6

Non-BPS non-Abelian strings

In this chapter we will review non-BPS non-Abelian strings. In particular, they appear in non-supersymmetric theories. We will see that, although for BPS strings in supersymmetric theories the transition from quasiclassical to quantum regimes in the world sheet theory on the string goes smoothly (see Section 4.4.4), for the non-Abelian strings in non-supersymmetric theories these two regimes are separated by a phase transition.

Next, we will show that the same behavior is typical for non-BPS strings in supersymmetric gauge theories. As an example we consider non-Abelian strings in the so-called $\mathcal{N} = 1^*$ theory which is a deformed $\mathcal{N} = 4$ supersymmetric theory with supersymmetry broken down to $\mathcal{N} = 1$ in a special way.

6.1 Non-Abelian strings in non-supersymmetric theories

In this section we will review some results reported in [154, 164] treating non-Abelian strings in non-supersymmetric gauge theories. The theory studied in [154] is essentially a bosonic part of $\mathcal{N} = 2$ supersymmetric QCD with the gauge group SU(N)×U(1) described in Chapter 4 in the supersymmetric setting.[1] The action of this model is

$$S = \int d^4x \left\{ \frac{1}{4g_2^2} \left(F^a_{\mu\nu}\right)^2 + \frac{1}{4g_1^2} \left(F_{\mu\nu}\right)^2 + \frac{1}{g_2^2} |D_\mu a^a|^2 \right.$$
$$+ |\nabla^\mu \varphi^A|^2 + \frac{g_2^2}{2} \left(\bar{\varphi}_A T^a \varphi^A\right)^2 + \frac{g_1^2}{8} \left(|\varphi^A|^2 - N\xi\right)^2$$
$$\left. + \frac{1}{2} \left| \left(a^a T^a + \sqrt{2} m_A\right) \varphi^A \right|^2 + \frac{i\,\theta}{32\,\pi^2}\, F^a_{\mu\nu} F^{*a}_{\mu\nu} \right\}, \qquad (6.1.1)$$

[1] In addition to the substitution (4.2.1) we discard the $f^{abc}\,\bar{a}^b\,a^c$ term in Eq. (4.1.9). This term plays no role in the consideration presented below.

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where $F^{*a}_{\mu\nu} = (1/2)\,\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$ and θ is the vacuum angle. This model is a bosonic part of the $\mathcal{N}=2$ supersymmetric theory (4.1.7) where, instead of two squark fields q^k and $\tilde{q}_k$, only one fundamental scalar φ^k is introduced for each flavor $A = 1,\ldots,N_f$, see the reduced model (4.2.2) in Section 4.2. We also limit ourselves to the case $N_f = N$ and drop the neutral scalar field a present in (4.1.7) as it plays no role in the string solutions. To keep the theory at weak coupling we consider large values of the parameter ξ in (6.1.1), $\xi \gg \Lambda_{\mathrm{SU}(N)}$.

We assume here that

$$\sum_{A=1}^{N} m_A = 0. \tag{6.1.2}$$

Later on it will be convenient to make a specific choice of the parameters m_A, namely,

$$m^A = m \times \mathrm{diag}\left\{e^{2\pi i/N},\, e^{4\pi i/N},\ldots, e^{2(N-1)\pi i/N},\, 1\right\}, \tag{6.1.3}$$

where m is a single common parameter. Then the constraint (6.1.2) is automatically satisfied. We can (and will) assume m to be real and positive. We also introduce a θ term in the model (6.1.1).

Clearly the vacuum structure of the model (6.1.1) is the same as of the theory (4.1.7), see Section 4.1. Moreover, the Z_N string solutions are the same; they are given in Eq. (4.2.6). The adjoint field plays no role in this solution and is given by its VEV (4.1.11). The tensions of these strings are given classically by Eq. (4.2.12). However, in contrast with the supersymmetric theory, now the tensions of Z_N strings acquire quantum corrections in loops.

If masses of the fundamental matter vanish in (6.1.1) this theory has unbroken $\mathrm{SU}(N)_{C+F}$ much in the same way as the theory (4.1.7). In this limit the Z_N strings acquire orientational zero modes and become non-Abelian. The corresponding solution for the elementary non-Abelian string is given by Eq. (4.3.1). Below we will consider two-dimensional effective low-energy theory on the world sheet of such non-Abelian string. Its physics appears to be quite different as compared with the one in the supersymmetric case.

6.1.1 World-sheet theory

Derivation of the effective world-sheet theory for the non-Abelian string in the model (6.1.1) can be carried out much in the same way as in the supersymmetric case [154], see Section 4.4. The world-sheet theory now is two-dimensional non-supersymmetric CP($N-1$) model (4.4.9). Its coupling constant β is given by the coupling constant g_2^2 of the bulk theory via the relation (4.4.10). Classically

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the normalization integral I is given by (4.4.11). Then it follows that $I = 1$ as in supersymmetric case. However, now we expect quantum corrections to modify this result. In particular, I can become a function of N in quantum theory.

