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How to determine a curve singularity

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Abstract. We characterize the finite codimension sub-k-algebras of k[[t]] as the solutions of a computable finite family of higher differential operators. For this end, we establish a duality between such a sub-algebras and the finite codimension k-vector spaces of k[u], this ring acts on k[[t]] by differentiation.

1 Introduction

It is well-known that the normalization of a curve *X* is a non-singular curve *Y*. Serre considers in [26, Chapter IV] the opposite direction, he showed how to construct a curve *X* from a given non-singular curve *Y* such that this curve is the normalization of *X*. This idea appears in several different contexts. For instance, in [17, 18, 23] and the references therein, is studied how to determine the finite codimension sub-**k**-algebras *B* of **k**[*t*]. Notice that, in this case, X = Spec(B) is an algebraic curve and the affine line $Y = \text{Spec}(\mathbf{k}[t])$ is its normalization. These sub-algebras are defined recursively on the codimension by linear and higher differential conditions. Only for low codimensions, explicit conditions are known. Since not all higher differential conditions define sub-algebras of **k**[*t*], it is an open problem for the characterization of families of linear higher differential operators defining finite codimension sub-**k**-algebras of **k**[*t*] (see [18]).

In the search of one-dimensional reduced local rings with locally decreasing Hilbert function, Roberts constructed such a local rings as connex, finite codimension sub-**k**-algebras of $\prod_{i=1}^{r} \mathbf{k}[t_i]$ defined by linear and first-order differentials conditions (see [19]). See [11] for the proof of Sally's conjecture on the monotony of Hilbert functions of one-dimensional Cohen–Macaulay local rings.

In this paper, we consider the local complete case. We characterize the finite codimension sub-**k**-algebras *B* of $\Gamma = \mathbf{k}[[t]]$ as the solutions of a computable finite codimension **k**-vector space $B^{\perp} \subset \Delta = \mathbf{k}[u]$ of higher differential operators (see Theorem 3.9). For this purpose, we establish a Macaulay-like duality between finite codimension sub-**k**-algebras *B* of Γ and finite codimension **k**-vector subspaces B^{\perp} , so-called algebra-forming vector spaces, of the polynomial ring Δ . The polynomial ring Δ acts on Γ by differentiation as in Macaulay's duality (see [14–16, 20]). At the end



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of Section 3, we describe the linear maps $B_2^{\perp} \rightarrow B_1^{\perp}$ induced by **k**-algebra morphisms $B_1 \rightarrow B_2$ between two finite codimension **k**-algebras B_1 , B_2 .

In Section 4, we study the algebra-forming vector spaces, showing that such a condition can be checked effectively (see Proposition 4.1). After this, we prove that for any finite codimension $\delta \mathbf{k}$ -algebra *B* there exist a finite filtration of \mathbf{k} -algebras, so-called standard filtration of *B*, $B = B_0 \subset B_1 \subset \cdots \subset B_{\delta} = \Gamma$ such that dim_k(B_{i+1}/B_i) = 1 for $i = 0, \ldots, \delta - 1$. As corollary of this construction, we get that we only need to consider algebra-forming single elements in order to define recursively a finite codimension \mathbf{k} -algebras. Moreover, we show how to recover the standard filtration by considering recursively derivations of the local rings appearing in the filtration (see Corollary 4.6).

Section 5 is devoted to study the inverse system of monomial **k**-algebras and the special case of monomial Gorenstein algebras. We end the section relating the inverse system of a curve singularity with its generic plane projection and its saturation.

In the last section, we link B^{\perp} with the canonical module of *B* (see Proposition 6.1).

The computations of this paper are performed by using the computer algebra system singular (see [8]).

2 Preliminaries

Let *R* denote the power series ring $\mathbf{k}[[x_1, ..., x_n]]$ over an algebraically closed characteristic zero field \mathbf{k} and we denote by max = $(x_1, ..., x_n)$ its maximal ideal.

Let *A* be a one-dimensional local ring with maximal ideal max. We denote by HF_A the Hilbert function of *A*, i.e., $HF_A(i) = \text{Length}_A(\max^i / \max^{i+1})$, $i \ge 0$. It is well-known that $HF_A^0(i) = e_0(A)$, $i \gg 0$, where $e_0(A)$ is the multiplicity of *A*. The first integral of HF_A is defined by, $i \ge 0$,

$$\operatorname{HF}_{A}^{1}(i) = \sum_{j=0}^{i} \operatorname{HF}_{A}(j) = \operatorname{Length}_{A}(A/\operatorname{max}^{i+1}).$$

We write $\operatorname{HF}_{A}^{0} = \operatorname{HF}_{A}$. There exists an integer $e_{1}(A)$ such that $\operatorname{HF}_{A}^{1}(i) = e_{0}(A)(i+1) - e_{1}(A)$ for $i \gg 0$; the (first) Hilbert polynomial is $\operatorname{HP}_{A}^{1}(T) = e_{0}(A)(T+1) - e_{1}(A)$. See [22, Chapter XII] for the basic properties of the Hilbert functions of one-dimensional Cohen–Macaulay local rings.

A branch *X* is an irreducible curve singularity of $(\mathbf{k}^n, 0) = \text{Spec}(R)$, i.e., *X* is a onedimensional, integral scheme X = Spec(R/I); we write $\mathcal{O}_X = R/I$ and I(X) = I.

Let $v: \overline{X} = \operatorname{Spec}(\overline{\mathbb{O}_X}) \longrightarrow (X, 0)$ be the normalization of (X, 0), where $\overline{\mathbb{O}_X} \cong \mathbf{k}[[t]]$ is the integral closure of \mathcal{O}_X on its full field of fractions $\operatorname{tot}(\mathcal{O}_X)$. The singularity order of X is $\delta(X) = \dim_k (\mathcal{O}_{\overline{X}}/\mathcal{O}_X)$. We denote by C the conductor of the finite extension $v^* : \mathcal{O}_X \hookrightarrow \overline{\mathcal{O}_X}$ and by c(X) the dimension of $\overline{\mathcal{O}_X}/\mathcal{C}$.

