Forum of Mathematics, Sigma (2020), **Vol. 8**:e57, 1–39
doi:10.1017/fms.2020.37

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Noncommutative strong maximals and almost uniform convergence in several directions

José M. Conde-Alonso[1], Adrián M. González-Pérez[2] and Javier Parcet[3]

[1]UAM - Departamento de Matemáticas, 7 Francisco Tomás y Valiente, 28049 Madrid, Spain; E-mail: jose.conde@uam.es.
[2]IMPAN (Instytut Matematyczny Polskiej Akademii Nauk), ul. Sniadekich 8, 00-656 Warsaw;
E-mail: agonzalez-perez@impan.pl.
[3]Consejo Superior de Investigaciones Científicas - ICMAT, 23 Nicolás Cabrera, 28049 Madrid, Spain; E-mail: parcet@icmat.es.

Received: 4 December 2019; **Revised:** 22 May 2020; **Accepted:** 6 June 2020

Keywords and phrases: von Neumann algebras; noncommutative Lp-martingales; ergodic mean; maximal operators

2020 Mathematics Subject Classification: *Primary* – 42B99, 47C15; *Secondary* – 28D99, 47D07, 60B99

Abstract

Our first result is a noncommutative form of the Jessen-Marcinkiewicz-Zygmund theorem for the maximal limit of multiparametric martingales or ergodic means. It implies bilateral almost uniform convergence (a noncommutative analogue of almost everywhere convergence) with initial data in the expected Orlicz spaces. A key ingredient is the introduction of the L_p-norm of the lim sup of a sequence of operators as a localized version of a ℓ_∞/c_0-valued L_p-space. In particular, our main result gives a strong L_1-estimate for the lim sup—as opposed to the usual weak $L_{1,\infty}$-estimate for the sup—with interesting consequences for the free group algebra.

Let $\mathcal{L}\mathbf{F}_2$ denote the free group algebra with 2 generators, and consider the free Poisson semigroup generated by the usual length function. It is an open problem to determine the largest class inside $L_1(\mathcal{L}\mathbf{F}_2)$ for which the free Poisson semigroup converges to the initial data. Currently, the best known result is $L\log^2 L(\mathcal{L}\mathbf{F}_2)$. We improve this result by adding to it the operators in $L_1(\mathcal{L}\mathbf{F}_2)$ spanned by words without signs changes. Contrary to other related results in the literature, this set grows exponentially with length. The proof relies on our estimates for the noncommutative lim sup together with new transference techniques.

We also establish a noncommutative form of Córdoba/Feffermann/Guzmán inequality for the strong maximal: more precisely, a weak (Φ,Φ) inequality—as opposed to weak $(\Phi,1)$—for noncommutative multiparametric martingales and $\Phi(s) = s(1+\log_+ s)^{2+\varepsilon}$. This logarithmic power is an ε-perturbation of the expected optimal one. The proof combines a refinement of Cuculescu's construction with a quantum probabilistic interpretation of M. de Guzmán's original argument. The commutative form of our argument gives the simplest known proof of this classical inequality. A few interesting consequences are derived for Cuculescu's projections.

Contents

1 Preliminaries 8
2 The limit maximal function 18
3 The Jessen-Marcinkiewicz-Zygmund inequality 24
4 Almost uniform convergence in $\mathbf{F}_2$ 28
5 The strong maximal inequality 31

© The Author(s), 2020. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Introduction

Given $f \in L_1(\mathbf{R}^n)$, Lebesgue's differentiation theorem gives

$$\lim_{r\to 0} A_r f(x) := \lim_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} f(y)\,dy = f(x) \quad \text{a.e. } x \in \mathbf{R}^n.$$

In one dimension, this is a general form of the fundamental theorem of calculus. Almost everywhere convergence to original data is also known to hold in other scenarios when replacing Euclidean balls by other averaging processes. In probability theory, we instead use conditional expectations onto martingale filtrations of σ-subalgebras. In ergodic theory, a similar result holds as well for ergodic means or subordinate Markovian semigroups. In all these settings, the strategy reduces to prove a *maximal inequality*: that is, a quantitative estimate for the maximal operator $Mf(x) = \sup_\alpha |A_\alpha(f)|$, where $(A_\alpha)_\alpha$ is an averaging process like those described above. The underlying maximal inequality behind Lebesgue differentiation is the celebrated Hardy-Littlewood maximal theorem [17]. Almost everywhere convergence of martingale approximations can be obtained from Doob's maximal inequality [9]. In ergodic theory, maximal inequalities for ergodic means and Markovian semigroups were established by, respectively, Yosida and Kakutani [53, 54] and Dunford and Schwartz [10].

In all the scenarios laid down before, the maximal operator is bounded as a map $L_1 \to L_{1,\infty}$ that is usually referred to as an operator of weak type $(1,1)$. In particular, we obtain almost everywhere convergence $A_\alpha(f) \to f$ for every $f \in L_1$, which is best possible in the context of one single parameter α. The problem above makes sense for multiple directions, though. Indeed, given a function $f \in L_1(\mathbf{R}^2)$, one can ask whether the two-parameter averages

$$(A_n \otimes A_m) f(x,y) = \frac{n}{2}\frac{m}{2}\int_{x-\frac{1}{n}}^{x+\frac{1}{n}}\int_{y-\frac{1}{m}}^{y+\frac{1}{m}} f(s,t)\,ds\,dt = \fint_{x-\frac{1}{n}}^{x+\frac{1}{n}}\fint_{y-\frac{1}{m}}^{y+\frac{1}{m}} f(s,t)\,ds\,dt$$

converge almost everywhere (a.e.) to f as $n, m \to \infty$. This is easy to prove when f is locally in L_p for some $p > 1$, since the associated maximal operator—usually referred to as the *strong maximal operator*—is bounded from L_p to L_p by a Fubini type argument and Marcinkiewicz interpolation. This technique fails in the quasi-Banach space $L_{1,\infty}$, where tensor product arguments become inefficient since one can produce simple tensors in $L_{1,\infty}(\mathbf{R}) \otimes_{\text{alg}} L_{1,\infty}(\mathbf{R})$ that are not in $L_{1,\infty}(\mathbf{R}^2)$. This difficulty is not an artifice of the proof. Indeed, in 1933, Saks constructed the first integrable function $f \in L_1(\mathbf{R}^2)$ for which $(A_n \otimes A_m)f$ fails to converge a.e. to f. The multiparametric problem was thus reoriented toward finding the largest class inside L_1 for which there is almost everywhere convergence in several directions. The first positive result was obtained by Jessen, Marcinkiewicz and Zygmund in [23], where they proved that $L\log^{d-1} L$ suffices in d parameters. They also showed that this space was optimal in terms of Orlicz classes. Their main result for $d = 2$ states that replacing the sup by a lim sup in the definition of the maximal operator gives a sublinear map $L\log L \to L_1$

$$\Big\| \limsup_{n,m\to\infty} \big|(A_n \otimes A_m) f\big| \Big\|_1 \lesssim \|f\|_{L\log L}. \tag{JMZ}$$

The crucial point here is that replacing sups by lim sups allows for strong L_1-estimates (instead of weak ones) where Fubini-type arguments are available. This is not weaker or stronger than the usual maximal $L_{1,\infty}$-estimates, and it still yields a.e. convergence to initial data. Moreover, their functional-analytic approach is very flexible, and their argument works verbatim for multiparametric martingales and ergodic means. Later, Córdoba and Fefferman and, independently, M. de Guzmán proved in [5, 14] the following weak type inequality for the strong maximal, whose restricted type form goes back to

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

[23]. For every $\lambda > 0$ and positive $f \in L\log L$, it holds that

$$\Big|\Big\{\sup_{n,m\geq 1}(A_n\otimes A_m)f(x,y) > \lambda\Big\}\Big| \lesssim \iint \frac{f}{\lambda}\Big(1+\log_+\Big(\frac{f}{\lambda}\Big)\Big)\,dx\,dy. \tag{CFG}$$

Inequalities (JMZ) and (CFG) extend to d variables just by working with the Orlicz spaces $L\log^{d-1}L$.

Noncommutative maximal inequalities

The theory of von Neumann algebras extends measure theory to noncommutative analogues of L_∞-spaces. Given the local nature of the results above, we shall be concerned with quantum probability spaces: in other words, with pairs $(\mathcal{M},\tau)$ given by a finite von Neumann algebra $\mathcal{M}$ with a normalized trace τ. Many measure-theoretical and probabilistic notions admit a natural analogue in this setting. Noncommutative L_p-spaces, noncommutative martingales and ergodic analogues will be of particular interest for us. Avoiding by now precise definitions for these objects, the strategy remains to prove "almost everywhere convergence 'of certain averaging processes by establishing appropriate "maximal inequalities' in optimal endpoint spaces. The lack of points after quantization forces us, though, to introduce more involved notions for the noncommutative analogues of the expressions just quoted. The notion of almost everywhere convergence admits natural generalizations that go back at least to Lance [30]. Egorov's theorem in a classical probability space (Ω,μ) gives that $f_n \to f$ a.e. iff there exist measurable sets E of arbitrarily small measure such that $(f-f_n)\mathbf{1}_{\Omega\setminus E} \to 0$ uniformly. This alternative form of convergence is usually called *almost uniform convergence*. It generalizes to von Neumann algebras replacing the sets E by projections $e \in \mathcal{M}$ of arbitrarily small trace. By noncommutativity, the left and/or right position of $\mathbf{1}-e$ with respect to the sequence $f-f_n$ matters. The bilateral choice $(\mathbf{1}-e)(f-f_n)(\mathbf{1}-e)$ is the right one with original data $f \in L_1 \setminus L_2$, as we shall recall in the body of the paper.

The formulation of noncommutative maximal inequalities is already subtle, since it is not possible to define the supremum of a family of noncommuting operators in a meaningful way. Namely, as in the Introduction of [28], one can produce examples of 2×2 noncommuting matrices A_1, A_2, A_3 for which no 2×2 matrix A satisfies $\langle \xi, A\xi\rangle = \max\{\langle \xi, A_j\xi\rangle : j=1,2,3\}$ for all $\xi \in \mathbf{R}^2$. The trick to overcome this for weak type maximal inequalities is to use

$$\Big\{\sup_{n\geq 1} f_n > \lambda\Big\} = \Omega \setminus \bigcap_{n\geq 1}\Big\{f_n \leq \lambda\Big\}.$$

That is, the λ-level set of the maximal function is the complement of the set where all entries f_n are bounded above by λ. An analogous formulation is possible for von Neumann algebras. Let $\mathscr{P}(\mathcal{M})$ be the lattice of projections in $\mathcal{M}$. Given $f \in L_1(\mathcal{M})_+$ and $\lambda > 0$, the goal is to find a projection $q(\lambda) \in \mathscr{P}(\mathcal{M})$ such that

(i) $q(\lambda)\,A_n(f)\,q(\lambda) \leq \lambda\, q(\lambda)$.