Now, let us discuss the impact of the θ term which we introduced in our bulk theory (6.1.1). At first sight, seemingly it cannot produce any effect because our string is magnetic. However, if one allows for slow variations of n^l in z and t, one immediately observes that the electric field is generated via $A_{0,3}$ in Eq. (4.4.5). Substituting F_{ki} from (4.4.7) into the θ term in the action (6.1.1) and taking into account the contribution from F_{kn} times F_{ij} ($k, n = 0, 3$ and $i, j = 1, 2$) we get the topological term in the effective CP($N-1$) model (4.4.9) in the form

$$S^{(1+1)} = \int dt\, dz \left\{ 2\beta \left[(\partial_\alpha n^* \partial_\alpha n) + (n^* \partial_\alpha n)^2 \right] - \frac{\theta}{2\pi} I_\theta \varepsilon_{\alpha\gamma} (\partial_\alpha n^* \partial_\gamma n) \right\}, \tag{6.1.4}$$

where I_θ is another normalizing integral given by the formula

$$\begin{aligned} I_\theta &= -\int dr \left\{ 2 f_{NA} (1-\rho) \frac{d\rho}{dr} + (2\rho - \rho^2) \frac{df}{dr} \right\} \\ &= \int dr \frac{d}{dr} \{ 2 f_{NA} \rho - \rho^2 f_{NA} \}. \end{aligned} \tag{6.1.5}$$

As is clearly seen, the integrand here reduces to a total derivative, and is determined by the boundary conditions for the profile functions ρ and f_{NA}. Substituting (4.4.6), (4.4.8), and (4.2.8), (4.2.7) we get

$$I_\theta = 1, \tag{6.1.6}$$

independently of the form of the profile functions. This latter circumstance is perfectly natural for the topological term.

The additional term in the CP($N-1$) model (6.1.4) we have just derived is the θ term in the standard normalization. The result (6.1.6) could have been expected since physics is 2π-periodic with respect to θ both in the four-dimensional bulk gauge theory and in the two-dimensional world sheet CP($N-1$) model. The result (6.1.6) is not sensitive to the presence of supersymmetry. It will hold in supersymmetric models as well. Note that the complexified bulk coupling constant converts into the complexified world sheet coupling constant,

$$\tau = \frac{4\pi}{g_2^2} + i \frac{\theta}{2\pi} \to 2\beta + i \frac{\theta}{2\pi}. \tag{6.1.7}$$

Now let us introduce small masses for the fundamental matter in (6.1.1). Clearly the diagonal color-flavor group SU$(N)_{C+F}$ is now broken by adjoint VEV's down

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to $\mathrm{U}(1)^{N-1} \times Z_N$. Still, the solutions for the Abelian (or Z_N) strings are the same as was discussed in Section 4.4.4 since the adjoint field does not enter these solutions. In particular, we have N distinct Z_N string solutions depending on what particular squark winds at infinity, see Section 4.4.4. Say, the string solution with the winding last flavor is still given by Eq. (4.2.6).

What is changed with the color-flavor $\mathrm{SU}(N)_{C+F}$ explicitly broken by $m_A \neq 0$, is that the rotations (4.3.1) no longer generate zero modes. In other words, the fields n^ℓ become quasimoduli: a shallow potential (4.4.49) for the quasi-moduli n^l on the string world sheet is generated [132, 133, 154]. Note that we can replace $\tilde{m}_A$ by m_A due to the condition (6.1.2). This potential is shallow as long as $m_A \ll \sqrt{\xi}$.

The potential simplifies if the mass terms are chosen according to (6.1.3),

$$V_{\mathrm{CP}(N-1)} = 2\beta\, m^2 \left\{ 1 - \left| \sum_{\ell=1}^{N} e^{2\pi i\, \ell/N}\, |n^\ell|^2 \right|^2 \right\}. \tag{6.1.8}$$

This potential is obviously invariant under the cyclic Z_N substitution

$$\ell \to \ell + k, \quad n^\ell \to n^{\ell+k}, \quad \forall \ell, \tag{6.1.9}$$

with k fixed. This property will be exploited below.

Now our effective two-dimensional theory on the string world sheet becomes a massive $\mathrm{CP}(N-1)$ model (see Appendix B). As in the supersymmetric case the potential (6.1.8) has N vacua at

$$n^\ell = \delta^{\ell \ell_0}, \quad \ell_0 = 1, 2, \ldots, N. \tag{6.1.10}$$

These vacua correspond to N distinct Abelian Z_N strings with $\varphi^{\ell_0 \ell_0}$ winding at infinity, see Eq. (4.4.4).

6.1.2 Physics in the large-N limit

The massless non-supersymmetric $\mathrm{CP}(N-1)$ model (6.1.4) was solved a long time ago by Witten in the large-N limit [159]. The massive case with the potential (6.1.8) was considered at large N in [154, 164] in connection with the non-Abelian strings. Here we will briefly review this analysis.

As was discussed in Section 4.4.4, the $\mathrm{CP}(N-1)$ model can be understood as a strong coupling limit of a U(1) gauge theory. The action has the form

$$S = \int d^2x \left\{ 2\beta\, |\nabla_k n^\ell|^2 + \frac{1}{4e^2} F_{kp}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 - \frac{\theta}{2\pi} \varepsilon_{kp} \partial_k A_p \right.$$
$$\left. + 4\beta \left| \sigma - \frac{\tilde{m}_\ell}{\sqrt{2}} \right|^2 |n^\ell|^2 + 2e^2 \beta^2 (|n^\ell|^2 - 1)^2 \right\}, \tag{6.1.11}$$

where we also included the θ term. As in the supersymmetric case, in the limit $e^2 \to \infty$ the σ field can be eliminated via the algebraic equation of motion which leads to the theory (6.1.4) with the potential (4.4.49).