Given a set of nonnegative integers $1 \le a_1 < \cdots < a_n$, we consider the monomial curve singularity $X(a_1, \ldots, a_n)$ defined by the parameterization

$$\begin{array}{cccc} \gamma : & R & \longrightarrow & \mathbf{k}[[t]] \\ & x_i & \mapsto & t^{a_i}, \end{array}$$

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i.e.,
$$I(X(a_1,...,a_n)) = \ker(\gamma)$$
. If $gcd(a_1,...,a_n) = 1$, then the induced map
 $\gamma : R/I(X(a_1,...,a_n)) \longrightarrow \mathbf{k}[[t]]$

is the normalization map of $\mathcal{O}_{X(a_1,\ldots,a_n)} = R/I(X(a_1,\ldots,a_n)) = \mathbf{k}[[t^{a_1},\ldots,t^{a_n}]].$

We denote by D_X the semigroup of values of X: the set of integers $v_t(f) = ord_t(t)$ where $f \in \mathcal{O}_X \setminus \{0\}$. It is easy to see that $\delta(X) = #(\mathbb{N} \setminus D_X)$. If B is a finite codimension sub-**k**-algebra of Γ then $X = \operatorname{Spec}(B)$ is branch. We write $D_B = D_X$.

Let ω_X be the dualizing module of *X*; we can consider the composition of \mathcal{O}_X -module morphisms

$$\gamma_X : \Omega_X \longrightarrow \nu_* \Omega_{\overline{X}} \cong \nu_* \omega_{\overline{X}} \longrightarrow \omega_X.$$

Let $d: \mathcal{O}_X \longrightarrow \Omega_X$ the universal derivation, then we have a **k**-linear map $\gamma_X d$ that we also denote by $d: \mathcal{O}_X \longrightarrow \omega_X$. Recall that the Milnor number of *X* is $\mu(X) = \dim_k(\omega_X/d\mathcal{O}_X)$, [5]. Since we only consider branches we have that $\mu(X) = 2\delta(X)$ (see [5, Proposition 1.2.1]). Notice that *X* is non-singular iff $\mu(X) = 0$ iff $\delta(X) = 0$ iff c(X) = 0.

We denote by $\pi : Bl(X) \longrightarrow X$ the blowing-up of *X* on its closed point. The fiber of the closed point of *X* has a finite number of closed points: the so-called points of the first neighborhood of *X*. We can iterate the process of blowing-up until we get the normalization of *X* (see [7, 24]). We denote by Inf(X) the set of infinitely near points of *X*. The curve singularity defined by an infinitely point *p* of *X* will be denote by (X, p); we set (X, 0) = X.

Proposition 2.1 Let X be a branch. Then (i)

$$\delta(X) = \sum_{p \in \mathrm{Inf}(X)} e_i(X, p).$$

(ii) It holds

$$e_0(X) - 1 \le e_1(X) \le \delta(X) \le \mu(X)$$

and $e_1(X) \leq \binom{e_0(X)}{2} - \binom{n-1}{2}$.

(iii) If X is singular, then $\delta(X) + 1 \le c(X) \le 2\delta(X)$, and $c(X) = 2\delta(X)$ if and only if \mathcal{O}_X is a Gorenstein ring.

Proof (i) [25]. (ii) [5, Proposition 1.2.4(i)] and [10, 12, 25]. (iii) [26, Proposition 7, page 80] and [2]. ■

3 Macaulay-like duality

In this section, we establish a Macaulay-like duality for the family of sub-**k**-algebras *B* of $\Gamma = \mathbf{k}[[t]]$ of finite codimension. For the classical Macaulay's duality, see [20], [14], and for the generalization to higher dimension of Macaulay's duality, see [15]. Recall that Macaulay's duality is a particular case of Matlis' duality (see [4]).

We write $\Delta = \mathbf{k}[u]$; Γ is a Δ -module with Δ acting on Γ by derivation. This action denoted by \circ is defined by

$$\begin{array}{cccc} \circ : \Delta \times \Gamma & \longrightarrow & \Gamma \\ (g, f) & \rightarrow & g \circ f = g(\partial_t)(f), \end{array}$$

where ∂_t denotes the derivative with respect to *t*. This action induces a non-singular **k**-bilinear perfect pairing:

(1)
$$\begin{array}{cccc} \bot : & \Delta \times \Gamma & \longrightarrow & \mathbf{k} \\ & & (g, f) & \mapsto & g \perp f = (g \circ f)(0). \end{array}$$

Definition 3.1 Given a sub-**k**-algebra B of $\Gamma = \mathbf{k}[[t]]$ we define B^{\perp} as the set of $g \in \Delta$ such that $g \perp f = 0$ for all $f \in B$. Notice that B^{\perp} is a **k**-vector subspace of Δ , this is, following the classic Macaulay's duality terminology, the inverse system of B. Given a **k**-vector subspace $V \subset \Delta$ we consider $\operatorname{Ann}(V) \subset \Gamma$ as the set of power series $f \in \Gamma$ such that $g \perp f = 0$ for all $g \in V$.

Let *B* be a finite codimension sub-**k**-algebra of Γ . Then we have a non-singular **k**-bilinear perfect pairing:

(2)
$$\begin{array}{cccc} \bot : & B^{\perp} \times \frac{\Gamma}{B} & \longrightarrow & \mathbf{k} \\ & & & & & \\ & & & & & g \perp f \end{array}$$

We denote by Perp(B), the **k**-vector space of maps

$$\begin{array}{ccccc} g^{\perp} : & B & \longrightarrow & \mathbf{k} \\ & f & \mapsto & g \perp f \end{array}$$

for all $g \in \Delta$. These maps are the elements of the dual space of *B* with finite support: $g^{\perp}(\max_{B}^{d}) = 0$ for $d > \deg(g)$. We denote by $Der_{\mathbf{k}}(B)$ the **k**-vector space of **k**-derivations of *B*. Since $Der_{\mathbf{k}}(B) \cong (\max_{B} / \max_{B}^{2})^{*}$, we can identify $Der_{\mathbf{k}}(B)$ with the **k**-vector space of elements σ of the dual space of *B* such that $\sigma(\max_{B}^{2}) = 0$.

We have $Der_{\mathbf{k}}(B) \subset Perp(B)$, this inclusion is strict. Let us consider the codimension 8 algebra $B = \mathbf{k}[[t^4, t^7, t^{17}]]$. The linear map $(u^{11})^{\perp} : B \longrightarrow \mathbf{k}$ is not a derivation since $t^{11} \in \max_B^2$ and $(u^{11})^{\perp}(t^{11}) = 11! \neq 0$.