(ii) $\tau\big(q(\lambda)^\perp\big) = \tau\big(\mathbf{1}-q(\lambda)\big) \lesssim \dfrac{\|f\|_1}{\lambda}$.

The noncommutative form of Yosida-Kakutani maximal theorem for ergodic averages was obtained by Yeadon [51, 52] following earlier results of Lance [30]. The noncommutative endpoint for Doob's maximal inequality was proved by Cuculescu in his seminal paper [6]. As in the classical case, these inequalities can be used to prove that $A_n(f) \to f$ bilaterally almost uniformly with initial data in $L_1(\mathcal{M})$. It is known to experts, though, that one should regard these inequalities as 'weakened forms' of their classical counterparts. As a byproduct of our results, we shall be able to quantify to what extent this happens in the context of Cuculescu's construction of $q(\lambda)$.

By Marcinkiewicz interpolation, the alluded maximal operators are L_p-bounded over classical measure spaces (Ω,μ) for all $p > 1$. However, it took more than 25 years to get noncommutative

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

L_p-maximal inequalities in the line of Cuculescu and Yeadon theorems. Indeed, the $L_p(\Omega)$-norm of a maximal function $Mf = \sup_n f_n$ must be understood as the $L_p(\Omega;\ell_\infty)$-norm of the sequence $(f_n)_n$ to be meaningful in noncommutative algebras. The *mixed-norm spaces* $L_p(\mathcal{M};\ell_\infty)$ were introduced by Pisier for hyperfinite algebras [36]—later by Junge in full generality—and required the full strength of operator space theory. It was Junge who extended in 2002 Doob's L_p-maximal inequality for noncommutative martingales with an ad hoc argument heavily relying on Hilbert module theory and duality [24]. A few years later, Junge and Xu obtained L_p-maximal inequalities for ergodic means and subordinated Markovian semigroups [28]. The key point was a novel interpolation theorem for families of positive preserving maps—as a substitute for Marcinkiewicz interpolation—which allows one to infer results from Cuculescu and Yeadon's 'extra-weak' inequalities. The subtle price for this is that the interpolation constants grow like $(p-1)^{-2}$ as $p \to 1^+$, the square of the classical growth rate. This different quantitative behavior translates into qualitative properties. Indeed, using a noncommutative version of Yano's extrapolation, it was shown in [20] that $(A_n)_n : L\log^2 L(\mathcal{M}) \to L_1(\mathcal{M};\ell_\infty)$, where the extra exponent in the logarithm is directly connected to the extra exponent in the L_p-operator norm. Both quantitative and qualitative results above are the best possible. Again, one could argue in hindsight that such deviations from the classical results arise from the extra-weak nature of Cuculescu's and Yeadon's projections. We shall find additional evidence of this here.

Our primary goal in this paper is to investigate noncommutative strong maximals, almost uniform convergence in several directions to initial data and applications in noncommutative harmonic analysis. More precisely, we shall prove noncommutative forms of the Jessen-Marcinkiewicz-Zygmund theorem (JMZ) and Córdoba-Fefferman-Guzmán inequality (CFG) for quite general processes. Beyond the (expected) tensor products of martingale filtrations, ergodic means or subordinated Markov semigroups, our examples include free products and other (noncommuting) compositions. A result of independent interest in the free group algebra will also be proved. Namely, combining our techniques toward noncommutative (JMZ) with new transference methods, we shall enlarge the largest known subspace of the predual of the free group algebra for which the free Poisson semigroup $\lambda_\omega \mapsto \exp(-t|w|)\lambda_\omega$ converges bilaterally almost uniformly to initial data as $t \to 0^+$.

A. The noncommutative Jessen-Marcinkiewicz-Zygmund theorem

Let us fix in what follows a quantum probability space $(\mathcal{M},\tau)$. To generalize (JMZ), we start by introducing a notion of noncommutative lim sup for finite von Neumann algebras. Let $c_0 \subset \ell_\infty$ be the subspace of sequences converging to 0. Following the intuition that the lim sup of a scalar-valued sequence $(a_n)_n$ satisfies the identity

$$\limsup_{n\to\infty} |a_n| = \|(a_n)_n\|_{\ell_\infty/c_0},$$

it would be natural to define the L_p-norm of the noncommutative lim sup as the seminorm associated with $L_p[\ell_\infty]/L_p[c_0]$ for $1 \le p \le \infty$. Nevertheless, this definition does not work as intended—not even for sequences of functions in classical measure spaces, as we explain in Remark 2.2. This imposes working with a localized version of it. Given $(f_n)_n \subset L_p(\mathcal{M})$, and recalling that we write $\mathscr{P}(\mathcal{M})$ for the projection lattice in $\mathcal{M}$, set

$$\left\| \limsup_{n\to\infty}{}^{+} f_n \right\|_p = \sup_{\varepsilon>0} \inf_{\substack{e\in\mathscr{P}(\mathcal{M}) \\ \tau(e)<\varepsilon}} \inf_{N\ge 1} \left\{ \left\| (e^\perp f_n e^\perp)_{n>N} \right\|_{L_p[\ell_\infty]} \right\},$$

where the $\limsup^+$ is just a suggestive notation and does not correspond to a well-defined operator. The mixed-norm space obtained by closure in that seminorm is denoted $L_p^{\mathrm{loc}}(\mathcal{M};\ell_\infty/c_0)$, where the loc stems from the fact that we localize in corners by projections with trace arbitrarily close to 1. In fact, this space does arise as an actual quotient of $L_p(\mathcal{M};\ell_\infty)$; further details will be given in the body of the

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

paper. As in the classical case, boundedness of the noncommutative $\limsup^+$ can be used as a tool to obtain bilateral almost uniform convergence (b.a.u.). But, contrary to the case of the supremum or the noncommutative $\sup^+$, a partial converse holds:

$$\operatorname*{bau-lim}_{n\to\infty} f_n = f \implies \left\|\limsup_{n\to\infty}{}^{+} f_n\right\|_p \le \|f\|_p.$$

Propositions 2.4 and 2.6 give further explanations. Let $\Phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a convex increasing function with $\Phi(0) = 0$ and $\Phi(\infty) = \infty$, and L_Φ its associated noncommutative Orlicz space (see Section 1). Our main technical result for the $\limsup^+$ is as follows.

Theorem A1. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra equipped with a tracial state τ. Let us consider two families $(A_n)_n$ and $(B_m)_m$ of positivity-preserving operators in $L_1(\mathcal{M})$. Assume the following conditions hold*:

(i) $\left\|(B_m)_m : L_\Phi(\mathcal{M}) \to L_1(\mathcal{M}; \ell_\infty)\right\| < \infty$.
(ii) *There is an operator $F : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ such that $A_n(f) \to F(f)$ b.a.u. for $f \in L_1(\mathcal{M})$.*

Then the following inequality holds as well

$$\left\|\limsup_{n,m\to\infty}{}^{+} A_n \circ B_m(f)\right\|_1 \le \|F\| \left\|(B_m)_m : L_\Phi(\mathcal{M}) \to L_1(\mathcal{M}; \ell_\infty)\right\| \|f\|_\Phi.$$

Many noncommutative forms of (JMZ) arise easily from Theorem A1. Their proofs are elementary and follow after combining Theorem A1 with noncommutative extrapolation and bilateral almost uniform convergence for elements of $L_1(\mathcal{M})$. Let us illustrate what it gives for noncommutative martingales. We first give the most general formulation for the abstract composition of a couple of not necessarily commuting martingale filtrations. Then we make explicit two cases of particular interest. In the first case, we pick filtrations living in different components of a tensor product, which gives rise to the closest generalization of the Jessen-Marcinkiewicz-Zygmund theorem. In the second case, we just replace tensor products by free products, which only makes sense in the noncommutative realm. As an illustration, the free product of two dyadic filtrations in the one-dimensional torus $\mathbf{T}$ can only be defined in the free group algebra; the result below gives an estimate for the L_1-norm of the noncommutative lim sup and almost uniform convergence in two directions with initial data in $L\log^2 L$.

Corollary A2. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and $(\mathcal{M}_n^{[i]})_n$ be nested sequences of von Neumann subalgebras $\mathcal{M}_n^{[i]} \subset \mathcal{M}_{n+1}^{[i]} \subset \mathcal{M}$ for $i = 1, 2$. Let us denote the associated conditional expectations as $\mathbf{E}_n^{[i]} : \mathcal{M} \to \mathcal{M}_n^{[i]}$. Then*

(i) *It holds that*

$$\left\|\limsup_{n,m\to\infty}{}^{+} \left(\mathbf{E}_n^{[1]} \circ \mathbf{E}_m^{[2]}\right)(f)\right\|_1 \lesssim \|f\|_{L\log^2 L}.$$

(ii) *Let $(\mathcal{M}_n^{[i]})_n$ be a weak-$*$ dense filtration of $\mathcal{M}^{[i]} \subset \mathcal{M}$ for $i = 1, 2$. Assume in addition that $(\mathcal{M}, \tau)$ is isomorphic either to $(\mathcal{M}^{[1]} \overline{\otimes} \mathcal{M}^{[2]}, \tau_1 \otimes \tau_2)$ or to $(\mathcal{M}^{[1]} * \mathcal{M}^{[2]}, \tau_1 * \tau_2)$. Then we get, respectively,*

$$\left\|\limsup_{n,m\to\infty}{}^{+} \left(\mathbf{E}_n^{[1]} \otimes \mathbf{E}_m^{[2]}\right)(f)\right\|_1 \lesssim \|f\|_{L\log^2 L},$$

$$\left\|\limsup_{n,m\to\infty}{}^{+} \left(\mathbf{E}_n^{[1]} * \mathbf{E}_m^{[2]}\right)(f)\right\|_1 \lesssim \|f\|_{L\log^2 L}.$$

In both cases, we get b.a.u. convergence to f as $n, m \to \infty$, for every $f \in L\log^2 L(\mathcal{M})$.

Although formulated in the biparametric case, Corollary A2 extends to d-parameters just by working with the Orlicz class $L\log^{2(d-1)} L$ instead. The additional logarithmic powers—compared to $L\log^{d-1} L$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

in the classical form of (JMZ)—are expected from the sharp known results for noncommutative maximals and Yano's extrapolation recalled above. In fact, we believe this is best possible. Given Markovian maps $T_i : \mathcal{M} \to \mathcal{M}$, we recall that an analogous statement holds true, replacing the above conditional expectations by (multiparametric) ergodic means

$$M_n(T_i) = \frac{1}{n+1} \sum_{k=0}^{n} T_i^k$$

or subordinated Markovian semigroups (see Corollary 3.1). As another illustration, we may also consider free products of convolutions in $\mathbf{T}$ against arbitrarily small balls $B_r(0)$, in the line of the original formulation of Jessen, Marcinkiewicz and Zygmund. This gives another form of (JMZ) in the free group algebra $\mathcal{L}\mathbf{F}_n = L_\infty(\mathbf{T}) * \cdots * L_\infty(\mathbf{T})$, with no analog for general von Neumann algebras.