The Z_N-cyclic symmetry (6.1.9) now takes the form

$$\sigma \to e^{i\frac{2\pi k}{N}}\sigma, \quad n^\ell \to n^{\ell+k}, \quad \forall \ell, \tag{6.1.12}$$

where k is fixed.

It turns out that the non-supersymmetric version of the massive CP($N-1$) model (6.1.11) has two phases separated by a phase transition [154, 164]. At large values of the mass parameter m we have the Higgs phase while at small m the theory is in the Coulomb/confining phase.

The Higgs phase

At large m, $m \gg \Lambda_\sigma$, the renormalization group flow of the coupling constant β in (6.1.11) is frozen at the scale m. Thus, the model at hand is at weak coupling and the quasiclassical analysis is applicable. The potential (6.1.8) has N degenerate vacua which are labeled by the order parameter $\langle\sigma\rangle$, the vacuum configuration being

$$n^\ell = \delta^{\ell\ell_0}, \quad \sigma = \frac{\tilde{m}_{\ell_0}}{\sqrt{2}}, \quad \ell_0 = 1, \ldots, N, \tag{6.1.13}$$

as in the supersymmetric case, see (4.4.53). In each given vacuum the Z_N symmetry (6.1.12) is spontaneously broken.

These vacua correspond to Abelian Z_N strings of the bulk theory. N vacua of the world-sheet theory have strictly degenerate vacuum energies. From the four-dimensional point of view this means that we have N strictly degenerate Z_N strings.

There are $2(N-1)$ elementary excitations. Here we count real degrees of freedom. The action (6.1.11) contains N complex fields n^ℓ. The common phase of n^{ℓ_0} is gauged away. The condition $|n^\ell|^2 = 1$ eliminates one more field. These elementary excitations have physical masses

$$M_\ell = |m_\ell - m_{\ell_0}|, \quad \ell \neq \ell_0. \tag{6.1.14}$$

Besides, there are kinks (domain "walls" which are particles in two dimensions) interpolating between these vacua. Their masses scale as

$$M_\ell^{\text{kink}} \sim \beta\, M_\ell. \tag{6.1.15}$$

The kinks are much heavier than elementary excitations at weak coupling. Note that they have nothing to do with Witten's n solitons [159] identified as solitons at strong coupling. The point of phase transition separates these two classes of solitons.

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As was already discussed in the supersymmetric case (see Section 4.5) the flux of the Abelian 't Hooft–Polyakov monopole is the difference of the fluxes of two "neighboring" strings, see (4.5.1). Therefore, the confined monopole in this regime is obviously a junction of two distinct Z_N strings. It is seen as a quasiclassical kink interpolating between the "neighboring" ℓ_0th and $(\ell_0 + 1)$th vacua of the effective massive CP$(N-1)$ model on the string world sheet. A monopole can move freely along the string as both attached strings are tension-degenerate.

The Coulomb/confining phase

Now let us discuss the Coulomb/confining phase of the theory occurring at small m. As was mentioned, at $m = 0$ the CP$(N-1)$ model was solved by Witten in the large-N limit [159]. The model at small m is very similar to Witten's solution. (In fact, in the large-N limit it is just the same.) The paper [164] presents a generalization of Witten's analysis to the massive case which is then used to study the phase transition between the Z_N asymmetric and symmetric phases. Here we will briefly summarize Witten's results for the massless model.

The non-supersymmetric CP$(N-1)$ model is asymptotically free (as its supersymmetric version) and develops its own scale Λ_σ. If $m = 0$, classically the field n^ℓ can have arbitrary direction; therefore, one might naively expect spontaneous breaking of SU(N) and the occurrence of massless Goldstone modes. This cannot happen in two dimensions. Quantum effects restore the full symmetry making the vacuum unique. Moreover, the condition $|n^\ell|^2 = 1$ gets in effect relaxed. Due to strong coupling we have more degrees of freedom than in the original Lagrangian, namely all N fields n become dynamical and acquire masses Λ_σ.

This is not the end of the story, however. In addition, one gets another composite degree of freedom. The U(1) gauge field A_k acquires a standard kinetic term at one-loop level,[2] of the form

$$N\,\Lambda^{-2}\;F_{kp}F_{kp}. \tag{6.1.16}$$

Comparing Eq. (6.1.16) with (6.1.11) we see that the charge of the n fields with respect to this photon is $1/\sqrt{N}$. The Coulomb potential between two charges in two dimensions is linear in separation between these charges. The linear potential scales as

$$V(R) \sim \frac{\Lambda_\sigma^2}{N}\,R, \tag{6.1.17}$$

where R is separation. The force is attractive for pairs $\bar{n}$ and n, leading to formation of weakly coupled bound states (weak coupling is the manifestation of the $1/N$

[2] By loops here we mean perturbative expansion in $1/N$ perturbation theory.