Next step is to characterize the vector **k**-vector subspaces B^{\perp} of Δ , where *B* ranges the family of finite codimension sub-**k**-algebras of Γ . First, we give some properties of B^{\perp} that we will use along the paper.

Given a polynomial $g = \sum_{i=0}^{d} a_i u^i \in \Delta$ we denote by Supp(g) the support of g: the finite set of integers *i* such that $a_i \neq 0$.

Proposition 3.2 Let $B \subset \Gamma$ be a codimension δ sub-**k**-algebra B of Γ , and let $C = (t^c)$ be the conductor of the extension $B \subset \Gamma$. Then:

- (1) $\dim_{\mathbf{k}}(B^{\perp}) = \delta$.
- (2) For all $g \in B^{\perp}$, we have $\operatorname{Supp}(g) \subset [1, c-1]$, and

$$u^{[1,e_0(B)-1]} = \{ u^i; i \in [1,e_0(B)-1] \} \subset B^{\perp} \subset (u,u^2,\ldots,u^{c-1}).$$

(3) The following conditions are equivalent:

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(i)
$$\delta = 0$$
,
(ii) $B = \Gamma$,
(iii) $B^{\perp} = 0$,
(iv) $B^{\perp} \subset \langle u^2, u^3, \dots \rangle$.

Proof (1) Since \perp is a **k**-bilinear perfect pairing, we get $\dim_k(B^{\perp}) = \delta$, see the equation (2).

(2) Since *B* is a **k**-algebra, we have $1 \in B$, so if $g = \sum_{j\geq 0} a_i u^i \in B^{\perp}$, then $0 = g \perp 1 = a_0$. Hence $B^{\perp} \subset \langle u, u^2, \ldots \rangle$. We know that $(t^c) \subset B$ so for all $g = \sum_{j\geq 0} a_i u^i \in B^{\perp}$, we have

$$0 = g \perp t^{c+i} = (c+i)!a_{c+i}$$

 $i \ge 0$. Hence, if $g \in B^{\perp}$, then $\deg(g) \le c - 1$. From this, we deduce that $B^{\perp} \subset \langle u, u^2, \ldots, u^{c-1} \rangle$.

Notice that $v_t(f) \ge e_0(B)$ for all $f \in B \setminus \{1\}$, so given $i \in [1, e_0(B) - 1]$ we have $u^i \perp f = 0$. Hence $u^i \in B^{\perp}$ and then $u^{[1, e_0(B) - 1]} \subset B^{\perp}$.

(3) The condition of (*i*) is equivalent to (*ii*). (*ii*) trivially implies (*iii*) and this implies (*iv*). If $B^{\perp} \subset \langle u^2, u^3, \ldots \rangle$, then $t \in B$, since *B* is a **k**-algebra, we get (*ii*).

For all power series $f = \sum_{i\geq 0} b_i t^i \in \Gamma$ and given a nonnegative integer $s \in \mathbb{N}$, we denote by $[f]_{\leq s}$ the truncated polynomial $[f]_{\leq s} = \sum_{i\geq 0}^{s} b_i t^i$.

Let *B* be a finite codimension sub-**k**-algebra of Γ with conductor *c*. Then *B* is a finitely generated **k**-algebra; let f_1, \ldots, f_r be a system of generators of *B* as **k**-algebra. We denote by $|_{B,d}$, $d \ge c - 1$, the finite set of polynomials $[f_1^{l_1} \ldots f_r^{l_r}]_{\le d}$ with $l_i \ge 0$, $i = 1, \ldots, r$, and $l_1 + \cdots + l_r \le d$. We denote by $W(\{f_1, \ldots, f_r\}, d) \subset \Delta$ the **k**-vector space generated by the polynomials of $|_{B,d}$. Notice that $W(\{f_1, \ldots, f_r\}, d) + \langle t^{d+1} \rangle = W(\{f_1, \ldots, f_r\}, d+1)$.

Proposition 3.3 Let B be a finite codimension sub-k-algebra of Γ with conductor c. Then B^{\perp} is the set of $g \in \Delta$ of degree at most c - 1 and such that $g \perp h = 0$ for all $h \in \mathfrak{g}_{B,c-1}$.

Proof Let f_1, \ldots, f_r be a system of generators of *B* as **k**-algebra, and let $|_{B,c-1}$ be the associated set of polynomials.

If $g \in B^{\perp}$, then deg $(g) \leq c - 1$, Proposition 3.2(2), so

$$0 = g \perp \left(f_1^{l_1} \dots f_r^{l_r}\right) = g \perp \left[f_1^{l_1} \dots f_r^{l_r}\right]_{\leq c-1}.$$

Hence, $g \perp h = 0$ for all $h \in \mathfrak{g}_{B,c-1}$.

Let $g \in \Delta$ be a polynomial with $\deg(g) \le c - 1$ and such that $g \perp h = 0$ for all $h \in \mathfrak{g}_{B,c-1}$. Any $f \in B$ can be written as

$$f = \sum_{l_1,\ldots,l_r \in \mathbb{N}} c_{l_1,\ldots,l_r} f_1^{l_1} \ldots f_r^{l_r}$$

with $c_{l_1,\ldots,l_r} \in \mathbf{k}$. Since deg $(g) \leq c - 1$, we have

$$g \perp f = \sum_{l_1,...,l_r \in \mathbb{N}} c_{l_1,...,l_r} (g \perp f_1^{l_1} \dots f_r^{l_r}) = \sum_{l_1,...,l_r \in \mathbb{N}} c_{l_1,...,l_r} (g \perp [f_1^{l_1} \dots f_r^{l_r}]_{\leq c-1}) = 0,$$

so
$$g \in B^{\perp}$$

Remark 3.4 Notice that Proposition 3.3 shows that the computation of B^{\perp} is effective. In fact, in the set $\downarrow_{B,c-1}$, there are involved a finite number of monomials and we only have to consider polynomials *g* of degree at most c - 1.

