_0^{P'} F\,(ct + 2l\xi'\eta' - l\zeta^2)\,d\zeta,$$

where $P' = \xi' + \eta'$, and then differentiating (3), $\square_0^2$ being invariant with respect to the rotation of axes in the xy plane.

3. *Boundary Conditions.*

Under the transformation used above, the half-plane $y = 0, x > 0$ goes into $\eta = 0$.

Let Ψ be the quantity that is under consideration. For an ѕlectromagnetic case we have

$$\Psi = E_z \qquad \text{or} \qquad \Psi = H_z.$$

In the first case the boundary condition on the half-plane is $\partial\Psi/\partial y = 0$, and in the second case $\Psi = 0$, the half-plane being a perfect conductor in both cases. If we have a sound field and Ψ is the excess pressure, then $\partial\Psi/\partial y = 0$ corresponds to the case of a perfectly rigid body occupying the half-plane and $\Psi = 0$ would correspond to a free surface in the half-plane (were it realisable).

The result of applying the transformation is to make the boundary $\eta = 0$ and the boundary conditions

$$\frac{\partial\Psi}{\partial\eta} = 0 \qquad \text{or} \qquad \Psi = 0 \qquad\qquad \text{respectively.}$$

The other boundary condition that applies is that when $x \to -\infty$, *i.e.* $\eta \to \infty$, then $\Psi \sim \Psi_0$, where Ψ_0 is the incident disturbance.

4. *Satisfaction of the Boundary Conditions.*

For simplicity, we will only consider the case where $\Psi = 0$ on $\eta = 0$.

The incident field is of some arbitrary form Ψ_0 travelling in the direction $a + \pi$.

Let us consider the function

$$\Psi = \tfrac{1}{2}\Psi_0\,(ct + l\,\overline{\xi^2 - \eta^2}\,\cos a + 2l\,\xi\eta \sin a) - \tfrac{1}{2}\,\Psi_0\,(ct + l\,\overline{\xi^2 - \eta^2}\,\cos a$$
$$- 2l\xi\eta \sin a)$$
$$+ \int_0^P F\,(ct + l\overline{\xi^2 - \eta^2}\,\cos a + 2l\xi\eta \sin a - l\zeta^2)\,d\zeta$$
$$- \int_0^Q F\,(ct + l\overline{\xi^2 - \eta^2}\,\cos a - 2l\xi\eta \sin a - l\zeta^2)\,d\zeta,$$

$$P = \{\,\xi^2 + \eta^2 + \overline{\xi^2 - \eta^2}\,\cos\alpha + 2\xi\eta\,\sin\alpha\}^{\frac{1}{4}} \to \infty \text{ as } \eta \to \infty,$$

$$Q = \{\xi^2 + \eta^2 + \overline{\xi^2 - \eta^2}\,\cos\alpha - 2\xi\eta\,\sin\alpha\}^{\frac{1}{4}} \to -\infty \text{ as } \eta \to \infty.$$

It is obvious by inspection that

$$\Psi \text{ satisfies } \Box\,{}^2_0\Psi = 0, \qquad \text{with } \Psi = 0 \text{ on } \eta = 0.$$

In order to satisfy the condition $\Psi \sim \Psi_0$ as $\eta \to \infty$ we must have

$$\Psi_0\,(ct + l\overline{\xi^2 - \eta^2}\,\cos\alpha + 2l\xi\eta\,\sin\alpha)$$

$$= \tfrac{1}{2}\Psi_0\,(ct + l\overline{\xi^2 - \eta^2}\,\cos\alpha + 2l\xi\eta\,\sin\alpha) - \tfrac{1}{2}\,\Psi_0\,(ct + l\overline{\xi^2 - \eta^2}\,\cos\alpha$$
$$- 2l\xi\eta\,\sin\alpha)$$

$$+ \int_0^\infty F\,(ct + l\,\overline{\xi^2 - \eta^2}\,\cos\alpha + 2l\xi\eta\,\sin\alpha - l\zeta^2)\,d\zeta$$

$$- \int_0^\infty F\,(ct + l\overline{\xi^2 - \eta^2}\,\cos\alpha - 2l\xi\eta\,\sin\alpha - l\zeta^2)\,d\zeta$$

for $\eta \to \infty$.