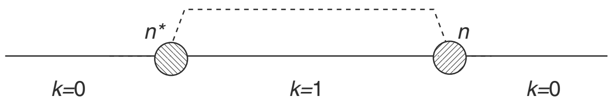

Figure 6.1. Linear confinement of the n-n^* pair. The solid straight line represents the string. The dashed line shows the vacuum energy density (normalizing $\mathcal{E}_0$ to zero).

suppression of the confining potential). Charged states are eliminated from the spectrum. This is the reason why the n fields were called "quarks" by Witten. The spectrum of the theory consists of $\bar{n}n$- "mesons." The picture of confinement of n's is shown in Fig. 6.1.

The validity of the above consideration rests on large N. If N is not large Witten's solution [159] ceases to be applicable. It remains valid in the qualitative sense, however. Indeed, at $N = 2$ the model was solved exactly [204, 205] (see also [206]). Zamolodchikovs found that the spectrum of the O(3) model consists of a triplet of degenerate states (with mass $\sim \Lambda_\sigma$). At $N = 2$ the action (6.1.11) is built of doublets. In this sense one can say that Zamolodchikovs' solution exhibits confinement of doublets. This is in qualitative accord with the large-N solution [159].

Inside the $\bar{n}\,n$ mesons, we have a constant electric field, see Fig. 6.1. Therefore the spatial interval between $\bar{n}$ and n has a higher energy density than the domains outside the meson.

Modern understanding of the vacuum structure of the massless CP($N-1$) model [207] (see also [208]) allows one to reinterpret confining dynamics of the n fields in different terms [155, 154]. Indeed, at large N, along with the unique ground state, the model has $\sim N$ quasi-stable local minima, quasi-vacua, which become absolutely stable at $N = \infty$. The relative splittings between the values of the energy density in the adjacent minima are of the order of $1/N$, while the probability of the false vacuum decay is proportional to $N^{-1}\exp(-N)$ [207, 208]. The n quanta (n quarks-solitons) interpolate between the adjacent minima.

The existence of a large family of quasi-vacua can be inferred from the study of the θ evolution of the theory. Consider the topological susceptibility, i.e. the correlation function of two topological densities

$$\int d^2x\,\langle Q(x),\; Q(0)\rangle, \tag{6.1.18}$$

where

$$Q = \frac{i}{2\pi}\varepsilon_{kp}\partial_k A_p = \frac{1}{2\pi}\varepsilon_{kp}\big(\partial_k n^*_\ell\;\partial_p n^\ell\big). \tag{6.1.19}$$

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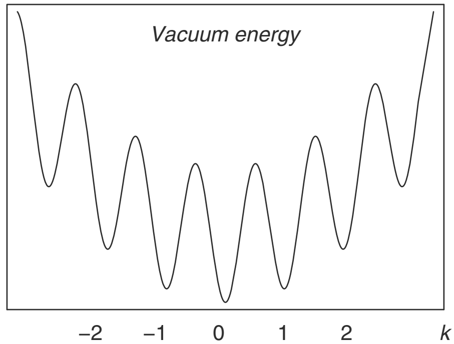

Figure 6.2. The vacuum structure of CP($N-1$) model at $\theta = 0$.

The correlation function (6.1.18) is proportional to the second derivative of the vacuum energy with respect to the θ angle. From (6.1.19) it is not difficult to deduce that this correlation function scales as $1/N$ in the large N limit. The vacuum energy by itself scales as N. Thus, we conclude that, in fact, the vacuum energy should be a function of θ/N.

On the other hand, on general grounds, the vacuum energy must be a 2π-periodic function of θ. These two requirements are seemingly self-contradictory. A way out reconciling the above facts is as follows. Assume that we have a family of quasi-vacua with energies

$$E_k(\theta) \sim N\,\Lambda_\sigma^2 \left\{1 + \text{const}\left(\frac{2\pi k + \theta}{N}\right)^2\right\}, \quad k = 0, \ldots, N-1. \quad (6.1.20)$$

A schematic picture of these vacua is given in Fig. 6.2. All these minima are entangled in the θ evolution. If we vary θ continuously from 0 to 2π the depths of the minima "breathe." At $\theta = \pi$ two vacua become degenerate, while for larger values of θ the former global minimum becomes local while the adjacent local minimum becomes global. It is obvious that for the neighboring vacua which are not too far from the global minimum

$$E_{k+1} - E_k \sim \frac{\Lambda_\sigma^2}{N}. \quad (6.1.21)$$

This is also the confining force acting between n and $\bar{n}$.

One could introduce order parameters that would distinguish between distinct vacua from the vacuum family. An obvious choice is the expectation value of the topological charge. The kinks n^ℓ interpolate, say, between the global minimum and the first local one on the right-hand side. Then $\bar{n}$'s interpolate between the first local

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minimum and the global one. Note that the vacuum energy splitting is an effect suppressed by $1/N$. At the same time, these kinks have masses which scale as N^0,

$$M_\ell^{\rm kink} \sim \Lambda_\sigma. \tag{6.1.22}$$

The multiplicity of such kinks is N [67], they form an N-plet of SU(N). This is in full accord with the fact that the large-N solution of (6.1.11) exhibits N quanta of the complex field n^ℓ.