The main result in [19] establishes bilateral almost uniform convergence for certain d-parametric ergodic means. However, the given argument extends to Markovian ergodic means and martingale approximations as well. The underlying 'maximal inequality' is, however, very different from ours since they prove an inequality of the form

$$\Big\| \sup_{n,m\geq 1}{}^{+} \underbrace{M_n(T_1) \circ M_m(T_2)(f)}_{f_{n,m}} \Big\|_{1,\infty} := \sup_{\lambda>0} \inf \Big\{ \lambda\, \tau\big(\mathbf{1} - q(\lambda)\big) : q(\lambda) f_{n,m} q(\lambda) \leq \lambda \mathbf{1} \Big\} \lesssim \|f\|_{L\log^2 L}. \tag{HS}$$

Although their formulation uses pairwise commuting ergodic means, their technique works without requiring commutativity. The advantage of replacing the noncommutative sup by our noncommutative lim sup in Corollary A2 is that we get an L_1 endpoint space, instead of weak L_1. This is particularly useful for using Fubini-type arguments, which will be crucial in our analysis of the free Poisson semigroup in the free group algebra below.

B. The noncommutative Córdoba-Fefferman inequality

As we already explained for (JMZ), it is natural to expect that the noncommutative form (CFG) should hold with data in $L\log^{2(d-1)} L$ in d-parameters. As usual, we fix $d = 2$ for simplicity and assume that $\mathcal{M}$ is the tensor or free product of $\mathcal{M}^{[i]}$ $(i = 1, 2)$, both equipped with a weak-$*$ dense martingale filtration with associated conditional expectations $\mathbf{E}_n^{[i]}$. Let $f_{n,m} = \mathbf{E}_n^{[1]} \otimes \mathbf{E}_m^{[2]}(f)$ or $f_{n,m} = \mathbf{E}_n^{[1]} * \mathbf{E}_m^{[2]}(f)$ accordingly. Then we conjecture that the following holds for every $f \in L\log^2 L(\mathcal{M})$:

Conjecture B1. *Given $\lambda > 0$ and $f \geq 0$, there exists $q(\lambda) \in \mathscr{P}(\mathcal{M})$ satisfying*

(i) $q(\lambda) f_{n,m} q(\lambda) \leq \lambda\, q(\lambda)$.

(ii) $\tau\big(\mathbf{1} - q(\lambda)\big) \lesssim \tau\left\{ \dfrac{f}{\lambda} \left(1 + \log_+ \left(\dfrac{f}{\lambda} \right) \right)^2 \right\}$.

We also expect that the 2 in the exponent of the logarithm is optimal. The conjectured inequality above can be understood as a noncommutative formulation of the weak Orlicz type (Φ, Φ), as defined in [14], where $\Phi(t) = t(1 + \log_+ t)^2$. Indeed, after fixing $\Phi_1, \Phi_2 : \mathbf{R}_+ \to \mathbf{R}_+$, it is said that an operator $f \mapsto f_n = A_n(f)$ is of *weak type* (Φ_1, Φ_2) iff for every $f \in L_{\Phi_1}(\mathcal{M})_+$ and $\lambda > 0$, there is a $q(\lambda) \in \mathscr{P}(\mathcal{M})$ satisfying $q(\lambda) f_{n,m} q(\lambda) \leq \lambda q(\lambda)$ and that

$$\tau\big(q(\lambda)^\perp\big) \lesssim \tau \circ \Phi_2\Big(\frac{f}{\lambda}\Big).$$

Although Conjecture B1 remains open, we have been able to obtain an ε-perturbation.

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Theorem B2. *Given $\lambda > 0$ and $f \geq 0$, there exists $q(\lambda) \in \mathscr{P}(\mathcal{M})$ satisfying*

(i) $q(\lambda) f_{n,m} q(\lambda) \lesssim_{(\varepsilon)} \lambda\, q(\lambda)$.

(ii) $\tau\big(\mathbf{1} - q(\lambda)\big) \lesssim_{(\varepsilon)} \tau\left\{ \frac{f}{\lambda}\Big(1 + \log_+\Big(\frac{f}{\lambda}\Big)\Big)\Big(1 + \log_+\Big(\frac{f}{\lambda^{1-\varepsilon}}\Big)\Big)^{1+\varepsilon} \right\}$.

Theorem B2 generalizes to any other filtrations satisfying $\mathbf{E}_n^{[1]} \circ \mathbf{E}_m^{[2]} = \mathbf{E}_m^{[2]} \circ \mathbf{E}_n^{[1]}$. Let us recall that Hong and Sun's inequality (HS) holds as well for commuting martingale filtrations. Their proof combines Cuculescu's theorem for 1-parameter martingales with a well-known decomposition of $L \log^{\alpha} L$ into atoms. Let us note that (HS) gives

$$\lambda\, \tau\big(\mathbf{1} - q(\lambda)\big) \lesssim \|f\|_{L \log^2 L}.$$

Their result may seem stronger, since it holds for every $f \in L \log^2 L(\mathcal{M})$, while our result works only for f in the smaller space $L \log^{2+\varepsilon} L(\mathcal{M})$. However, neither is stronger than the other since our inequality gives a faster rate of decay for $\lambda \mapsto \tau(\mathbf{1} - q(\lambda))$ when $\lambda > 1$. Indeed, the proof of (HS) requires going through $L_1[\ell_\infty]$, which forces the loss of some factor with respect to the optimal size of the projections $q(\lambda)$ as λ grows. On the other hand, our proof uses a subtle refinement of Cuculescu's construction together with a smart choice for $q(\lambda)$ in terms of the two martingale filtrations. More details on the relation between both results will be given in Remark 5.4. To understand the importance of that difference, recall that Conjecture B1 and Theorem B2 give weak (Φ, Φ)-type inequalities, while (HS) *is* a weak type $(\Phi, 1)$ inequality that has mixed homogeneity. Having the correct weak Orlicz type may be of importance in order to apply real interpolation between Orlicz spaces. Beyond (HS), we suspect that Theorem B2 holds true for multiparametric ergodic means, but we have not investigated this topic. We also suspect that $L \log^2 L(\mathcal{M})$ is optimal in terms of b.a.u. convergence and weak Orlicz type.

Conjecture B3. *Let $f_{n,m} = \mathbf{E}_n^{[1]} \otimes \mathbf{E}_m^{[2]}(f)$ be as above*:

1. *If $f \mapsto (f_{n,m})_{n,m}$ is of weak type (Φ, Φ), then $\Phi(t) \geq c\, t(1 + \log_+ t)^2$ for large t.*
2. *If $f_{n,m} \to f$ b.a.u. for every $f \in L_\Phi(\mathcal{M})$, then $\Phi(t) \geq c\, t(1 + \log_+ t)^2$ for large t.*

The point **2** has a clear precedent in [23, Theorem 8/Lemma F] in the Abelian case.

C. Applications to the free group

One of our original motivations to study multiparametric bilateral almost uniform convergence comes from a problem in the free group algebra. Let $\mathbf{F}_2 = \langle a, b \rangle$ be the free group on two generators, and let $\omega \mapsto |\omega|$ be its standard word length. It is a very well known observation due to Haagerup [16] that the function $\omega \mapsto e^{-t|\omega|}$ is positive definite and therefore the *free Poisson semigroup* $S_t : \mathcal{L}\mathbf{F}_2 \to L\mathbf{F}_2$ given by

$$S_t\Big(\sum_{\omega \in \mathbf{F}_2} a_\omega\, \lambda_\omega \Big) = \sum_{\omega \in \mathbf{F}_2} e^{-t|\omega|}\, a_w\, \lambda_\omega$$

is a symmetric Markovian semigroup. The following natural problem remains open:

Problem C1.

1. *Does $S_t(f) \to f$ bilaterally almost uniformly for every $f \in L_1(\mathcal{L}\mathbf{F}_2)$ as $t \to 0$?*
2. *If not, what is the largest subclass of $L_1(\mathcal{L}\mathbf{F}_2)$ over which $S_t(f) \to f$ b.a.u. as $t \to 0$?*

Currently, the largest known class is $L \log^2 L(\mathcal{L}\mathbf{F}_2)$, and it was obtained in [20]. It is far from clear that the answer to the first question above is positive. In fact, indirectly related work by Ornstein and Tao [34, 47] indicates that b.a.u. convergence might fail in the whole $L_1(\mathcal{L}\mathbf{F}_2)$. Our methods in this paper will provide a significantly larger subspace of $L_1(\mathcal{L}\mathbf{F}_2)$ where bilateral almost uniform convergence holds. Recall that, after identifying $\mathcal{L}\mathbf{F}_2$ with $\mathcal{L}(\mathbf{Z} * \mathbf{Z}) \cong L_\infty(\mathbf{T}) * L_\infty(\mathbf{T})$, the free Poisson semigroup factorizes as the free product $S_t = P_t * P_t$, where P_t is the classical Poisson semigroup in $\mathbf{T}$. Theorem A1

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

easily gives

$$\operatorname*{bau-lim}_{s,t\to 0^+} (P_s * P_t) f = f \qquad \text{for every } f \in L\log^2 L(\mathcal{L}\mathbf{F}_2).$$

It is tempting to use that both free components of $\mathcal{L}\mathbf{F}_2$ are Abelian—so that there is almost uniform convergence for $f \in L\log L(\mathbf{T})$—to bootstrap the result above, proving b.a.u. convergence for every $f \in L\log L(\mathcal{L}\mathbf{F}_2)$. This does not work as intended, but we still obtain an improvement. Let $\Sigma \subset \mathbf{F}_2$ be the subset of reduced words $s_1^{n_1} s_2^{n_2} \dots s_r^{n_r}$ with no sign changes in the exponents of the generators a and b. The usefulness of Σ is that the group homomorphism $\mathbf{F}_2 \to \mathbf{Z}\times\mathbf{Z}$ given by removing the a and b, respectively, *detects* the length $|\omega|$ if $\omega \in \Sigma$. This facilitates the application of transference techniques for elements $f \in \mathcal{L}\mathbf{F}_2$ supported in Σ. Let us define

$$L_1(\mathcal{L}\mathbf{F}_2)|_\Sigma = \Big\{ f \in L_1(\mathcal{L}\mathbf{F}_2) : \tau(f\,\lambda_\omega^*) = 0, \forall \omega \notin \Sigma \Big\}.$$

Theorem C2. *Consider the space*

$$\mathcal{C} = L_1(\mathcal{L}\mathbf{F}_2)|_\Sigma + L\log^2 L(\mathcal{L}\mathbf{F}_2),$$

$$\|f\|_{\mathcal{C}} = \inf\Big\{ \|g\|_1 + \|h\|_{L\log^2 L} : f = g + h \text{ and } g \in L_1(\mathcal{L}\mathbf{F}_2)|_\Sigma \Big\}.$$

Then, if $t_n \to 0^+$, we have that $(S_{t_n})_n$ satisfies

$$\Big\| \limsup_{n\to\infty}{}^{+} S_{t_n}(f) \Big\|_1 \lesssim \|f\|_{\mathcal{C}}. \tag{0.1}$$

In particular, $S_t(f) \to f$ bilaterally almost uniformly as $t \to 0^+$ for every element $f \in \mathcal{C}$.