The functions Ψ_0 and F must therefore be related by

$$\tfrac{1}{2}\Psi_0\,(\mu) = \int_0^\infty F\,(\mu - l\zeta^2)\,d\zeta + C. \tag{A}$$

As explained in Friedlander's paper (3) the arbitrary constant can be ignored, and the solution of the integral equation is given by

$$F\,(u) = \frac{1}{\pi}\,l^{\frac{1}{4}} \int_{-\infty}^u \frac{\Psi_0'\,(z)\,dz}{(u - z)^{\frac{1}{4}}} = \frac{2l^{\frac{1}{4}}}{\pi} \int_0^\infty \Psi_0'\,(u - t^2)\,dt \tag{B}$$

provided that $\qquad\qquad\qquad F(-\infty) = 0.$ (C)

It is clear by inspection of (A) that if we have a continuous wave of circular frequency $\omega = kc$, $\Psi_0 = \exp\,(ik\mu)$, then the solution is

$$F\,(u) = \sqrt{2}\,\exp\,(i\pi/4)\,\exp\,(ik\mu),$$

which in fact satisfies (B) even though (C) is not true.

If we take $\Psi_0\,(\mu) = \exp\,(ik\mu)$, our incident wave is

$$\exp\,\{ik\,(ct + l\overline{\xi^2 - \eta^2}\,\cos\alpha + 2l\zeta\eta\,\sin\alpha)\},$$

which is a plane wave travelling in the direction $\alpha + \pi$,

$$\Psi = \tfrac{1}{2}\exp\,\{ik\rho\,\cos\,(\phi - \alpha)\} - \tfrac{1}{2}\exp\,\{ik\rho\,\cos\,(\phi + \alpha)\}$$

$$+ \exp\,\{ik\rho\,\cos\,(\phi - \alpha)\}\sqrt{2}\,\exp\,(i\pi/4)\int_0^P \exp\,(-ikl\zeta^2)\,d\zeta$$

$$- \exp\,\{ik\rho\,\cos\,(\phi + \alpha)\}\,\sqrt{2}\,\exp\,(i\pi/4)\int_0^Q \exp\,(-ikl\zeta^2)\,d\zeta,$$

the time factor having been dropped.

Up till now l has been an arbitrary length. It will be convenient to take it as the wavelength $(kl = k\lambda = 2\pi)$,

$$P = \{\xi^2 - \eta^2 + \overline{\xi^2 - \eta^2} \cos \alpha + 2\xi\eta \sin \alpha\}^{\frac{1}{2}}$$
$$= (\rho + x \cos \alpha + y \sin \alpha)^{\frac{1}{2}}/\lambda^{\frac{1}{2}}$$
$$= (\rho/\lambda)^{\frac{1}{2}} [1 + \cos (\phi - \alpha)]^{\frac{1}{2}} = (2\rho/\lambda)^{\frac{1}{2}} \cos \tfrac{1}{2} (\phi - \alpha).$$

In a similar manner we find that

$$Q = (2\rho/\lambda)^{\frac{1}{2}} \cos \tfrac{1}{2} (\phi + \alpha).$$

By a little manipulation we get

$$\Psi = \tfrac{1}{2} \exp \{ik\rho \cos (\phi - \alpha)\} - \tfrac{1}{2} \exp \{ik\rho \cos (\phi + \alpha)\}$$
$$+ \exp \{ik\rho \cos (\phi - \sigma)\} \frac{1}{\sqrt{\pi}} \exp (i\pi/4) \int_0^p \exp (- i\gamma^2) \, d\gamma$$
$$- \exp \{ik\rho \cos (\phi + \alpha)\} \frac{1}{\sqrt{\pi}} \exp (i\pi/4) \int_0^q \exp (- i\gamma^2) \, d\gamma$$

where $\quad p = \sqrt{2k\rho} \cos \tfrac{1}{2} (\phi - \alpha), \qquad q = \sqrt{2k\rho} \cos \tfrac{1}{2} (\phi + \alpha).$

By making use of the fact that

$$1 = \frac{2}{\sqrt{-(i\pi)}} \int_{-\infty}^0 \exp (- i\gamma^2) \, d\gamma$$

we have immediately

$$\beta\Psi = \exp \{ik\rho \cos (\phi - \alpha)\} \int_{-\infty}^p \exp (- i\gamma^2) \, d\gamma$$
$$-\exp \{ik\rho \cos (\phi + \alpha)\} \int_{-\infty}^q \exp (- i\gamma^2) \, d\gamma$$

where $\beta = \sqrt{\pi} \exp (- i\pi/4)$.