Remark 3.5 Although B^{\perp} is a k-vector subspace of Δ for any sub-k-algebra B of Γ , not all $\operatorname{Ann}(V)$ is a k-algebra for a given k-vector subspace $V \subset \Delta$. In fact, let us consider the k-vector subspace $V \subset \Delta$ generated by u^2 . Then $\operatorname{Ann}(V)$ is the set of $f = \sum_{i\geq 0} a_i t^i \in \Gamma$ such that $a_2 = 0$. This is not a k-algebra because $u^2 \perp t = 0$, so $t \in \operatorname{Ann}(V)$ and $u^2 \perp t^2 = 2 \neq 0$, so $t^2 \notin \operatorname{Ann}(V)$.

Definition 3.6 A finite dimensional **k**-vector subspace $V \subset \Delta$ is so-called algebraforming with respect to a **k**-algebra $B \subset \Gamma$ iff the following conditions hold:

(a) g(0) = 0 for all $g \in V$ and,

(b) for all $f \in B$ such that $g \perp f = 0$ for all $g \in V$ it holds $g \perp f^2 = 0$ for all $g \in V$.

An element $g \in \Delta$ is so-called algebra-forming with respect to *B* if $V = \langle g \rangle$ is algebra-forming with respect to *B*.

Example 3.7 Let us consider the codimension $\delta = 4$ algebra $B = \mathbf{k}[[t^3 + t^4, t^5]]$ of Γ . The conductor of B is c = 8. Then B^{\perp} is the set of polynomials $g \in \Delta$ of degree at most 7 such that $g \perp f = 0$ for $f \in {}_{\downarrow B,c-1} = \{t^3 + t^4, t^5, t^6 + 2t^7\}$. A simple computation shows that B^{\perp} is the **k**-vector space generated by the four linear independent polynomials $u, u^2, u^3 - \frac{1}{4}u^4, u^6 - \frac{1}{27}u^7$. Let us consider

$$B_2 = \mathbf{k}[[t^3, t^4, t^5]] \subset B_3 = \mathbf{k}[[t^2, t^3]],$$

then we have $B_2 = \text{Ann}\langle u^2 \rangle \cap B_3$, i.e., u^2 is an algebra-forming element with respect to B_2 .

In the following result, we prove that, in fact, if $V \subset \Delta$ is algebra-forming with respect to a **k** algebra $B \subset \Gamma$, then Ann $(V) \cap B$ is a sub-**k**-algebra of Γ .

Proposition 3.8 Let $V \subset \Delta$ be an algebra-forming **k**-vector subspace with respect to a **k**-algebra $B \subset \Gamma$. Then Ann $(V) \cap B$ is a sub-**k**-algebra of Γ .

Proof Clearly $C = Ann(V) \cap B$ is a k-vector subspace of Γ . Given $f_1, f_2 \in C$ we have that $f_1 + f_2 \in C$ and from

$$f_1f_2 = \frac{1}{2}((f_1 + f_2)^2 - f_1^2 - f_2^2),$$

we deduce that $g \perp (f_1 f_2) = 0$, i.e., $f_1 f_2 \in C$. Since g(0) = 0 for all $g \in V$ we get $1 \in C$, so *C* is a sub-**k**-algebra of Γ .

The following result is an extension of Macaulay's duality to finite codimension sub-**k**-algebras $B \subset \Gamma$.

Theorem 3.9 Given a nonnegative integers $\delta > 0$ and $c \ge \delta + 1$, there is a one-to-one correspondence \perp between the following sets:

- (1) sub-**k**-algebras B of Γ of codimension δ as **k**-vector spaces such that the conductor of $B \subset \Gamma$ is (t^c) ,
- (2) algebra forming, with respect to Γ , **k**-vector subspace $V \subset \Delta$ of dimension δ , generated by polynomials of degree at most c 1 and such that there is a polynomial $g \in V$ with deg(g) = c 1.

This correspondence is inclusion reversing: given two sub-**k**-algebras B_1 and B_2 of Γ , $B_1 \subset B_2$ if and only if $B_2^{\perp} \subset B_1^{\perp}$.

Proof Let *B* be a sub-**k**-algebra *B* of Γ . Since we have a non-singular **k**-bilinear pairing:

we get that B^{\perp} is a **k**-vector subspace of dimension δ of Δ . By definition B^{\perp} is algebraforming with respect to Γ . Being *c* the conductor we have $(t^c) \subset B$, so deg $(g) \leq c - 1$ for all $g \in B^{\perp}$ and there exist $g \in B^{\perp}$ of degree c - 1.

Let *V* be an algebra forming, with respect to Γ , **k**-vector subspace satisfying the conditions of (2). Let us consider the **k**-algebra $B = \operatorname{Ann}(V)$. From the perfect pairing (1), we get that the codimension of *B* in Γ is δ . Since *V* is generated by polynomials of degree at most c - 1 we have that $(t^c) \subset B$, so the conductor of *B* is at most *c*. Furthermore, since there is $g \in V$ with deg(g) = c - 1 we deduce that *c* is the conductor of *B*.

It is straightforward to prove the inclusion reversing from the definition of the inverse system B^{\perp} .

We end this section by describing the **k**-linear maps $B_2^{\perp} \longrightarrow B_1^{\perp}$ induced by **k**-algebra isomorphisms $B_1 \longrightarrow B_2$ between two finite codimension **k**-algebras B_1 and B_2 of Γ . Let *c* be an integer bigger than the conductors of B_1 and B_2 .

The perfect pairing (1) induce a perfect pairing

$$\begin{array}{cccc} \bot : & \Delta_{\leq c-1} \times \frac{\Gamma}{(t^c)} & \longrightarrow & \mathbf{k} \\ & & (g,\overline{f}) & \mapsto & g \perp f = (g \circ f)(0), \end{array}$$

where $\Delta_{\leq c-1}$ is the **k**-vector space of polynomials of degree at most c-1. We consider the usual **k**-vector basis of $\Gamma/(t^c)$ of the cosets of t^i , i = 0, ..., c-1. Its dual basis is $\frac{1}{i!}u^i$, i = 0, ..., c-1, since

$$\left(\frac{1}{i!}u^i\right) \perp t^j = \delta_{i,j}$$

 $1 \leq i, j \leq c - 1$.

The **k**-algebra B_i has conductor at most c so we can consider that $B_i \subset \Gamma/(t^c)$, i = 1, 2. On the other hand, from Proposition 3.2, we have that $B_i^{\perp} \subset \Delta_{\leq c-1}$, i = 1, 2.