Theorem C2 is related to a few recent results in the literature [26, 32, 33] that have established L_p-boundedness of Fourier multipliers over special subsets of $\mathbf{F}_2$. In [26], it was proved that the Hörmander-Mikhlin multiplier theorem and Littlewood-Paley square function inequalities admit a generalization when we restrict to arbitrary branches of the free group algebra. Next, these square function estimates were greatly improved in [32] after proving the L_p-boundedness of the truncation into branches with new Hilbert transform methods. Finally, in [33], the authors proved that certain specific multipliers are L_p-bounded iff the restriction of the Fourier multiplier to the torus $\mathbf{T}$—the branch containing the integer powers of one generator—is L_p-bounded. The price for the equivalence with the classical theory is that the admissible multipliers are constant in large regions of $\mathbf{F}_2$ and do not include natural operators like the free Poisson semigroup. In fact, these multipliers are determined by its value in a linearly growing subset of $\mathbf{F}_2$. In conclusion, in these previous results, the reference subsets over which the theory was built were relatively simple in terms of their geometry, like branches or other linear growth subsets. In that respect, it is worth noting that our reference set Σ grows exponentially with respect to word length:

$$\big|\Sigma \cap \{\omega \in \mathbf{F}_2 : |\omega| \le n\}\big| \sim 2^n.$$

1. Preliminaries

In this section, we will briefly review basic definitions and results that will be used throughout the article.

Noncommutative integration

We denote by $\mathcal{B}(H)$ the $*$-algebra of bounded operators on a Hilbert space H. A *von Neumann algebra* is a $*$-subalgebra $\mathcal{M} \subset \mathcal{B}(H)$ that is closed in the weak-$*$ topology of $\mathcal{B}(H)$. Naturally, von Neumann algebras are dual algebras that come equipped with a weak-$*$ topology inherited from that of $\mathcal{B}(H)$. The interested reader can look for more on the basic theory of von Neumann algebras in the many texts

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

available, such as [44, 45]. A contractive and positivity-preserving linear map in $\mathcal{M}^*$ is called a *state*. We will say that it is *normal* when it is weak-$*$ continuous. A *finite trace* or simply a *trace* will be a normal state $\tau : \mathcal{M} \to \mathbf{C}$ satisfying $\tau(fg) = \tau(gf)$ for every $f, g \in \mathcal{M}$. We will say that such a state is *faithful* whenever $\tau(f) = 0$ implies that $f = 0$ for $f \in \mathcal{M}_+ = \{g \in \mathcal{M} : g \geq 0\}$. A von Neumann algebra admitting a faithful finite state is said to be *finite*. Throughout this text, we will work in the context of finite von Neumann algebras.

Many measure-theoretical notions extend naturally to the context of finite von Neumann algebras. We will start with the construction of noncommutative L_p-spaces and Orlicz spaces, which can be consulted in detail in [29, 38, 48]. Recall that the p-norm is given by

$$\|f\|_p = \tau(|f|^p)^{\frac{1}{p}}.$$

We can define $L_p(\mathcal{M}, \tau)$ as the metric closure of $\mathcal{M}$ with respect to the metric induced by the p-norm above. There is an alternative characterization of the noncommutative L_p-spaces. For that, it is necessary to construct the space of τ-measurable unbounded operators $L_0(\mathcal{M}, \tau)$, a locally convex space analogous to the classical space of measurable operators with the topology of convergence in measure. Then the L_p-spaces can be described as the subset of the τ-measurable operators satisfying that $\tau(|f|^p) < \infty$ (see [48]). In the cases $p = 1$ and $p = 2$, $L_p(\mathcal{M}, \tau)$ can be easily characterized as the predual of $\mathcal{M}$ —denoted by $\mathcal{M}_*$—and the GNS construction associated with τ, respectively.

We will also work throughout this text with *noncommutative Orlicz spaces*. Let $\Phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a convex increasing function with $\Phi(0) = 0$ and $\Phi(\infty) = \lim_{t\to\infty} \Phi(t) = \infty$. Given such a function, we define $L_\Phi(\mathcal{M}, \tau)$ as

$$L_\Phi(\mathcal{M}, \tau) = \left\{ f \in L_0(\mathcal{M}, \tau) : \exists \lambda \geq 0,\ \tau \circ \Phi\left(\frac{|f|}{\lambda}\right) < \infty \right\}.$$

Those are Banach spaces when endowed with the norm given by

$$\|f\|_{L_\Phi} = \inf \left\{ \lambda \geq 0 : \tau \circ \Phi\left(\frac{|f|}{\lambda}\right) \leq 1 \right\}.$$

We will omit the dependency on τ whenever it is clear from the context. Among all Orlicz spaces, we will use primarily those where Φ is of the form $\Phi(t) = t\,(1 + \log_+ t)^\alpha$, where $\log_+(t) = \max\{0, \log(t)\}$. In that case, we will denote the space as $L \log^\alpha L(\mathcal{M}, \tau)$.

One of the most common examples of finite von Neumann algebras is given by $\mathcal{B}(\ell_2^m)$ with its unique trace Tr. In that case, the associated L_p-spaces are known as *Schatten classes* S_p^m.

Throughout this text, we will make brief use of the theory of operator spaces, especially around the discussion before Problem 4.4. The category of operator spaces is that which is obtained when one takes the category of Banach spaces and replaces bounded maps by *completely bounded maps*: that is, maps whose operator norms are stable by matrix amplifications. The interested reader can look at some of the already available texts [37, 11] as well as the beginning of [3]. One key observation about operator spaces that we will use is that the complete norm of a map can be detected by operator space valued Schatten classes $S_p[E]$ (see [36] for the construction). Indeed, let $\phi : E \to F$ be a completely bounded map; for every $1 \leq p \leq \infty$, we have that

$$\|\phi : E \to F\|_{\mathrm{cb}} = \sup_{m \geq 1} \left\{ \left\| \mathrm{id} \otimes \phi : S_p^m[E] \to S_p^m[E] \right\| \right\}. \tag{1.1}$$

One consequence of the identity above is that the operator space structure of an operator space E is completely defined by the family of norms of $S_p^m[E]$. We will use this often since for $L_p(\mathcal{M})$, we have that $S_p^m[L_p(\mathcal{M})] = L_p(M_m(\mathbf{C}) \otimes \mathcal{M})$. Other spaces related to L_p admit an equally explicit description.

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Noncommutative maximal inequalities

The theory of maximal inequalities over noncommutative L_p-spaces is not without subtleties. The main one is that, given a family of positive noncommuting operators $(f_n)_n$, there is no natural means of defining its supremum in a way that commutes with taking vector states (see [28, page 386]). Nevertheless, it is possible to define a quantity that behaves like the L_p norm of the supremum of such a family by working with the mixed norm spaces $L_p(\mathcal{M};\ell_\infty)$. In [36], Pisier introduced a way of defining spaces of the form $L_p(\mathcal{M};E)$ for hyperfinite $\mathcal{M}$, generalizing the Banach-valued L_p spaces over a measure space. His method involves taking a direct limit of the E-valued Schatten classes $S_p^m[E]$ introduced in the previous paragraph. Later, this notion was extended to the setting of QWEP von Neumann algebras by Junge [25]. In the case of $E = \ell_\infty$, an ad hoc definition requiring a factorization was introduced in [24] (see also [28]). In this case, one does not need any approximation properties of the von Neumann algebra. This is the case that we will review here. Fix $1 \le p \le \infty$ and a semifinite von Neumann algebra $\mathcal{M}$ with a n.s.f. trace τ.

Definition 1.1. *Given a bounded sequence $(f_n)_n \subset L_p(\mathcal{M})$, it is said that $(f_n)_n \in L_p(M;\ell_\infty)$ iff there exists a factorization $f_n = \alpha\, g_n\, \beta$ such that $\alpha,\beta \in L_{2p}(\mathcal{M})$ and $g_n \in \ell_\infty \overline{\otimes} \mathcal{M} = \ell_\infty[\mathcal{M}]$. The norm in $L_p(\mathcal{M};\ell_\infty)$ is given by*

$$\big\|(f_n)_n\big\|_{L_p[\ell_\infty]} := \inf\Big\{\|\alpha\|_{2p}\ \sup_n \|h_n\|_\infty\ \|\beta\|_{2p}\Big\}, \tag{1.2}$$

where the infimum is taken over all possible decompositions $f_n = \alpha\, h_n\, \beta$.

We define $L_p(\mathcal{M};c_0)$ as the norm-closure of the finitely supported sequences $(f_n)_{0\le n\le N}$ inside $L_p(\mathcal{M};\ell_\infty)$. As a shorthand notation, we will write

$$L_p(\mathcal{M};\ell_\infty) = L_{2p}(\mathcal{M})\ (\ell_\infty \overline{\otimes} \mathcal{M})\ L_{2p}(\mathcal{M}).$$

The same convention will be employed whenever a norm is given by the infimum over products. Note that it is easy to replace ℓ_∞ by a general von Neumann algebra $\mathcal{N}$ in the above expression to get $L_p(\mathcal{M};\mathcal{N}) = L_{2p}(\mathcal{M})\ (\mathcal{M}\overline{\otimes}\mathcal{N})\ L_{2p}(\mathcal{M})$. One interesting case is $\mathcal{N} = L_\infty(\mathbf{R}_+)$. This is the space that we will use to formulate maximal inequalities with respect to a continuous parameter. For purely conventional reasons, we will often denote

$$\Big\|\sup_n{}^{+} f_n\Big\|_p := \|(f_n)_n\|_{L_p[\ell_\infty]}.$$

Here, $\sup^+$ is just an evocative notation, since we already said that the noncommutative supremum may not be well defined. We will frequently use the following well-known characterization of the $L_p[\ell_\infty]$-norm for $f_n \ge 0$:

$$\|(f_n)_n\|_{L_p[\ell_\infty]} = \inf_{f_n \le g} \|g\|_p. \tag{1.3}$$

It is possible to characterize the natural operator space structure of $L_p(\mathcal{M};\ell_\infty)$ as the only one satisfying that

$$S_p^m\big[L_p(\mathcal{M};\ell_\infty)\big] = L_p(M_m \otimes \mathcal{M};\ell_\infty) \tag{1.4}$$