Thus we see that the CP($N-1$) model has a fine structure of "vacua" which are split, with the splitting of the order of Λ_σ^2/N. In four-dimensional bulk theory these "vacua" correspond to elementary non-Abelian strings. Classically all these strings have the same tension (4.2.12). Due to quantum effects in the world sheet theory the degeneracy is lifted: the elementary strings become split, with the tensions

$$T = 2\pi\xi + c_1\, N\, \Lambda_\sigma^2 \left\{ 1 + c_2 \left(\frac{2\pi k + \theta}{N} \right)^2 \right\}, \tag{6.1.23}$$

where c_1 and c_2 are numerical coefficients. Note that (i) the splitting does not appear to any finite order in the coupling constant; (ii) since $\xi \gg \Lambda_\sigma$, the splitting is suppressed in both parameters, $\Lambda_\sigma/\sqrt{\xi}$ and $1/N$.

Kinks of the world-sheet theory represent confined monopoles (string junctions) in the four-dimensional bulk theory. Therefore kink confinement in CP($N-1$) model can be interpreted as follows [155, 154]. The non-Abelian monopoles, in addition to the four-dimensional confinement (which ensures that the monopoles are attached to the strings) acquire a two-dimensional confinement along the string: a monopole–antimonopole forms a meson-like configuration, with necessity, see Fig. 6.1.

In summary, the CP($N-1$) model in the Coulomb/confining phase, at small m, has a vacuum family with a fine structure. For each given θ (except $\theta = \pi, 3\pi$, etc.) the true ground state is unique, but there is a large number of "almost" degenerate ground states. The Z_N symmetry is unbroken. The classical condition (4.4.3) is replaced by $\langle n^\ell \rangle = 0$. The spectrum of physically observable states consists of kink-anti-kink mesons which form the adjoint representation of SU(N).

Instead, at large m the theory is in the Higgs phase; it has N strictly degenerate vacua (6.1.13); the Z_N symmetry is broken. We have $N-1$ elementary excitations n^ℓ with masses given by Eq. (6.1.14). Thus we conclude that these two regimes should be separated by a phase transition at some critical value m_* [154, 164]. This phase transition is associated with the Z_N symmetry breaking: in the Higgs phase the Z_N symmetry is spontaneously broken, while in the Coulomb phase it is restored. For $N = 2$ we deal with Z_2 which makes the situation akin to the Ising model.

In the world-sheet theory this is a phase transition between the Higgs and Coulomb/confining phase. In the bulk theory it can be interpreted as a phase

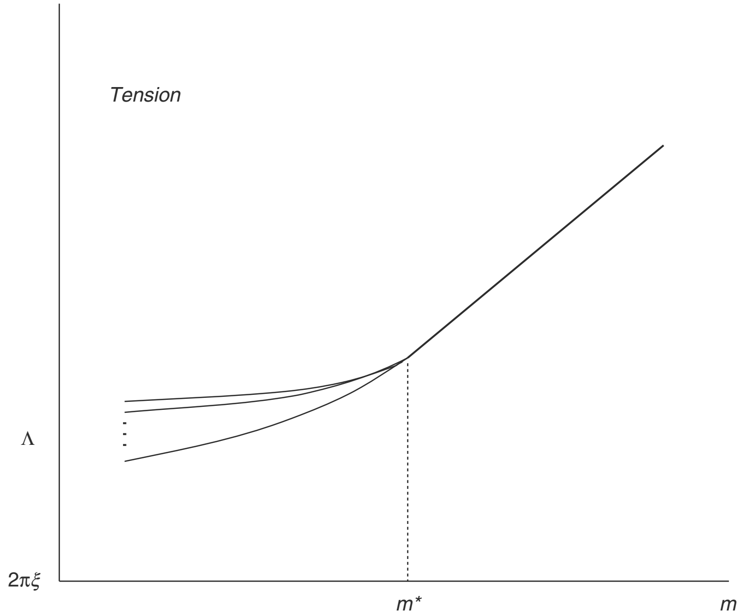


Figure 6.3. Schematic dependence of string tensions on the mass parameter m. At small m in the non-Abelian confinement phase the tensions are split while in the Abelian confinement phase at large m they are degenerative.

transition between the Abelian and non-Abelian confinement. In the Abelian confinement phase at large m, the Z_N symmetry is spontaneously broken, all N strings are strictly degenerate, and there is no two-dimensional confinement of the 4D-confined monopoles. In contrast, in the non-Abelian confinement phase occurring at small m, the Z_N symmetry is fully restored, all N elementary strings are split, and the 4D-confined monopoles combine with antimonopoles to form a meson-like configuration on the string, see Fig. 6.1. We show schematically the dependence of the string tensions on m in these two phases in Fig. 6.3.

In [164] the phase transition point is found using large-N methods developed by Witten in [159]. It turns out that the critical point is

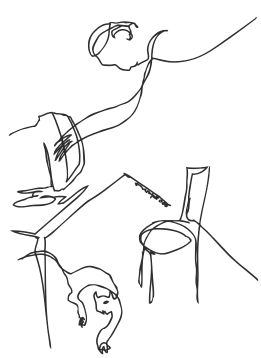
$$m_* = \Lambda_\sigma . \tag{6.1.24}$$

The vacuum energy is calculated in both phases and is shown to be continuous at the critical point. If one approaches the critical point, say, from the Higgs phase some composite states of the world sheet theory (6.1.11), such as the photon and the kinks, become light. One is tempted to believe that these states become massless at the critical point (6.1.24). However, this happens only in a very narrow vicinity of the phase transition point where $1/N$ expansion fails. Thus, the large-N approximation is not powerful enough to determine the critical behavior.