If B_1 is isomorphic to B_2 by ϕ , then their normalizations are isomorphic:

$$\Gamma = \overline{B_1} \stackrel{\overline{\phi}}{\cong} \overline{B_2} = \Gamma.$$

This automorphism is determined by a power series $h(t) \in (t)$ such that $u \perp h \neq 0$ and

$$\overline{\phi} : \begin{array}{ccc} \Gamma & \longrightarrow & \Gamma \\ f & \mapsto & f(h). \end{array}$$

Then we have an isomorphism of k-vector spaces

$$\frac{\Gamma}{B_1} \xrightarrow{\overline{\phi}} \frac{\Gamma}{B_2}$$

and the perfect pairing induces a k-vector isomorphism

$$\phi^*: B_2^{\perp} \longrightarrow B_1^{\perp}.$$

The matrix M_{ϕ} associated with ϕ in the basis t^i , $i = 0, \ldots, c-1$, is the $c \times c$ matrix whose columns are the coefficients of $\phi(t^i) = h^i$, $i = 0, \ldots, c-1$, with respect to this basis. Hence, the matrix of $\phi^* : B_2^* = B_2^\perp \longrightarrow B_1^* = B_1^\perp$ with respect to the basis $\frac{1}{i!}u^i$, $i = 0, \ldots, c-1$, is the transpose matrix ${}^{\tau}M_{\phi}$ of M_{ϕ} .

Example 3.10 Let $B_2 \subset \Gamma$ be a **k**-algebra generated by two elements f_1 , f_2 with $v_t(f_1) = 2$ and $v_t(f_2) = 7$. We may assume that $f_1 = t^2 + \text{monomials of higher degree}$. Then B_2 is of finite codimension $\delta = 3$ and conductor c = 6.

Since Γ is complete there exist a power series $h \in (t)$ such that $h^2 = f_1$; we write $h = t + h_2 t^2 + \dots + h_5 t^5 + \dots$ Notice that $\Gamma = \mathbf{k}[[h]]$.

Let ϕ the automorphism of Γ defined by h, i.e., $\phi(f) = f(h)$. Then $\phi^{-1}(B_2)$ is a **k**-algebra B_1 generated by $f'_1 = t^2$ and $f'_2(h)$ such that $v_h(f'_2) = 7$. After a change of generators B_1 is generated by $f'_1 = t^2$ and $f'_2 = t^7$.

The induced isomorphism $\phi : B_1 \longrightarrow B_2$ has the following 6×6 associated matrix with respect the basis t^i , $i = 0, \dots, 5$,

$$M_{\phi} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 1 & 0 & 0 & 0 \\ 0 & h_3 & 2h_2 & 1 & 0 & 0 \\ 0 & h_4 & 2h_3 + h_2^2 & 3h_2 & 1 & 0 \\ 0 & h_5 & 2b_4 + 2h_2h_3 & 3h_3 + 3h_2^2 & 4h_2 & 1 \end{pmatrix}.$$

Then the matrix of the isomorphism $\phi^* : B_2^{\perp} \longrightarrow B_1^{\perp}$ with respect to $\frac{1}{i!}u^i$, $i = 0, \ldots, 5$, is M_{ϕ}^{τ} . Since B_1 is the monomial **k**-algebra $\mathbf{k}[[t^2, t^7]]$, the **k**-vector space B_1^{\perp} is generated by u, u^3, u^5 . From this, we can compute B_2^{\perp} by considering $({}^{\tau}M_{\phi})^{-1}$.

4 Algebra-forming vector spaces

The first goal of this section is to characterize the algebra-forming k-vector spaces.

Proposition 4.1 Let B be a k-sub-algebra of finite codimension of Γ with conductor c, and let f_1, \ldots, f_s be a system of generators of B. Given an integer $d \ge c - 1$, let h_1, \ldots, h_m be a system of generators of $W(\{f_1, \ldots, f_s\}, d)$.

Let V be a dimension δ **k**-vector subspace of $(u) \subset \Delta$ generated by polynomials of degree at most d - 1. Let $g_1, \ldots, g_{\delta} \in V$ be a basis of V.

Then V is algebra-forming with respect to B iff for all r-upla $(\lambda_1, \ldots, \lambda_m) \in \mathbf{k}^m$ such that

(3)
$$\sum_{j=1}^{m} \lambda_j (g_i \perp h_j) = 0$$

for all $i = 1, \ldots, \delta$, then

(4)
$$\sum_{j=1}^{m} \lambda_j^2 (g_i \perp h_j^2) + 2 \sum_{j=1, l=1, j \neq l}^{m} \lambda_j \lambda_j (g_i \perp h_j h_l) = 0$$

for all $i = 1, \ldots, \delta$.

Proof From Proposition 3.2, we have to prove that for all $f \in B$ such that $g \perp f = 0$ for all $g \in V$ we have that $g \perp f^2 = 0$ for all $g \in V$. Since the polynomials of *V* are of degree at most d - 1 we only have to prove that for all $f \in W = W(\{f_1, \ldots, f_s\}, d)$ such that $g \perp f = 0$ for all $g \in V$, we have that $g \perp f^2 = 0$ for all $g \in V$.

A general element of *W* can be written as $f = \sum_{j=1}^{m} \lambda_j h_j$. Hence the condition $g_i \perp f = 0$ is equivalent to

$$\sum_{j=1}^m \lambda_j (g_i \perp h_j) = 0$$

for all $i = 1, ..., \delta$. Similarly, the condition $g_i \perp f^2 = 0$ is equivalent to

$$\sum_{j=1}^m \lambda_j^2(g_i \perp h_j^2) + 2 \sum_{j=1,l=1,j\neq l}^m \lambda_j \lambda_j (g_i \perp h_j h_l) = 0$$

for all $i = 1, \ldots, \delta$.

Remark 4.2 The set of points $(\lambda_1, \ldots, \lambda_m) \in \mathbb{P}_k^{m-1}$ satisfying the identities of (3) form a linear subvariety L, and the points satisfying the identities of (4) defines a subvariety $Q \subset \mathbb{P}_k^{m-1}$ intersection of δ quadrics. Hence, V is algebra forming with respect to B iff $L \subset Q$. This is a computable condition.