(see [28]). Another property that will be useful is the fact that $L_p(\mathcal{M};\ell_\infty)$ is a bimodule with respect to $\mathcal{M}$: that is,

$$\big\|a\cdot(f_n)_n\cdot b\big\|_{L_p[\ell_\infty]} \le \|a\|_{\mathcal{M}}\,\big\|(f_n)\big\|_{L_p[\ell_\infty]}\,\|b\|_{\mathcal{M}}. \tag{1.5}$$

It also holds that $L_p(\mathcal{M};\ell_\infty)$ is a bimodule with respect to the central action of ℓ_∞ acting as pointwise multiplicators. That is, $\big\|(\eta_n\cdot f_n)_n\big\|_{L_p[\ell_\infty]} \le \|(\eta_n)_n\|_{\ell_\infty}\,\|(f_n)\|_{L_p[\ell_\infty]}$. Taking $\eta = \mathbf{1}_{[0,N]}$ gives the

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

following monotonicity property with respect to the index set, which will be useful in the next section:

$$\big\|(f_n)_{N>n}\big\|_{L_p[\ell_\infty]} \le \big\|(f_n)_n\big\|_{L_p[\ell_\infty]}. \tag{1.6}$$

One may also define ℓ_∞-valued Orlicz spaces $L_\Phi(\mathcal{M};\ell_\infty)$. Following the notation of [1, Definition 3.2], we will use the following notation:

$$\tau\circ\Phi\Big(\sup_n{}^{+} f_n\Big) = \inf\left\{\frac{\tau\circ\Phi(|\alpha|^2)+\tau\circ\Phi(|\beta|^2)}{2}\Big(\sup_n\|g_n\|_\infty\Big) : f_n = \alpha\, g_n\,\beta\right\}.$$

Then the norm of $L_\Phi[\ell_\infty]$ is given by

$$\big\|(f_n)_n\big\|_{L_\Phi[\ell_\infty]} = \inf\left\{\lambda\ge 0 : \tau\circ\Phi\Big(\frac{1}{\lambda}\sup_n{}^{+} f_n\Big)\le 1\right\}.$$

In this context, the identity (1.3) holds up to constants: if $f_n \ge 0$, then we have the following (see [1]):

$$\big\|(f_n)_n\big\|_{L_\Phi[\ell_\infty]} \sim \inf\big\{\|g\|_{L_\Phi} : f_n \le g\big\}. \tag{1.7}$$

Almost uniform convergence

Next we recall the definitions of almost uniform convergence in semifinite von Neumann algebras. The following definition goes back at least to [30] (see also [6] as well as [51, 52, 21, 22].

Definition 1.2. *Given $(f_n)_n \subset L_0(\mathcal{M},\tau)$, we will say that $f_n \to f$* bilaterally almost uniformly *(*b.a.u.*), iff for every $\epsilon > 0$, there is a projection $e \in \mathcal{P}(\mathcal{M})$ such that*

(i) $\tau(e) < \epsilon$.
(ii) $\big\|e^\perp (f_n - f)\, e^\perp\big\|_\infty \longrightarrow 0$ *as* $n\to\infty$.

The use of the term 'bilateral' in the definition above refers to the fact that the projection $e^\perp$ acts both on the right and on the left. One can similarly define the following stronger notions of almost uniform convergence:

$$\begin{aligned} f_n \to f \text{ column almost uniformly} &\iff \big\|(f_n - f)\, e^\perp\big\|_\infty \to 0,\\ f_n \to f \text{ row almost uniformly} &\iff \big\|e^\perp (f_n - f)\big\|_\infty \to 0. \end{aligned}$$

It is also useful to work with combinations of these modes of convergence. For example, we say that $f_n \to f$ converges column+row almost uniformly iff there is a decomposition $f_n - f = a_n + b_n$ such that a_n and b_n converge column and row almost uniformly respectively (see [7]). The terminology or *row* and *column* comes from the fact that

$$\begin{aligned} f_n \to f \text{ column almost uniformly} &\iff (f_n - f)^*(f_n - f) \to 0 \text{ bilaterally almost uniformly},\\ f_n \to f \text{ row almost uniformly} &\iff (f_n - f)(f_n - f)^* \to 0 \text{ bilaterally almost uniformly} \end{aligned}$$

and the analogous terminology used in the theory of operator spaces (see [11, Section 3.4]).

The above definitions generalize the notion of almost everywhere convergence in the case of finite measure spaces. Indeed, by Egorov's theorem (see [13, 2.33]), a sequence of measurable functions f_n converges almost everywhere to f iff for every $\epsilon > 0$ there is a measurable set E with $\mu(E) < \epsilon$ and such that $\mathbf{1}_{E^C}(f_n - f)$ converge uniformly to 0.

As in the classical case, the standard approach to prove that a family of operators $S_n(f)$ converge bilaterally almost uniformly for every f in a L_p or Orlicz space is to try to obtain a noncommutative maximal inequality for elements $(S_n(f))_n$: that is,

$$\big\|(S_n)_n : L_p(\mathcal{M}) \to L_p(\mathcal{M};\ell_\infty)\big\| < \infty.$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

We will use a modification of this principle to prove Theorem A1. Similarly, when the goal is to prove column/row almost uniform convergence, asymmetric versions of the space $L_p(\mathcal{M};\ell_\infty)$ given by

$$L_p(\mathcal{M};\ell_\infty^r) = L_p(\mathcal{M})\,(\ell_\infty \overline{\otimes}\, \mathcal{M}),$$
$$L_p(\mathcal{M};\ell_\infty^c) = (\ell_\infty \overline{\otimes}\, \mathcal{M})\, L_p(\mathcal{M})$$

have to be used (see [18] for the details).

Extrapolation and interpolation for maximal inequalities

Two tools that we are going to use repeatedly are the noncommutative analogues of Marcinkiewicz interpolation and Yano's extrapolation [50] in the case of maximal operators. The particular noncommutative analogue of Marcinkiewicz interpolation that we will use was obtained in [28] (see also [8]), while the noncommutative extrapolation was obtained in [20]. Let us start by recalling the following definition, which extends the classical notion of weak type (p,p) to von Neumann algebras.

Definition 1.3. *Let $(S_n)_n$ be a family of positivity preserving operators $S_n : L_p(\mathcal{M}) \to L_0(\mathcal{M})$. $(S_n)_n$ is said to be of* maximal weak type (p,p) *iff for every $f \in L_p(\mathcal{M})_+$ and $\lambda > 0$, there is a projection $e \in \mathcal{P}(\mathcal{M})$ such that*

(i) $\tau(e^\perp) \lesssim \dfrac{\|f\|_p^p}{\lambda^p}$.
(ii) $e\, S_n(f)\, e \leq \lambda\, e$ *for every n.*

Observe that every family that induces a bounded map $(S_n)_n : L_p(\mathcal{M}) \to L_p(\mathcal{M};\ell_\infty)$ is of maximal weak type (p,p). In the case of Abelian von Neumann algebras, this definition coincides with the associated maximal operator $\mathcal{R}(f) = \sup_n |S_n(f)|$ being of weak type (p,p).

Theorem 1.4 ([28, Theorem 3.1]). *Assume $(S_n)_n$ is a family of positivity preserving operators $S_n : L_{p_0}(\mathcal{M}) + L_{p_1}(\mathcal{M}) \to L_0(\mathcal{M})$ that are of maximal weak type (p_0,p_0) and (p_1,p_1) for $1 \leq p_0 < p_1 \leq \infty$. Then, for every $p_0 < p < p_1$,*

$$\big\|(S_n)_n : L_p(\mathcal{M}) \to L_p(\mathcal{M};\ell_\infty)\big\| \lesssim \max\left\{\left(\frac{1}{p}-\frac{1}{p_1}\right)^{-2}, \left(\frac{1}{p_0}-\frac{1}{p}\right)^{-2}\right\}.$$

It is very interesting to note that the constant appearing in the exponent depending of p, p_0 and p_1 is -2 instead of -1 as in the classical case. The order of magnitude of the constant was also shown to be optimal in [28], proving that the commutative and noncommutative theories diverge significantly.

We will use the result above in the cases in which $(S_n)_n$ is simultaneously of weak type $(1,1)$ and bounded as a map $(S_n)_n : \mathcal{M} \to \ell_\infty \overline{\otimes}\, \mathcal{M}$. In that case, the order of the norm of $(S_n)_n : L_p(\mathcal{M}) \to L_p(\mathcal{M};\ell_\infty)$ in terms of p is given by $(p-1)^{-2}$. It turns out that extrapolation results give a sort of reciprocal near $p = 1$. Indeed, the following holds:

Theorem 1.5 ([20, Theorem 2.3]). *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra with a faithful tracial state and $(S_n)_n : L_1(\mathcal{M}) \cap \mathcal{M} \to L_1(\mathcal{M}) + \mathcal{M}$ a family of positivity preserving operators satisfying that*

$$\big\|(S_n)_n : L_p(\mathcal{M}) \to L_p(\mathcal{M};\ell_\infty)\big\| \lesssim \max\left\{1, \left(\frac{1}{p-1}\right)^{\alpha}\right\}$$

for every $1 < p \leq \infty$. Then

$$\big\|(S_n)_n : L\log^\alpha L(\mathcal{M}) \to L_1(\mathcal{M};\ell_\infty)\big\| < \infty.$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Ergodic means

Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra. We consider the following properties for a linear operator $T : \mathcal{M} \to \mathcal{M}$:

(I) $T : \mathcal{M} \to \mathcal{M}$ is a normal contraction.
(II) T is *completely positive*.
(III) T is τ*-preserving*: that is, $\tau \circ T = \tau$.
(IV) T is *symmetric*: that is, $\tau(f^* T(g)) = \tau(T(f)^* g)$ for every $f, g \in \mathcal{M}$.

If T satisfies properties (I)-(III), we will say that it is a *Markovian* operator. If it satisfies (IV) as well, then we say that it is a *symmetric Markovian* operator. If T satisfies that $\tau \circ T \leq \tau$ instead if (III), then it is said to be *submarkovian*. It is an immediate consequence of the Riesz interpolation that symmetric Markovian operators extend as contractions to L_p spaces, $1 \leq p \leq \infty$. One can see that Markovian operators are also unital. Given an operator T, we define its *ergodic means* $M_n(T)$ by

$$M_n(T) = \frac{1}{n+1} \sum_{k=0}^{n} T^k. \tag{1.8}$$

Observe that $M_n(T)$ is Markovian when T is and symmetric when T is. Consider the subset of fixed points by T in $\mathcal{M}$:

$$\ker_\infty(T - \mathbf{1}) = \{f \in \mathcal{M} : T(f) = f\}.$$

Clearly, $\ker_\infty(T - \mathbf{1})$ is a weak-$*$ closed operator system. It is well known (see [28] and references therein) that it is complemented and induces a splitting

$$\mathcal{M} = \ker_\infty(T - \mathbf{1}) \oplus \overline{\mathrm{Im}[T - \mathbf{1}]}^{w^*}, \tag{1.9}$$

where the complement is the weak-$*$ closure of the image of $(\mathbf{1} - T)$. Denote by F the projection from $\mathcal{M}$ to $\ker_\infty(T - \mathbf{1})$. An important observation is that the decomposition in (1.9) extends to more general spaces associated with $\mathcal{M}$, like L_p or Orlicz spaces. We will abuse notation and denote by F the projection $L_p(\mathcal{M}) \to \ker_p(T - \mathbf{1})$ as well. The following proposition gathers a few known results.