To conclude this section we would like to stress that we encounter a crucial difference between the non-Abelian confinement in supersymmetric and non-supersymmetric gauge theories. For BPS strings in supersymmetric theories we have no phase transition separating the phase of the non-Abelian strings from that of the Abelian strings [132, 133]. Even for small values of the mass parameters supersymmetric theory strings are strictly degenerate, and the Z_N symmetry is spontaneously broken. In particular, at $\Delta m_A = 0$ the order parameter for the broken Z_N, which differentiates the N degenerate vacua of the supersymmetric CP$(N-1)$ model, is the bifermion condensate of two-dimensional fermions living on the string world sheet of the non-Abelian BPS string, see Section 4.4.3 and Section 5.1.3.

Moreover, the presence of the phase transition between Abelian and non-Abelian confinement in non-supersymmetric theories suggests a solution for the problem of enrichment of the hadronic spectrum mentioned in the beginning of Section 4, see also a more detailed discussion in Section 4.9. In the phase of Abelian confinement we have N strictly degenerative Abelian Z_N strings which give rise to too many "hadron" states, not present in actual QCD. Therefore, the Abelian Z_N strings can hardly play a role of prototypes for QCD confining strings. Although the BPS strings in supersymmetric theories become non-Abelian as we tune the mass parameters m_A to a common value, still there are N strictly degenerative non-Abelian strings and, therefore, still too many "hadron" states in the spectrum.

As was explained in this section, the situation in non-supersymmetric theories is quite different. As we make the mass parameters m_A equal we enter the non-Abelian confinement phase. In this phase N elementary non-Abelian strings are split. Say, at $\theta = 0$ we have only one lightest elementary string producing a single two-particle meson with the given flavor quantum numbers and spin, exactly as observed in nature. If N is large, the splitting is small, however. If N is not-so-large the splitting is of the order of Λ_σ^2. Therefore, the mesons produced by excited strings are unstable and may appear invisible experimentally.



6.2 Non-Abelian strings in $\mathcal{N} = 1^*$ theory

So far in our quest for the non-Abelian strings we focused on a particular model, with the SU(N)×U(1) gauge group and fundamental matter. However, it is known that solutions for Z_N strings were first found in simpler models, with the SU(N) gauge group and adjoint matter [134, 135, 136, 137] (in fact, the gauge group becomes SU(N)/Z_N if only adjoint matter is present in the theory). A natural question which immediately comes to one's mind is: can these Z_N strings under some special conditions develop orientational zero modes and become non-Abelian? The answer to this question is yes. Solutions for non-Abelian strings in the simplest theory with the SU(2) gauge group and adjoint matter were found in [155] (actually, the gauge group of this theory is SO(3)). Here we will briefly review the results of this paper.

Although the model considered in [155] is supersymmetric the price one has to pay for its simplicity is that the strings which appear in this model are not BPS. The reason is easy to understand. One cannot introduce the FI term in the theory with the gauge group SU(N) and, therefore, one cannot construct the string central charge [27].

The model considered in [155] is the so-called $\mathcal{N} = 1^*$ supersymmetric theory with the gauge group SU(2). It is a deformed $\mathcal{N} = 4$ theory with the mass terms for three $\mathcal{N} = 1$ chiral superfields. Let us take two equal masses, say $m_1 = m_2 = m$, while the third mass m_3 is assumed to be distinct. Generally speaking, $\mathcal{N} = 4$ supersymmetry is broken down to $\mathcal{N} = 1$, unless $m_3 = 0$. If $m_3 = 0$ the theory has $\mathcal{N} = 2$ supersymmetry. It exemplifies $\mathcal{N} = 2$ gauge theory with the adjoint matter (two $\mathcal{N} = 1$ flavors of adjoint matter with equal masses).

Classically the vacuum structure of this theory was studied in [209], while quantum effects were taken into account in [210]. The $\mathcal{N} = 4$ theory with the SU(2) gauge group has three vacua, and if the coupling constant of the $\mathcal{N} = 4$ theory is small,[3] $g^2 \ll 1$, one of these vacua is at weak coupling. All three adjoint scalars condense in this vacuum. Therefore, it is called the Higgs vacuum [209, 210]. Two other vacua of the theory are always at strong coupling. For small m_3 they correspond to the monopole and dyon vacua of the perturbed $\mathcal{N} = 2$ theory [2]. Here we will concentrate on the Higgs vacuum in the weak coupling regime.

In this vacuum the gauge group SU(2) is broken down to Z_2 by the adjoint scalar VEV's. Therefore there are stable Z_2 non-BPS strings associated with

$$\pi_1(\mathrm{SU}(2)/Z_2) = Z_2. \tag{6.2.1}$$

If we choose a special value of m_3,

$$m_3 = m,$$

[3] Note that the coupling of the unbroken $\mathcal{N} = 4$ theory g^2 does not run since the $\mathcal{N} = 4$ theory is conformal.

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there is a diagonal $O(3)_{C+F}$ subgroup of the global gauge group SU(2), and the flavor O(3) group, unbroken by vacuum condensates. In parallel with Refs. [131, 130, 132, 133] (see also Chapter 4), the presence of this group leads to emergence of orientational zero modes of the Z_2-strings associated with rotation of the color magnetic flux of the string inside the SU(2) gauge group, which converts the Z_2 string into non-Abelian.