Definition 4.3 Let *B* be a sub-**k**-algebra of finite codimension δ of Γ and conductor *c*. Let *D* be the semigroup of *B*; we write the set $t^{\mathbb{N} \setminus D_B} = \{t^i; i \in \mathbb{N} \setminus D_B\}$ as $g_1 = t^{c-1}, \ldots, g_{\delta} = t$. Then we define the so-called standard filtration of *B* as follows: B_i is the **k**-algebra generated by B and g_1, \ldots, g_i for $i = 1, \ldots, \delta$; we set $B_0 = B$. Notice

that $B_{\delta} = \Gamma$ and that we have

 $B = B_0 \subset B_1 \subset \cdots \subset B_\delta = \Gamma$

and $\dim_{\mathbf{k}}(B_{i+1}/B_i) = 1, i = 0, ..., \delta - 1.$

After the definition of standard filtration, we only have to consider algebra-forming elements $g \in \Delta$, with respect a suitable sub-**k**-algebras of Γ , in order to define a **k**-algebra recursively. The algebra-forming elements are not unique as the following example shows.

Example 4.4 Let us consider the Example 3.7. The standard filtration of *B* is

$$B = \mathbf{k}[[t^3 + t^4, t^5]] \subset B_1 = \mathbf{k}[[t^3 + t^4, t^5, t^7]] \subset B_2 = \mathbf{k}[[t^3, t^4, t^5]] \subset B_3 = \mathbf{k}[[t^2, t^3]] \subset \Gamma.$$

The chain of **k**-algebras is defined as follows. The cosets of t, t^2 , t^4 , t^7 in Γ/B form a basis of Γ/B as **k**-vector space. Then B_1 is the **k**-algebra generated by B and t^7 , B_2 is the **k**-algebra generated by B_1 and t^4 , B_3 is the **k**-algebra generated by B and t^2 , and finally Γ is the **k**-algebra generated by B and t.

We know that B^{\perp} is a four-dimensional **k**-vector space generated by $u, u^2, u^3 - \frac{1}{4}u^4, u^6 - \frac{1}{2.7}u^7$; we have $B_3 = \operatorname{Ann}\langle u \rangle$, $B_2 = \operatorname{Ann}\langle u^2 \rangle \cap B_3$, $B_1 = \operatorname{Ann}\langle u^3 - \frac{1}{4}u^4 \rangle \cap B_2$, $B = \operatorname{Ann}\langle u^6 - \frac{1}{2.7}u^7 \rangle \cap B_1$. On the other hand, the **k**-algebra $C_1 = \mathbf{k}[[t^3 + t^5, t^4]] \subset B_1$ can be obtained as

$$C_1 = \operatorname{Ann}\langle u^3 - \frac{1}{4.5}u^5\rangle \cap B_2,$$

i.e., $u^3 - \frac{1}{4.5}u^5$ is an algebra-forming element with respect to B_2 . Notice that B_1 and C_1 are non analytically isomorphic codimension one **k**-algebras of B_2 .

Next, we show how to build the standard filtration by using derivations.

Proposition 4.5 Let $C \subset B$ be two sub-k-algebras of Γ such that $\dim_k(B/C) = 1$. There exist $\alpha \in Der_k(B)$ such that $\ker(\alpha) = C$.

Proof If we denote by \max_B , the maximal ideal of *B* then $\max_C \subset \max_B$, $\dim_k(\max_B / \max_C) = 1$ and $\max_B^2 \subset \max_C$. Since we have

$$\frac{\max_C}{\max_B^2} \subset \frac{\max_B}{\max_B^2}$$

we deduce that there exists a linear form $\alpha : \frac{\max_B}{\max_B^2} \longrightarrow \mathbf{k}$ such that $\ker(\alpha) = \frac{\max_C}{\max_B^2}$. From this, we get the claim.

Corollary 4.6 Let B be a sub-k-algebra of finite codimension δ of Γ . Let us consider the standard filtration of B:

$$B = B_0 \subset B_1 \subset \cdots \subset B_{\delta} = \Gamma.$$

For all $i = 1, ..., \delta$, there exists a derivation $\partial_{l_i} \in Der_k(B_i)$, $l_i \in max_{B_i}$, such that $ker(\partial_{l_i}) = B_i$.

Example 4.7 Let us consider the Example 4.4. The element u^{\perp} corresponds to the derivation ∂_t of Γ defined by t, so $B_3 = \ker(\partial_t)$. The maximal ideal of B_3 is minimally generated by t^2 , t^3 , the element $(u^2)^{\perp}$ is the derivation $\partial_{t^2} \in Der_k(B_3)$, so $B_2 = \ker(\partial_{t^2})$. The maximal ideal of B_2 is minimally generated by t^3 , t^4 , t^5 . The element $(u^3 - \frac{1}{4}u^4)^{\perp}$ is the derivation $\partial_{t^3 - \frac{1}{4}t^4} \in Der_k(B_2)$, so $B_1 = \ker(\partial_{t^3 - \frac{1}{4}t^4})$. Finally, $\partial_{t^7} \in Der_k(B_1)$ and $B = \ker(\partial_{t^7})$.

5 Monomial algebras

In this section, we first compute the inverse system of a monomial **k**-algebra. After this, we characterize monomial Gorenstein curve singularities in terms of its inverse system. We end the section relating the inverse system of a curve singularity with its generic plane projection and its saturation.

The following result it is easy to deduce from the proof of the second part of Proposition 3.2(2).

Proposition 5.1 Let D be an additive sub-semigroup of \mathbb{N} with finite complement. Then B^{\perp} is the **k**-vector space generated by: $g_i = u^i$ for $i \in \mathbb{N} \setminus D$.

Example 5.2 Let *B* be a sub-**k**-algebra of **k**[[*t*]] of codimension $\delta = 1$. Then *B* is the **k**-algebra $B = \mathbf{k}[[D]]$, where *D* is the sub-semigroup of \mathbb{N} generated by 2, 3. Hence, B^{\perp} is the **k**-vector space generated by *u*, i.e., *B* is the set of power series $f = \sum_{i\geq 0} b_i t^i \in \mathbf{k}[[t]]$ with $u \perp f = b_1 = 0$ (see [26, Example b, Section 4 of Chapter IV] and [18, Section 22]).