It is fairly obvious that for the boundary condition $\partial\Psi/\partial\eta = 0$ on $\eta = 0$ we carry through an almost identical analysis beginning with the quantity

$$\Psi = \tfrac{1}{2} \Psi_0 (ct + l\overline{\xi^2 - \eta^2} \cos \alpha + 2l\xi\eta \sin \alpha) + \tfrac{1}{2} \Psi_0 (ct + l\overline{\xi^2 - \eta^2} \cos \alpha$$
$$- 2l\xi\eta \sin \alpha)$$
$$+ \int_0^P F (ct + l\overline{\xi^2 - \eta^2} \cos \alpha + 2l\xi\eta \sin \alpha - l\zeta^2) \, d\zeta$$
$$+ \int_0^Q F (ct + l \overline{\xi^2 - \eta^2 \cos} \alpha - 2l\xi\eta \sin \alpha - l\zeta^2) \, d\zeta.$$

We arrive at the result

$$\beta\Psi = \exp \{ik\rho \cos (\phi - \alpha)\} \int_{-\infty}^p \exp (- i\gamma^2) \, d\gamma$$
$$+ \exp \{ik\rho \cos (\phi + \alpha)\} \int_{-\infty}^q \exp (- i\gamma^2) \, d\gamma,$$

P, Q, p, q, β having the same meanings as before.

5. *Expressions for other field components.*

The results of the foregoing analysis may be used to give formulæ for the field components which are not parallel to the edge of the half-plane.   The two cases of electric vector perpendicular and parallel to the edge will be considered.   In order to write down formulæ for these components, it is first necessary to write down the results of certain differentiations.   Since

$$\frac{\partial F}{\partial x} = \cos\phi\,\frac{\partial F}{\partial\rho} - \sin\phi\,\frac{1}{\rho}\frac{\partial F}{\partial\phi}, \qquad \frac{\partial F}{\partial y} = \sin\phi\,\frac{\partial F}{\partial\rho} + \cos\phi\,\frac{1}{\rho}\frac{\partial F}{\partial\phi},$$

the values of $\dfrac{\partial G}{\partial x}, \dfrac{\partial G}{\partial y}$ can be written down fairly easily, where

$$G = \exp\{ik\rho\cos(\phi+\chi)\}\int_{-\infty}^{m}\exp(-i\gamma^2)\,d\gamma$$

and $$m = \sqrt{2k\rho}\,\cos\tfrac{1}{2}(\phi+\chi).$$

These values are given by

$$\frac{1}{ik}\frac{\partial G}{\partial x} = G\cos\chi + \sqrt{1/2k\rho}\,\exp\{-i[k\rho+\tfrac{1}{4}\pi]\}\cos\tfrac{1}{2}(\phi-\chi)$$

and

$$-\frac{1}{ik}\frac{\partial G}{\partial y} = G\sin\chi + \sqrt{1/2k\rho}\,\exp\{-i[k\rho-\tfrac{1}{4}\pi]\}\sin\tfrac{1}{2}(\phi-\chi).$$

We again write $$\beta = \sqrt{\pi}\exp(-i\pi/4).$$
We write also

$$G_+ = \exp\{ik\rho\cos(\phi+a)\}\int_{-\infty}^{q}\exp(-i\gamma^2)\,d\gamma$$

$$G_- = \exp\{ik\rho\cos(\phi-a)\}\int_{-\infty}^{p}\exp(-i\gamma^2)\,d\gamma$$

$$p = \sqrt{2k\rho}\,\cos\tfrac{1}{2}(\phi-a), \qquad q = \sqrt{2k\rho}\,\cos\tfrac{1}{2}(\phi+a).$$

(A) *Electric vector perpendicular to edge.*

The components which are of interest in this case are related by

$$E_x = Z_0\frac{1}{ik}\frac{\partial H_z}{\partial y}, \qquad E_y = -Z_0\frac{1}{ik}\frac{\partial H_z}{\partial x}.$$

The incident field is defined by

$$Z_0 H_z = E_0\exp\{ik(x\cos a + y\sin a)\}$$
$$= E_0\exp\{ik\rho\cos(\phi-a)\}$$

$$E_x = E_0 \exp\{ik\rho \cos(\phi - a)\} \sin a$$
$$E_y = -E_0 \exp\{ik\rho \cos(\phi - a)\} \cos a.$$

The diffracted field is defined by

$$\beta Z_0 H_z = E_0 (G_- + G_+),$$

and performing the necessary differentiations we deduce that

$$\beta E_x/E_0 = \sin a (G_- - G_+)$$
$$- \sqrt{2/k\rho} \sin \tfrac{1}{2}\phi \cos \tfrac{1}{2}a \exp\{-i(k\rho - \tfrac{1}{2}\pi)\}$$
$$\beta E_y/E_0 = -\cos a (G_- + G_+)$$
$$- \sqrt{2/k\rho} \exp\{-i(k\rho + \tfrac{1}{2}\pi)\} \cos \tfrac{1}{2}\phi \cos \tfrac{1}{2}a.$$

(B) *Electric vector parallel to edge.*

The components which are of interest in this case are related by

$$H_x = -Z_0^{-1} \frac{1}{ik} \frac{\partial E_z}{\partial y}, \qquad H_y = Z_0^{-1} \frac{1}{ik} \frac{\partial E_z}{\partial x}.$$