Proposition 1.6. *Let $T : \mathcal{M} \to \mathcal{M}$ be a Markovian operator.*

(i) *For every $f \in L_1(\mathcal{M})$, $\|M_n(T)(f) - F(f)\|_1 \to 0$.*
(ii) *For every $f \in \mathcal{M}$, $M_n(T)(f) \to F(f)$ in the weak-$*$ topology.*
(iii) *There is a weak-$*$ dense class $\mathcal{S} \subset \mathcal{M}$ that is norm dense in $L_p(\mathcal{M}, \tau)$ for $1 \leq p < \infty$ and in $L\log^\alpha L(\mathcal{M})$, such that $\|M_n(T)(f) - F(f)\|_\infty \to 0$ for every $f \in \mathcal{S}$.*

Let us justify (iii). We can use the decomposition in (1.9) and the fact that the complement of $\ker_\infty(T - \mathbf{1})$ is the weak-$*$ closure of $(\mathbf{1} - T)[\mathcal{M}]$ to define $S = \ker_\infty(T - \mathbf{1}) + (\mathbf{1} - T)[\mathcal{M}]$. Now, if $g \in \mathcal{S}$ is of the form $g = g_1 + (g_2 - T(g_2))$ with $g_1 \in \mathcal{F}_\infty(T)$ and $g_2 \in \mathcal{M}$, we have

$$M_n(T)(g) = g_1 + \frac{1}{n+1}(g_2 - T^{n+1}(g_2))$$

and therefore

$$\|F(g) - M_n(T)(g)\|_\infty \leq 2\frac{\|g_2\|_\infty}{n+1},$$

which shows the convergence in $\mathcal{M}$. The norm density result for spaces of the form $L\log^\alpha L(\mathcal{M})$ and for $L_p(\mathcal{M})$ with $1 \leq p < \infty$ follows similarly.

Based on earlier results for the finite case that appeared in [30], it was proved in [51, 52] that the ergodic means $(M_m(T))_m$ are maximally of weak type $(1, 1)$. Since they are trivially bounded

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

as a map $(S_n)_n : \mathcal{M} \to \ell_\infty \overline{\otimes} \mathcal{M}$, we can apply Theorem 1.4 to get that $(M_m(T))_m$ is bounded as a map $(M_m(T))_m : L_p(\mathcal{M}) \to L_p(\mathcal{M}; \ell_\infty)$. The combination of this result with the noncommutative extrapolation in Theorem 1.5 gives the following result, which was obtained in [28] and [20]:

Theorem 1.7. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra with a faithful tracial state and T a Markovian operator.*

(i) *For every $1 < p \leq \infty$,*

$$\big\|(M_n(T))_n : L_p(\mathcal{M}) \to L_p(\mathcal{M}; \ell_\infty)\big\| \lesssim \max\left\{1, \left(\frac{1}{p-1}\right)^2\right\}.$$

(ii) $\big\|(M_n(T))_n : L\log^2 L(\mathcal{M}) \to L_1(\mathcal{M}; \ell_\infty)\big\| < \infty$

(iii) *For every $f \in L_1(\mathcal{M})$, $M_m(T)(f) \to F(f)$ bilaterally almost uniformly as $m \to \infty$.*

(iv) *If T is symmetric Markovian, then (i) and (ii) hold with T^m instead of $M_m(T)$.*

It is worth noting that the analogue of (iii) is false when $M_m(T)$ is replaced by T^m even if T is symmetric. The reason is that T^m may not satisfy a maximal weak $(1,1)$ inequality, and the passage from $M_m(T)$ to T^m requires an argument based on interpolation due originally to Stein (see [43]). Moreover, there are Abelian examples of Markovian operators such that $T^m(f)$ diverges almost everywhere for some $f \in L_1$ (see [34]). Nevertheless, it holds that $T^m(f) \to f$ for every $f \in L\log^2 L(\mathcal{M})$, when $\mathcal{T}$ is symmetric.

Semigroups

Recall that a semigroup of operators is a family $(S_t)_{t\geq 0}$ that satisfies $S_t \circ S_s = S_{t+s}$ and $S_0 = \mathrm{id}$.

Definition 1.8. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra. A semigroup $(S_t)_{t\geq 0}$ is called Markovian iff each $S_t : \mathcal{M} \to \mathcal{M}$ is symmetric Markovian and the map $t \mapsto S_t$ is pointwise weak-$*$ continuous.*

The theory of Markovian semigroups and their ergodic theory can be understood as a continuous analogue of that of Markovian operators. With that in mind, given $s \geq 0$, we define its ergodic mean associated with the semigroup $\mathcal{S} = (S_t)_{t\geq 0}$ by

$$M_s(\mathcal{S})(f) = \frac{1}{s}\int_0^s S_t(f)\, dt = \fint_0^s S_t(f)\, dt.$$

The Dunford-Schwartz ergodic theorem can be extended to the noncommutative setting: the family $(M_s(\mathcal{S}))_{s\geq 0}$ is of maximal weak type $(1,1)$ (see [28]). As a consequence of that, we have that there is bilateral almost uniform convergence of $M_s(\mathcal{S})f$ to f as $s \to 0^+$. In contrast, the problem of determining whether $S_t(f) \to f$ bilaterally almost uniformly for $f \in L_1(\mathcal{M})$ is far more difficult, since the operators (S_t) may not satisfy a maximal inequality. Nevertheless, there is a class of semigroups that have bilateral almost uniform convergence for every element in $L_1(\mathcal{M})$, the so-called *subordinated semigroups*. Given semigroups $\mathcal{S} = (S_t)_{t\geq 0}$ and $\mathcal{T} = (T_t)_{t\geq 0}$, $\mathcal{S}$ is subordinated to $\mathcal{T}$ iff there is a family of functions $\phi_t : \mathbf{R}_+ \to \mathbf{R}_+$ such that

- $\phi_{t_1+t_2} = \phi_{t_1} * \phi_{t_2}$.
- For every $f \in \mathcal{M}$ and $t \geq 0$

$$S_t(f) = \int_0^\infty T_s(f)\phi_t(s)\, d\, s.$$

Additionally, when the functions ϕ_t are of bounded variation and satisfy that $\lim_{s\to\infty} \phi_t(s) = 0$ for every s, we have

$$S_t = \int_0^\infty T_s \phi_t(s)\, ds = -\int_0^\infty T_s\left(\int_s^\infty d\phi_t(z)\right) ds = \quad -\int_0^\infty \left(\fint_0^z T_s\right) z\, d\phi_t(z), \tag{1.10}$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

where the signed measure $d\phi_t(z)$ is the Lebesgue-Stieltjes derivative of ϕ_t. If

$$\sup_{t\geq 0}\int_0^\infty z\, d|\phi_t|(z) < \infty,$$

where $d|\phi_t|(z)$ is the total variation associated with the signed measure above, we have that the bilateral almost uniform convergence of $(S_s)_{s\geq 0}$ is implied by that of $(M_s(\mathcal{T}))_{t\geq 0}$.

One important class of subordinated semigroups are the *subordinated Poisson semigroups*. Since $\mathcal{S} = (S_t)_{t\geq 0}$ extends to $L_2(\mathcal{M})$, we can use the spectral theorem to express S_t as e^{-tA}, when A is an unbounded self adjoint operator. A result that goes back at least to Stein [43] implies that $P_s = e^{-sA^{\frac{1}{2}}}$ is also a Markovian semigroup, and the following subordination formula holds

$$P_s(f) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \underbrace{s e^{-\frac{s^2}{4v}} v^{-\frac{3}{2}}}_{\phi_s(v)} S_v(f)\, dv. \tag{1.11}$$

The formula above follows from an explicit computation of the inverse Laplace transform of $e^{-s\eta^{\frac{1}{2}}}$. The name of Poisson stems from the fact that when S_t is taken to be the heat semigroup in $\mathbf{R}^n$ or $\mathbf{T}^n$, P_s recovers the classical Poisson semigroup. A simple computation shows that the functions $\phi_s(v)$ satisfy

$$\sup_{s>0}\left\| s\frac{\partial}{\partial s}\phi_s(v)\right\|_1 < \infty.$$

As a consequence, we obtain the following:

Proposition 1.9. *Let $\mathcal{S} = (S_t)_{t\geq 0}$ be a Markovian semigroup and $P_s(f)$ its subordinated Poisson semigroup. For every $f \in L_1(\mathcal{M})$, we have that $P_s(f) \to f$ bilaterally almost uniformly as $s \to 0$.*

Martingales

Here we are going to give basic facts and definitions regarding the theory of noncommutative martingale sequences (see also [39, 40] for more details). Let $\mathcal{N} \subset \mathcal{M}$ be two finite von Neumann algebras, and assume that the inclusion i is unital. Since $\tau(i(f)) = \tau(f)$, i extends to the L_1-spaces $L_1(\mathcal{N}) \subset L_1(\mathcal{M})$. Let us denote this extended inclusion by i_1. Its dual $i_1^* = \mathbf{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N} \subset \mathcal{M}$ is the conditional expectation onto $\mathcal{N}$: that is, the map $\mathbf{E}_{\mathcal{N}}$ is a surjective and completely positive projection that is furthermore $\mathcal{N}$-bimodular. Given a sequence of finite von Neumann subalgebras $(\mathcal{M}_m)_n$ of $\mathcal{M}$, we will say that they are nested if $\mathcal{M}_m \subset \mathcal{M}_{m+1}$. We will also assume in all of the examples that $(\mathcal{M}_m)_m$ approximates $\mathcal{M}$ is the sense that

$$\overline{\bigcup_{n\geq 1}\mathcal{M}_n}^{\,w^*} = \mathcal{M}.$$

The sequence of algebras $(\mathcal{M}_m)_m$ is sometimes referred to as a *filtration* and the associated sequence of operators $f_m = \mathbf{E}_m(f)$ as a *martingale sequence*. We will denote by $\mathcal{C}$ the union of all the algebras $\mathcal{M}_m$. Clearly $\mathcal{S}$ is norm dense in $L_p(\mathcal{M})$ with $1 \leq p < \infty$. We also have that the analogue of Proposition 1.6 holds for the weak-$*$ dense algebra $\mathcal{S}$ given by the union of $\mathcal{M}_n$.