Example 5.3 Assume now that *B* is sub-k-algebra of $\mathbf{k}[[t]]$ of codimension $\delta = 2$. Then its semi-group D_B is $D_1 = \langle 2, 5 \rangle$ or $D_2 = \langle 3, 4 \rangle$. In the first case, *B* is generated as k-algebra by $f_1 = t^2 + b_3 t^3$ and $f_2 = t^5$. The conductor is c = 4. Then B^{\perp} is generated by $g_1 = u$, $g_2 = 6b_3u^2 + u^3$. In the second case, *B* is the monomial k-algebra $B = \mathbf{k}[[D_2]]$ so B^{\perp} is the sub-k-algebra generated by $g_1 = u$ and $g_2 = u^2$. The conductor is c = 5 (see [18, Section 23]). It is known that the algebras of the first case are all analytically isomorphic to $\mathbf{k}[[D_1]]$.

The inverse system of a monomial Gorenstein \mathbf{k} -algebra case can be handled. Let us recall the definition of symmetric semi-group and the celebrate result of Kunz.

Definition 5.4 We say that a sub-semigroup D of \mathbb{N} such that $\#(\mathbb{N} \setminus D) < \infty$ and with conductor c is symmetric if the condition $t \in D$ is equivalent to $c - 1 - t \notin D$.

Kunz proved that the ring $\mathbf{k}[[D]]$ is Gorenstein ring if and only if *D* is a symmetric semigroup,[21]. This symmetry is inherited by B^{\perp} .

Proposition 5.5 Let D be a sub-semigroup of \mathbb{N} such that $\#(\mathbb{N} \setminus D) < \infty$ and conductor c. The following conditions are equivalent:

- (1) $\mathbf{k}[[D]]$ is Gorenstein,
- (2) for all $g \in \mathbf{k}[[D]]^{\perp}$ it holds $t^{c-1}g(1/t) \in \mathbf{k}[[D]]$.

Proof Since $B = \mathbf{k}[[D]]$ is a monomial **k**-algebra we know that B^{\perp} is generated by $g = \sum_{i=1}^{c-1} a_i u^i$ such that $a_i = 0$ for $i \in D$ (see Proposition 5.1). Then the exponents of the nonzero terms of $t^{c-1}g(1/t)$ are c - 1 - i with $i \notin D$. Then the claim is equivalent to the symmetry of D, i.e., the Gorensteinness of B.

Example 5.6 Let *D* be the semigroup generated by 4, 6, and 9. This is a symmetric semigroup with conductor c = 12. The algebra $B = \mathbf{k}[[D]]$ is Gorenstein and isomorphic to $\mathbf{k}[[x, y, z]]/I$, where $I = (x^3 - y^2, y^3 - z^2)$. Then B^{\perp} is generated by the polynomials $g = a_1u + a_2u^2 + a_3u^3 + a_5u^5 + a_7u^7$, $a_i \in \mathbf{k}$. The polynomials $t^{11}g(1/t) = a_1t^{10} + a_2t^9 + a_3u^8 + a_4u^6 + a_5u^4$ have all exponents in *D*. The **k**-vector space B^{\perp} is generated by the following elements $g_1 = u$, $g_2 = u^2$, $g_3 = u^3$, $g_4 = u^5$, $g_5 = u^7$.

Given a finite codimension subalgebra *B* of Γ , we consider the curve singularity X = Spec(B) defined by *B*. Let *X'* be the generic plane projection of *X*, [3], and let \widetilde{X} be the saturation of *X*, [28] and the references therein. We have

$$\mathcal{O}_{X'} \subset \mathcal{O}_X = B \subset \mathcal{O}_{\widetilde{X}} \subset \Gamma,$$

and then

$$\mathcal{O}_{\widetilde{Y}}^{\perp} \subset B^{\perp} \subset \mathcal{O}_{X'}^{\perp}.$$

We have, [9],

$$\delta(\widetilde{X}) \le \delta(X) \le \delta(X') \le (e_0(X) - 1)\delta(\widetilde{X}) - {e_0(X) - 1 \choose 2}$$

From [27, Proposition 1.6, page 971], we know that \widetilde{X} is also the saturation of X'.

On the other hand, \widetilde{X} is a monomial curve singularity. Assume that the coset of x_1 in *B* is t^{e_0} with e_0 the multiplicity of *B*. Since the rings are complete and the ground field is algebraically closed, we can assumed it after a suitable election of the uniformization parameter of Γ . Let $\{e_0; \beta_1, \ldots, \beta_g\}$ be the characteristic of X', [28, Section 3, page 993], then $\mathcal{O}_{\widetilde{X}}$ is the monomial subalgebra with generators:

$$\begin{cases} t^{e_0}, \\ t^{s_v n_{v+1}...n_g}, & m_v \le s_v \le [m_{v+1}/n_{v+1}], v = 1, ..., g-1, \\ t^{m_g+i}, & 0 \le i \le e_0 - 1, \end{cases}$$

where $\beta_v/e_0 = m_v/n_1 \dots n_v$ is the *v*th characteristic exponent of X', $v = 1, \dots, g-1$, and $gcd(m_i, n_i) = 1$ for all $i = 1, \dots, g$ (see [28, Section 3, page 995]). The facts $\mathcal{O}_{\widetilde{X}}^{\perp} \subset B^{\perp}$ and Proposition 5.2 can be useful in order to simplify the

The facts $\mathcal{O}_{\widetilde{X}}^{\perp} \subset B^{\perp}$ and Proposition 5.2 can be useful in order to simplify the computation of B^{\perp} as the next example shows.

Example 5.7 Let us consider the k-algebra $B = \mathbf{k}[[t^6, t^8 + t^{11}, t^{10} + t^{13}]]$; its saturation is $\widetilde{B} = \mathbf{k}[[t^6, t^8, t^{10}, t^{11}, t^{13}, t^{15}]]$ (see [6, Example 2.5.1]). The sequence of multiplicities of the resolution of X = Spec(B) is $\{6, 2, 2, 2, 2, 1, ...\}$. We can compute $\delta(X)$ by computing $e_1(C)$, where *C* ranges the local rings of the resolution process, in this case, we get $\{8, 1, 1, 1, 1, 0, ...\}$, so $\delta(X) = 12$. The semigroup of *B* is $D = \{0, 6, 8, 10, 12, 14, 16, 18, 19, 20, 22 \rightarrow\}$, i.e., the conductor of *D* is 22.