Let us discuss this model in more detail. In terms of $\mathcal{N} = 1$ supermultiplets, the $\mathcal{N} = 4$ supersymmetric gauge theory with the SU(2) gauge group contains a vector multiplet, consisting of the gauge field A^a_μ and gaugino $\lambda^{\alpha a}$, and three chiral multiplets Φ^a_A, $A = 1, 2, 3$, all in the adjoint representation of the gauge group, with $a = 1, 2, 3$ being the SU(2) color index. The superpotential of the $\mathcal{N} = 4$ gauge theory is

$$W_{\mathcal{N}=4} = -\frac{\sqrt{2}}{g^2}\varepsilon_{abc}\Phi^a_1\Phi^b_2\Phi^c_3. \tag{6.2.2}$$

One can deform this theory, breaking $\mathcal{N} = 4$ supersymmetry down to $\mathcal{N} = 2$, by adding two equal mass terms m, say, for the first two flavors of the adjoint matter,

$$W_{\mathcal{N}=2} = \frac{m}{2g^2}\sum_{A=1,2}\left(\Phi^a_A\right)^2. \tag{6.2.3}$$

Then, the third flavor combines with the vector multiplet to form an $\mathcal{N} = 2$ vector supermultiplet, while the first two flavors (6.2.3) can be treated as $\mathcal{N} = 2$ massive adjoint matter. If one wishes, one can further break supersymmetry down to $\mathcal{N} = 1$, by adding a mass term to the Φ_3 multiplet,

$$W_{\mathcal{N}=1^*} = \frac{m_3}{2g^2}\left(\Phi^a_3\right)^2. \tag{6.2.4}$$

The bosonic part of the action is

$$\begin{aligned} S_{\mathcal{N}=1^*} = \frac{1}{g^2}\int d^4x \Bigg(& \frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \sum_A \left|D_\mu\, \Phi^a_A\right|^2 \\ & + \frac{1}{2}\sum_{A,B}\left[(\bar\Phi_A\bar\Phi_B)(\Phi_A\Phi_B) - (\bar\Phi_A\Phi_B)(\bar\Phi_B\Phi_A)\right] \\ & + \sum_A\left|\frac{1}{\sqrt{2}}\varepsilon_{abc}\varepsilon^{ABC}\Phi^b_B\Phi^c_C - m_A\Phi^a_A\right|^2\Bigg), \end{aligned} \tag{6.2.5}$$

where $D_\mu\, \Phi^a_A = \partial_\mu\Phi^a_A + \varepsilon^{abc}A^b_\mu\Phi^c_A$, and we use the same notation Φ^a_A for the scalar components of the corresponding chiral superfields.

As was mentioned above, we are going to study the so-called Higgs vacuum of the theory (6.2.5), where all three adjoint scalars develop VEVs of the order of $m, \sqrt{mm_3}$. The scalar condensates Φ_A^a can be written in the form of the following 3×3 color-flavor matrix (convenient for the SU(2) gauge group and three chiral flavor superfields)

$$\langle \Phi_A^a \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{mm_3} & 0 & 0 \\ 0 & \sqrt{mm_3} & 0 \\ 0 & 0 & m \end{pmatrix}. \tag{6.2.6}$$

These VEV's break the SU(2) gauge group completely. The W-bosons masses are

$$m_{1,2}^2 = m^2 + mm_3 \tag{6.2.7}$$

for $A_\mu^{1,2}$, while the mass of the photon field A_μ^3 is

$$m_\gamma^2 = 2mm_3. \tag{6.2.8}$$

In what follows, we will be especially interested in a particular point in the parameter space: $m_3 = m$. For this value of m_3, (6.2.6) presents a symmetric color-flavor locked vacuum

$$\langle \Phi_A^a \rangle = \frac{m}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.2.9}$$

This symmetric vacuum respects the global $\mathrm{O}(3)_{C+F}$ symmetry,

$$\Phi \to O\Phi O^{-1}, \quad A_\mu^a \to O^{ab} A_\mu^b, \tag{6.2.10}$$

which combines transformations from the global color and flavor groups, similarly to the $\mathrm{SU}(N)_{C+F}$ group of the U(N) theories, see Chapter 4. It is this symmetry that is responsible for the presence of the non-Abelian strings in the vacuum (6.2.9).

Note that at $m_3 = m$ all gauge bosons have equal masses,

$$m_g^2 = 2m^2, \tag{6.2.11}$$

as is clearly seen from (6.2.7) and (6.2.8). This means, in particular, that in the point $m_3 = m$ we lose all traces of the "Abelization" in our theory, which are otherwise present at generic values of m_3.

Let us also emphasize that the coupling g^2 in Eq. (6.2.5) is the $\mathcal{N} = 4$ coupling constant. It does not run in the $\mathcal{N} = 4$ theory at scales above m, and we assume it to be small,

$$g^2 \ll 1. \tag{6.2.12}$$

At the scale m the gauge group SU(2) is broken in the vacuum (6.2.9) by the scalar VEVs. Much in the same way as in the U(N) theory (see Chapter 4), the running of the coupling constant below the scale m is determined by the β function of the effective two-dimensional sigma model on the world sheet of the non-Abelian string.