Cuculescu showed that the family $(\mathbf{E}_n)$ is maximally of weak type $(1,1)$ (see [6]). This result is a generalization of Doob's maximal inequality in the noncommutative setting. We have the following consequences, obtained through Theorems 1.5 and 1.4.

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Theorem 1.10. *Let* $(\mathcal{M},\tau)$, $(\mathcal{M}_m)_m$ *and* $(\mathbf{E}_m)_m$ *be as above.*

(i) *For every* $1<p<\infty$,

$$\big\|(\mathbf{E}_n)_n : L_p(\mathcal{M}) \to L_p(\mathcal{M};\ell_\infty)\big\| \lesssim \max\left\{1,\left(\frac{1}{p-1}\right)^2\right\}. \tag{1.12}$$

(ii) $\big\|(\mathbf{E}_n)_n : L\log^2 L(\mathcal{M}) \to L_1(\mathcal{M};\ell_\infty)\big\| < \infty$.

(iii) *As a consequence* $\mathbf{E}_n(f) \to f$ *bilaterally almost uniformly for every* $f \in L_1(\mathcal{M})$.

Free products of von Neumann algebras

It is easy to see that if $\phi_1 : \mathcal{M}_1 \to \mathcal{M}_1$ and $\phi_2 : \mathcal{M}_2 \to \mathcal{M}_2$ are completely positive normal maps over von Neumann algebras, their tensor product $\phi_1 \otimes \phi_2 : \mathcal{M}_1 \mathbin{\overline{\otimes}} \mathcal{M}_2 \to \mathcal{M}_1 \mathbin{\overline{\otimes}} \mathcal{M}_2$ is also completely positive and normal. Furthermore, if τ_1 and τ_2 are traces, we have that the tensor product of two Markovian operators $T_1 : \mathcal{M}_1 \to \mathcal{M}_1$ and $T_2 : \mathcal{M}_2 \to \mathcal{M}_2$ extends to a Markovian operator $T_1 \otimes T_2 : \mathcal{M}_1 \mathbin{\overline{\otimes}} \mathcal{M}_2 \to \mathcal{M}_1 \mathbin{\overline{\otimes}} \mathcal{M}_2$. The same result holds for $\mathbf{E}_1 \otimes \mathbf{E}_2 : \mathcal{M}_1 \mathbin{\overline{\otimes}} \mathcal{M}_2 \to \mathcal{N}_1 \mathbin{\overline{\otimes}} \mathcal{N}_2 \subset \mathcal{M}_1 \mathbin{\overline{\otimes}} \mathcal{M}_2$ whenever $\mathbf{E}_i : \mathcal{M}_i \to \mathcal{N}_i \subset \mathcal{M}_i$ are conditional expectations. We are going to recall a result from [4] that allows one to do the same for free products.

Let $(\mathcal{M}_i,\tau_i)$ for $i \in \{1,2\}$ be two finite von Neumann algebras, and assume that they are cyclically represented inside Hilbert spaces $\mathcal{M}_i \subset \mathcal{B}(H_i)$. We will fix cyclic vectors $\xi_i \in H_i$. The reader can assume without loss of generality that (H_i,ξ_i) is the GNS construction associated with τ_i. We will denote by (H,ξ) the *free product* of the pointed Hilbert spaces (H_i,ξ_i) given by

$$H = \mathbf{C}\xi \oplus \bigoplus_{n\geq 0} \bigoplus_{i_1\neq i_2 \dots \neq i_n} H_{i_1}^\circ \otimes H_{i_2}^\circ \otimes H_{i_3}^\circ \dots \otimes H_{i_n}^\circ,$$

where $H_i^\circ = H_i \ominus \mathbf{C}\xi_i = (\mathbf{C}\xi_i)^\perp$. To define the reduced free product $\mathcal{M}_1 * \mathcal{M}_2$, notice that there are two unitary maps $V_i : H(i) \otimes_2 H_i \to H$, where each $H(i) \subset H$ is given by

$$H(i) = \mathbf{C}\xi \oplus \bigoplus_{n\geq 0} \bigoplus_{i\neq i_2 \dots \neq i_n} H_i^\circ \otimes H_{i_2}^\circ \otimes H_{i_3}^\circ \dots \otimes H_{i_n}^\circ$$

(see [49] for the details). The unitaries above allow one to define two faithful representations $\pi_i : \mathcal{M}_i \to B(H)$ for $i \in \{1,2\}$ given by

$$\pi_i(a) = V_i\,(\mathbf{1}\otimes a)\,V_i^*.$$

We will denote by $\pi = \pi_1 * \pi_2$ the C^*-algebra representations defined over the universal free product representation of $\mathcal{M}_1$ and $\mathcal{M}_2$ as C^*-algebras. The image of π is a quotient of such universal C^*- algebra. We will denote by $\mathcal{M}_1 * \mathcal{M}_2$ the von Neumann algebra given by the weak-$*$ closure on the image of π. The vector $\xi \in H$ defines a natural trace that we will denote by $\tau_1 * \tau_2$. Notice that if we denote by $\mathcal{M}_i^\circ$ the weak-$*$ closed linear subspace given by

$$\mathcal{M}_i^\circ = \big\{f \in \mathcal{M}_i : \tau_i(f) = 0\big\},$$

then we can define a weak-$*$ dense unital $*$-algebra D inside $\mathcal{M}_1 * \mathcal{M}_2$ as

$$D = \mathbf{C}\mathbf{1} \oplus \bigoplus_{n\geq 0} \bigoplus_{i_1\neq i_2 \dots \neq i_n} \mathcal{M}_{i_1}^\circ \otimes \mathcal{M}_{i_2}^\circ \otimes \mathcal{M}_{i_3}^\circ \dots \otimes \mathcal{M}_{i_n}^\circ,$$

where the direct sums and tensor products are understood in an algebraic sense. The multiplication is given by associativity and the following rule for pure tensors:

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

$$\begin{aligned}&\big(x_1 \otimes x_2 \otimes \ldots \otimes x_n\big) \cdot \big(y_1 \otimes y_2 \ldots \otimes y_m\big)\\ &\quad = \begin{cases} x_1 \otimes x_2 \otimes \ldots \otimes x_n \otimes y_1 \otimes y_2 \ldots \otimes y_m & \text{when } x_n \in \mathcal{M}_{i_n}^{\circ},\ y_1 \in \mathcal{M}_{j_1}^{\circ} \\ & \text{and } i_n \neq j_1, \\ (x_1 \otimes x_2 \otimes \ldots \otimes x_{n-1}) \cdot \big((x_n\, y_1)^{\circ} + \tau(x_n\, y_1)\mathbf{1}\big) \otimes y_2 \ldots \otimes y_m & \text{otherwise.} \end{cases}\end{aligned}$$

In the formula above, $(x_n\, y_1)^{\circ}$ is notation for $x_n\, y_1 - \tau(x_n\, y_1)$. Given two unital maps T_1 and T_2 preserving the states τ_1 and τ_2, respectively, we can define the map $T_1 * T_2 : D \to D$ by linear extension of

$$(T_1 * T_2)\big(x_1 \otimes x_2 \otimes \ldots \otimes x_n\big) = T_{i_1}(x_1) \otimes T_{i_2}(x_2) \otimes \ldots \otimes T_{i_n}(x_n),$$

where $x_1 \otimes x_2 \otimes \ldots \otimes x_n \in \mathcal{M}_{i_1}^{\circ} \otimes \mathcal{M}_{i_2}^{\circ} \otimes \mathcal{M}_{i_3}^{\circ} \ldots \otimes \mathcal{M}_{i_n}^{\circ}$ with $i_1 \neq i_2 \ldots \neq i_n$. The following result states that we can extend the map $T_1 * T_2$ to the whole $\mathcal{M}_1 * \mathcal{M}_2$ when the T_i are normal u.c.p. and trace preserving.

Theorem 1.11 ([4, Proposition 2.1/Corollary 2.2]).

(i) *Let $T_i : \mathcal{M}_i \to \mathcal{M}_i$ for $i \in \{1, 2\}$ be Markovian maps; the map $T_1 * T_2$ extends to a Markovian map over $(\mathcal{M}_1 * \mathcal{M}_2, \tau_1 * \tau_1)$. Furthermore, $T_1 * T_2$ is symmetric when the maps T_i are.*

(ii) *Let $\mathcal{N}_i \subset \mathcal{M}_i$ be unital von Neumann subalgebras, and let $\mathbf{E}_i : \mathcal{M}_i \to \mathcal{N}_i \subset \mathcal{M}_i$ be their associated conditional expectations; $\mathbf{E}_1 * \mathbf{E}_2 : \mathcal{M}_1 * \mathcal{M}_2 \to \mathcal{N}_1 * \mathcal{N}_2 \subset \mathcal{M}_1 * \mathcal{M}_2$ is a conditional expectation.*

Group von Neumann algebras

We end this section with another interesting example of finite von Neumann algebra that will be used in Section 4. Let Γ be a discrete group and $\lambda : \Gamma \to \mathcal{B}(\ell_2\Gamma)$ the left regular representation: that is, the one given by $\lambda_g \delta_h = \delta_{gh}$, where $(\delta_h)_{h\in\Gamma}$ is the canonical orthogonal base of $\ell_2(\Gamma)$. We define the group von Neumann algebra $\mathcal{L}\Gamma$ by

$$\mathcal{L}\Gamma = \overline{\text{span}}^{\text{w}^*}\{\lambda_g\}_{g\in\Gamma} = \{\lambda_g\}''_{g\in\Gamma} \subset \mathcal{B}(\ell_2\Gamma).$$

$\mathcal{L}\Gamma$ has a natural faithful trace $\tau : \mathcal{L}\Gamma \to \mathbf{C}$ given by $\tau(f) = \langle \delta_e, f\, \delta_e \rangle$ (see [35]). It is worth noting that each $f \in \mathcal{L}\Gamma$ can be expressed as a sum

$$f = \sum_{g\in\Gamma} \widehat{f}(g)\, \lambda_g,$$

where $\widehat{f}(g) = \tau(f\, \lambda_g^*)$. The series above converges in the weak-$*$ topology. The notation $\widehat{f}$ is reminiscent of the case of Abelian groups, where $\mathcal{L}\Gamma = L_\infty(\widehat{\Gamma})$, with $\widehat{\Gamma}$ being the Pontryagin dual of Γ. In this setting, $\widehat{f}$, as defined above, coincides with the Fourier transform for Abelian groups (see [12, Chapter 4]).