On the other hand, the semigroup of $\mathcal{O}_{\widetilde{X}}$ is $\{0, 6, 8, 10 \longrightarrow\}$, its conductor is 10. Hence, $\mathcal{O}_{\widetilde{X}}^{\perp}$ is generated by u^i with $i \in \{1, 2, 3, 4, 5, 7, 9\}$, and B^{\perp} is the set of polynomials $g = \sum_{i=0}^{21} a_i u^i$ such that $a_6 = 0$, $990a_{11} - a_8 = 0$, $a_{12} = 0$, $1716a_{13} - a_{10} = 0$, $a_{16} = 0$, $4080a_{17} - a_{14} = 0$, $a_{18} = a_{19} = a_{20} = a_{21} = 0$.

6 The canonical module

As in the Artin case, we can relate the canonical module with the inverse system. In that case, we have that if *I* is an Artinian ideal, then $I^{\perp} \cong E_{R/I}(\mathbf{k}) \cong \omega_{R/I}$ (see [4, 14]). In the case of branches, we can determine the "negative" part of the canonical module.

Let X be a branch of $(\mathbf{k}^n, 0)$ and \overline{X} its normalization. We first describe the canonical module ω_X by using Rosenlicht's regular differential forms (see [26, Chapter IV 9], [5, Section 1], see also [13]). We denote by $\Omega_{\overline{X}}(p)$, the set of meromorphic forms in \overline{X} with a pole at most in $p = v^{-1}(0)$. Then Rosenlicht's differential forms are defined as follows: ω_X^R is the set of $v_*(\alpha)$, $\alpha \in \Omega_{\overline{X}}(p)$, such that for all $F \in \mathcal{O}_X$,

$$\operatorname{res}_{p}(F\alpha) = 0$$

Notice that we have a mapping that we also denote by

$$d_R: \mathfrak{O}_X \longrightarrow \Omega_X \longrightarrow \nu_* \Omega_{\overline{X}} \hookrightarrow \omega_X^R.$$

In [1, Chapter VIII], it is proved that $\omega_X \stackrel{\varphi}{\cong} \omega_X^R$ and $d_R = \phi d$, where $d : \mathfrak{O}_X \longrightarrow \omega_X$ is the map defined in the Section 1. Since \mathfrak{O}_X is a one-dimensional reduced ring, we know that $\omega_{(X,0)}$ is a sub- \mathfrak{O}_X -module of tot (\mathfrak{O}_X) (see [4, Proposition 3.3.18]). There is a perfect pairing, [26, Chapter IV],

$$\begin{array}{cccc} \frac{v_{\star} \mathcal{O}_{\overline{X}}}{\mathcal{O}_{X}} & \times & \frac{\omega_{(X,0)}}{v_{\star} \Omega_{\overline{X}}} & \stackrel{\eta}{\longrightarrow} & \mathbb{C} \\ F & \times & \alpha & \longrightarrow & \operatorname{res}_{p}(F\alpha) \end{array}$$

notice that for all $\lambda \in R$ it holds $\eta(\lambda F, \alpha) = \operatorname{res}_p(\lambda F \alpha) = \eta(F, \lambda \alpha)$.

Proposition 6.1 Let X be a branch of $(\mathbf{k}^n, 0)$ and \overline{X} its normalization. Then we have an isomorphism of the $\delta(X)$ dimensional **k**-vector spaces:

$$B^{\perp} \stackrel{\varepsilon}{\cong} \frac{\omega_X}{\nu_* \Omega_{\overline{X}}}$$

such that $\varepsilon(g)$ is the coset defined by $\alpha = \sum_{i=0}^{c-1} i! c_i t^{-i-1}$, for all $g = \sum_{i=0}^{c-1} c_i u^i \in B^{\perp}$.

Proof We write $B = \mathcal{O}_X$, $\Gamma = v_* \mathcal{O}_{\overline{X}}$, and $\Omega_{\overline{X}} = \Gamma dt$. Then ε is the composition of the isomorphisms induced by the above two perfect pairings

$$B^{\perp} \stackrel{\epsilon_1}{\cong} \left(\frac{\Gamma}{B}\right)^* \stackrel{\epsilon_2}{\cong} \frac{\omega_X}{\nu_* \Omega_{\overline{X}}}$$

Next, we describe both morphisms $\varepsilon_1, \varepsilon_2$. Given $g \in B^{\perp}$, we can write it as

$$g = c_0 + c_1 u + \dots, c_{c-1} u^{c-1},$$

so $\varepsilon_1(g)$ is the linear form induced by $\xi : \Gamma^* \longrightarrow \mathbf{k}$ defined by: if $f = \sum_{i \ge 0} a_i t^i \in \Gamma$, then

$$\xi(f) = \sum_{i=0}^{c-1} i! a_i c_i.$$

On the other hand, every $\alpha \in \omega_X$ can be written as $\alpha = t^n h(t) dt$ with $n \in \mathbb{Z}$ and $h(t) \in \Gamma$ an invertible series. From [13, Proposition 2.6], we get that $\alpha = \sum_{i \ge -c} e_i t^i$ such that $\operatorname{res}_0(\alpha F) = 0$ for all $f \in B$. Given $f = \sum_{i \ge 0} a_i t^i \in \Gamma$, we have

$$\operatorname{res}_{0}(f\alpha) = \sum_{i=0}^{c-1} a_{i} e_{-i-1}$$

so $\varepsilon_2^{-1}(\alpha)$ is the linear form induced by $\xi' : \Gamma^* \longrightarrow \mathbf{k}$ defined by

$$\xi'(f) = \sum_{i=0}^{c-1} a_i e_{-i-1}.$$

From this, we deduce that $e_{-i-1} = i!c_i$ for i = 0, ..., c - 1.

Example 6.2 [13, Example 2.7] Let us consider the monomial curve X with parameterization $x_1 = t^4$, $x_2 = t^7$, $x_3 = t^9$. We have c = 11, $\delta = 6$. Then ω_X is the **k**-vector space spanned by t^{-11} , t^{-7} , t^{-6} , t^{-4} , t^{-3} , t^{-2} , t^n , $n \ge 0$, and the quotient $\omega_X/\nu_*\Omega_{\overline{X}}$ admits as **k**-vector space base the cosets of t^{-11} , t^{-7} , t^{-6} , t^{-4} , t^{-3} , t^{-2} , and \mathcal{O}_X^{\perp} is the **k**-vector space with basis $u, u^2, u^3, u^5, u^6, u^{10}$.

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