The incident field is defined by

$$E_z = E_0 \exp\{ik(x \cos a + y \sin a)\}$$
$$= E_0 \exp\{ik\rho \cos(\phi - a)\}$$
$$Z_0 H_x = -E_0 \exp\{ik\rho \cos(\phi - a)\} \sin a$$
$$Z_0 H_y = E_0 \exp\{ik\rho \cos(\phi - a)\} \cos a.$$

The diffracted field is defined by

$$\beta E_z = E_0 (G_- - G_+),$$

and performing the necessary differentiations we deduce that

$$\beta Z_0 H_x/E_0 = \sin a (G_- + G_+)$$
$$+ \sqrt{2/k\rho} \exp\{-i(k\rho - \tfrac{1}{2}\pi)\} \cos \tfrac{1}{2}\phi \sin \tfrac{1}{2}a$$
$$\beta Z_0 H_y/E_0 = \cos a (G_- - G_+)$$
$$- \sqrt{2/k\rho} \exp\{-i(k\rho + \tfrac{1}{2}\pi)\} \sin \tfrac{1}{2}\phi \sin \tfrac{1}{2}a.$$

(C) *Fresnel approximation.*

The Fresnel approximation, being scalar, is applicable to any of the six components of the electromagnetic field.

If $\Psi_0 = \exp\{ik(x \cos a + y \sin a)\}$ defines any incident field

component in the absence of a screen, then, to the first order, the field
in $y < 0$ is given by

$$\beta \Psi = \Psi_0 \int_{-\infty}^{S} \exp\left(- i\gamma^2\right) d\gamma$$

where $S = \sqrt{2k\rho}\frac{1}{2}\chi$ and $\chi = \pi - (\phi - a)$, ρ being assumed large and
χ being assumed small.

Now if χ is small

$$\tfrac{1}{2}\chi \sim \sin \tfrac{1}{2}\chi = \cos \tfrac{1}{2}(\pi - \chi) = \cos \tfrac{1}{2}(\phi - a)$$

and so $\qquad\qquad S \sim \sqrt{2k\rho}\, \cos \tfrac{1}{2}(\phi - a).$

Thus the Fresnel approximation gives the approximate result that
$\beta \Psi = G_-.$

On examining the results obtained from the exact theory, we
observe that $G_- \beta^{-1}$ is common to all the expressions for the six
field components.

6. *Discussion.*

It is clear from the form of the result that, provided the integral
equation (A) is soluble, the method of solution will hold for any shape
to the incident pulse, as has been indicated by Friedlander.

Furthermore, it is possible, formally at any rate, to obtain the
diffracted field from any incident field; for if we have an incident
field

$$\Psi_0(x, y) = \int_C A(a) \exp\{ik(x \cos a + y \sin a)\} da$$

then the resultant field will be obtained by taking the diffracted field
for a plane wave, multiplying by A (a) and integrating with respect
to a over the proper contour C in the complex a-plane.

The use of parabolic co-ordinates in this problem of the half-
plane obviates the use of the two-valued function of Sommerfeld, and
the line $\eta = 0$ corresponds to the slit in the Riemann surface at
which the two sheets coalesce, negative η corresponding to the unused
sheet.

Another formulation involves the replacing of the plane (for an
electromagnetic field) by a sheet of magnetic or electric current and
writing down the condition that, for a certain component, the sum of
the incident field and the field due to the sheet vanishes along the
sheet, thereby giving an integral equation of the Wiener-Hopf type
for the current distribution (2). Such an integral equation is soluble
only with considerable difficulty.

Examination of the conditions $p \gtrless 0, q \gtrless 0$ gives us further insight into the physical conditions involved. It is clear that

$$\frac{1}{\bar{\beta}} \int_{-\infty}^{T} \exp(-i\gamma^2)\, d\gamma \to \begin{array}{ll} 1 & (T \to +\infty) \\ 0 & (E \to -\infty). \end{array}$$

If we now consider as an example the case where $\Psi = 0$ on $\eta = 0$ (the other case is very similar), we have three separate cases:
(1) $p > 0, q > 0$. At large distances from the origin the field is effectively

$$\exp\{ik\rho \cos(\phi - a)\} - \exp\{ik\rho \cos(\phi + a)\}$$

which is the same as that for reflection by a complete plane.

(2) $p > 0, q < 0$. At large distances from the origin the field is effectively

$$\exp\{ik\rho \cos(\phi - a)\}$$

which is the same as the incident field.

(3) $p < 0, q < 0$. This is the geometrical shadow region and at large distances from the origin the field is effectively zero, the region being screened by the half plane.

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