Skipping details we present here the solution for the non-Abelian string in the model (6.2.5) found in [155]. When m_3 approaches m, the theory acquires additional symmetry. In this case the scalar VEVs take the form (6.2.9), preserving the global combined color-flavor symmetry (4.1.15). On the other hand, the Z_2 string solution itself is not invariant under this symmetry. The symmetry (4.1.15) generates orientational zero modes of the string. The string solution in the singular gauge is

$$\begin{aligned}
\Phi^a_A &= O \begin{pmatrix} \frac{g}{\sqrt{2}}\phi & 0 & 0 \\ 0 & \frac{g}{\sqrt{2}}\phi & 0 \\ 0 & 0 & a_0 \end{pmatrix} O^{-1} \\
&= \frac{g}{\sqrt{2}}\phi\delta^a_A + S^a S^A \left(a_0 - \frac{g}{\sqrt{2}}\phi \right), \\
A^a_i &= S^a \, \frac{\varepsilon_{ij} x_j}{r^2} \, f(r), \quad i, j = 1, 2,
\end{aligned} \tag{6.2.13}$$

where we introduced the unit orientational vector S^a,

$$S^a = O^a_b \delta^{b3} = O^a_3. \tag{6.2.14}$$

It is easy to see that the orientational vector S^a defined above coincides with the one we introduced in Section 4, see Eq. (4.4.21). The solution (6.2.13) interpolates between the Abelian Z_2 strings for which $\vec{S} = \{0, 0, \pm 1\}$. We see that the string flux is determined now by an arbitrary vector S^a in the color space, much in the same way as for the non-Abelian strings in the U(N) theories.

Since this string is not BPS-saturated, the profile functions in (6.2.13) satisfy now the *second*-order differential equations,

$$\phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}f^2\phi = \phi\left(g^2\phi^2 - \sqrt{2}m_3 a_0\right) + 2\phi\left(a_0 - \frac{m}{\sqrt{2}}\right)^2,$$
$$a_0'' + \frac{1}{r}a_0' = -\frac{m_3}{\sqrt{2}}\left(g^2\phi^2 - \sqrt{2}m_3 a_0\right) + 2g^2\phi^2\left(a_0 - \frac{m}{\sqrt{2}}\right),$$
$$f'' - \frac{1}{r}f' = 2g^2 f\phi^2, \tag{6.2.15}$$

where the primes stand for derivatives with respect to r, and the boundary conditions are

$$\phi(0) = 0, \quad \phi(\infty) = \frac{\sqrt{mm_3}}{g},$$
$$a_0'(0) = 0, \quad a(\infty) = \frac{m}{\sqrt{2}},$$
$$f(0) = 1, \quad f(\infty) = 0. \tag{6.2.16}$$

The string tension is

$$T = 2\pi\int_0^\infty rdr\left[\frac{f'^2}{2g^2r^2} + \phi'^2 + \frac{a_0'^2}{g^2} + \frac{f^2\phi^2}{r^2} + \frac{g^2}{2}\left(\phi^2 - \frac{\sqrt{2}m_3}{g^2}a_0\right)^2 + 2\phi^2\left(a_0 - \frac{m}{\sqrt{2}}\right)^2\right]. \tag{6.2.17}$$

The second-order equations for the string profile functions were solved in [155] numerically and the string tension was found as a function of the mass ratio m_3/m. Note that for the BPS string (which appears in the limit $m_3 \to 0$) the tension is

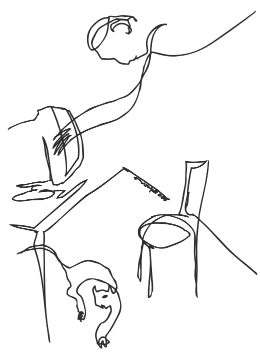
$$T_{\rm BPS} = 2\pi\, mm_3/g^2.$$

The effective world sheet theory for the non-Abelian string (6.2.13) was shown to be the non-supersymmetric CP(1) model [155]. Its coupling constant β is related to the coupling constant g^2 of the bulk theory via (4.4.10), where now the normalization integral

$$I \sim 0.78.$$

In this theory there is a 't Hooft–Polyakov monopole with the unit magnetic charge. Since the Z_2-string charge is $1/2$, it cannot end on the monopole, much in the same way as for the monopoles in the U(N) theories, see Section 4.5. Instead, the confined monopole appears to be a junction of the Z_2 string and anti-string. In the world sheet CP(1) model it is seen as a kink interpolating between the two vacua.

At small values of the mass difference $m_3 - m$ the world sheet theory is in the Coulomb/confining phase, see Section 6.1.2, although, strictly speaking, the large-N analysis is not applicable in this case. Still, the monopoles, in addition to four-dimensional confinement ensuring that they are attached to a string, also experience confinement in two dimensions, along the string [155]. This means that each monopole on the string must be accompanied by an antimonopole, with a linear potential between them along the string. As a result, they form a meson-like configuration, see Fig. 6.1. As was mentioned in Section 6.1.2, this follows from the exact solution of the CP(1) model [204, 205]: only the triplets of $\mathrm{SU}(2)_{C+F}$ are seen in the spectrum.



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