In this article, we will work over the algebra of the free group on two generators $\mathbf{F}_2$. Let us denote the generators by a and b, and let $\mathcal{L}\mathbf{F}_2$ be its reduced group von Neumann algebra. Over $\mathbf{F}_2$, there is a natural length $\omega \mapsto |\omega|$ given by

$$|\omega| = \min\big\{n : \omega = s_1 s_2 \ldots s_n \text{ with } s_i \in \{a, a^{-1}, b, b^{-1}\}\big\}.$$

It was shown in [15] that the word length generates a Markovian and symmetric semigroup of operators $\mathcal{S} = (S_t)_{t\geq 0}$. Indeed, given any element $f \in \mathcal{L}\mathbf{F}_2$, we define the maps $S_t : \mathcal{L}\mathbf{F}_2 \to \mathcal{L}\mathbf{F}_2$ given by

$$S_t\Big(\sum_{\omega\in\mathbf{F}_2} \widehat{f}(\omega)\, \lambda_\omega\Big) = \sum_{\omega\in\mathbf{F}_2} \widehat{f}(\omega)\, e^{-t|\omega|}\, \lambda_\omega.$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

2. The limit maximal function

In this section, we will introduce the L_p-norm of the lim sup operator for von Neumann algebras. Let us introduce the linear (nonclosed) space of bilaterally almost finite sequences $\mathcal{L}_{\mathrm{loc}} \subset L_p(\mathcal{M};\ell_\infty) \subset L_1(\mathcal{M};\ell_\infty)$. We will say that a sequence $(f_n)_n \in \mathcal{L}_{\mathrm{loc}}$ iff for every $\epsilon > 0$ there is a projection $e \in \mathscr{P}(\mathcal{M})$ and an integer N such that

- $\tau(e) < \epsilon$.
- $e^\perp f_n e^\perp = 0$, for every $n > N$.

When working in $L_p(\mathcal{M};\ell_\infty)$, the second condition can be interpreted by saying that $(e^\perp f_n e^\perp)_n \in L_p(\mathcal{M};\ell_\infty^N) = L_{2p}(\mathcal{M})\,(\ell_\infty^N \mathbin{\overline{\otimes}} \mathcal{M})\,L_{2p}(\mathcal{M})$.

It is relatively easy to see that $\mathcal{L}_{\mathrm{loc}}$ is a linear space. Indeed, it is trivial that $\mathcal{L}_{\mathrm{loc}}$ is closed under scalar products. To see that the space is closed under sums, assume that $(f_n)_n$ and $(g_n)_n$ are in $\mathcal{L}_{\mathrm{loc}}$, and take $\epsilon > 0$; by definition, we can find $p, q \in \mathscr{P}(\mathcal{M})$ and $N_1, N_2 \in \mathcal{N}$ such that

- $\tau(p) < \epsilon/2$ and $\tau(q) < \epsilon/2$.
- $p^\perp f_n p^\perp = 0$, for every $n > N_1$ and $q^\perp g_n q^\perp = 0$, for every $n > N_2$.

We have that $e = p \vee q \in \mathscr{P}(\mathcal{M})$ satisfies that $\tau(p \vee q) < \tau(p) + \tau(q) < \epsilon$ and that $e^\perp = p^\perp \wedge q^\perp$; therefore

$$e^\perp (f_n + g_n)\, e^\perp = e^\perp f_n e^\perp + e^\perp g_n e^\perp = e^\perp (p^\perp f_n p^\perp)\, e^\perp + e^\perp (q^\perp g_n q^\perp)\, e^\perp = 0,$$

for $n > \max\{N_1, N_2\}$.

Definition 2.1. *We define the space $L_p^{\mathrm{loc}}(\mathcal{M};c_0)$ as the closure of $\mathcal{L}_{\mathrm{loc}}$ inside $L_p(\mathcal{M};\ell_\infty)$. Similarly, we define*

$$L_p^{\mathrm{loc}}(\mathcal{M};\ell_\infty/c_0) = \frac{L_p(\mathcal{M};\ell_\infty)}{L_p^{\mathrm{loc}}(\mathcal{M};c_0)}. \tag{2.1}$$

Observe that given $(f_n)_n \in L_p(\mathcal{M};\ell_\infty)$, the quotient map $q : L_p(\mathcal{M};\ell_\infty) \to L_p^{\mathrm{loc}}(\mathcal{M};\ell_\infty/c_0)$ induces a seminorm on $L_p(\mathcal{M};\ell_\infty)$ after composing with the norm of $L_p^{\mathrm{loc}}(\mathcal{M};\ell_\infty/c_0)$. We will denote that seminorm as follows:

$$\left\| \limsup_{n\to\infty}{}^{+} f_n \right\|_p := \left\| q(f_n) \right\|_{L_p^{\mathrm{loc}}[\ell_\infty/c_0]}.$$

We will also omit the quotient map q when no ambiguity can arise. Notice also that, as in the case of the supremum, the lim sup$^+$ is purely a notation.

Remark 2.2. Recall that, given a sequence of scalars $(a_n)_n$, we have

$$\limsup_{n\to\infty} |a_n| = \|a_n\|_{\ell_\infty/c_0}.$$

Therefore, it would be tempting to define the L_p-norm of the limsup as the seminorm associated with the quotient

$$\frac{L_p(\mathcal{M};\ell_\infty)}{L_p(\mathcal{M};c_0)}.$$

Nevertheless, in the case of $p = 1$, such a definition does not recover the L_1-norm of the limsup—not even in the Abelian case. Indeed, notice that, for the sequence $(f_n)_n$ over $L_1([0,1])$, given by $f_n(t) = n\,\mathbf{1}_{[0,1/n]}(t)$, we have that

$$\left\| (f_n)_n \right\|_{\frac{L_1[\ell_\infty]}{L_1[c_0]}} = \inf_N \left\{ \left\| (f_n)_{n>N} \right\|_{L_1[\ell_\infty]} \right\}$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

$$= \inf_N \left\{ \left\| \sup_{n>N} n\mathbf{1}_{[0,1/n]} \right\|_1 \right\} \quad = \quad \inf_N \int_0^{\frac{1}{N+1}} \left\lceil \frac{1}{t} \right\rceil dt = \infty,$$

while the limsup of the sequence f_n is 0 almost everywhere. In this case, the sequence f_n does not converge in the L_1-norm, but there are sequences of functions $f_n \in L_1([0,1])$ such that f_n converges both in L_1-norm and almost everywhere and yet the seminorm associated with $L_1[\ell_\infty]/L_1[c_0]$ diverges. Such a family of functions can be constructed by choosing an integrable function $f \in L_1([0,1])$ such that their truncated Hardy-Littlewood maximal function

$$M(f)(t) = \sup_{0<r\leq 1} M_r(f)(t) = \sup_{0<r\leq 1} \frac{1}{r} \int_t^{t+r} |f(s)|\, ds$$

is not in L_1 and then taking f_n as a sequence of averages $M_{r_n}(f)$, with $r_n \to \infty$.

To understand what is happening, recall that the Banach space functor $E \mapsto L_1(\Omega) \otimes_\pi E \cong L_1(\Omega; E)$, where $\otimes_\pi$ is the projective tensor product in the category of Banach spaces [41], is both a projective and an injective functor. Therefore, we have

$$L_1(\Omega) \otimes_\pi \ell_\infty/c_0 = \frac{L_1(\Omega) \otimes_\pi \ell_\infty}{L_1(\Omega) \otimes_\pi c_0},$$

for a measure space Ω. But the nonseparability of ℓ_∞/c_0 makes $L_1(\Omega) \otimes_\pi \ell_\infty/c_0$ strictly smaller than the space of all measurable functions $F : \Omega \to \ell_\infty/c_0$ such that

$$\theta \mapsto \|F(\theta)\|_{\ell_\infty/c_0} = \limsup_{n\to\infty} |f_n(\theta)|,$$

is integrable, where $(f_n(\theta))_n$ is a representative of $F(\theta)$. This is due to the fact the equivalence between $L_1(\Omega) \otimes_\pi E$ and the measurable E-valued functions with integrable E-norm requires using Pettis' theorem (see [41, Section 2.3]), which may fail in the nonseparable case.

There is an alternative characterization of the norm of $L_p^{\mathrm{loc}}(\mathcal{M}; \ell_\infty/c_0)$ that would be useful afterward:

Proposition 2.3. *We have that*

$$\left\| \limsup_{n\to\infty}{}^{+} f_n \right\|_p = \sup_{\epsilon>0} \inf_{\substack{e\in\mathscr{P}(\mathcal{M}) \\ \tau(e)<\epsilon}} \inf_{0<N} \left\| (e^\perp f_n e^\perp)_{N<n} \right\|_{L_p[\ell_\infty]}. \tag{2.2}$$

Proof. By definition, the $\limsup^+$ is given by

$$\left\| \limsup_{n\to\infty}{}^{+} f_n \right\|_p = \inf \left\{ \|(f_n - h_n)_n\|_{L_p[\ell_\infty]} : (h_n)_n \in L_p^{\mathrm{loc}}(\mathcal{M}; c_0) \right\},$$

but, by the definition of $L_p^{\mathrm{loc}}(\mathcal{M}; c_0)$, the linear subspace $\mathcal{L}_{\mathrm{loc}}$ is dense; therefore, we can take the infimum of the right-hand side in $(h_n)_n \in \mathcal{L}_{\mathrm{loc}}$. The space $\mathcal{L}_{\mathrm{loc}}$ can be expressed as

$$\mathcal{L}_{\mathrm{loc}} = \bigcap_{\epsilon>0} \bigcup_{\substack{e\in\mathscr{P}(\mathcal{M}) \\ \tau(e)<\epsilon}} \bigcup_{N\in\mathbf{N}} \left\{ (h_n)_n : e^\perp h_n e^\perp = 0, \forall n > N \right\}.$$

Using the monotonicity of the infimum over the index sets, we obtain that

$$\left\| \limsup_{n\to\infty}{}^{+} f_n \right\|_p = \sup_{\epsilon>0} \inf_{\substack{e\in\mathscr{P}(\mathcal{M}) \\ \tau(e)<\epsilon}} \inf_N \inf \left\{ \|(f_n - h_n)_n\|_{L_p[\ell_\infty]} : e^\perp h_n e^\perp = 0, \forall n > N \right\}.$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Fix $\epsilon > 0$, e and N. In the innermost infimum, we can choose h_n as

$$h_n = \begin{cases} f_n & \text{when } n \leq N, \\ e\, f_n\, e + e\, f_n\, e^\perp + e^\perp\, f_n\, e & \text{when } n > N \end{cases}$$

and a calculation gives that

$$\|f_n - h_n\|_{L_p[\ell_\infty]} = \big\|(e^\perp\, f_n\, e^\perp)_{n>N}\big\|_{L_p[\ell_\infty]}.$$

This implies the inequality ($\leq$) in equation (2.2). For the other direction, we will need to use the module properties (1.5) and (1.6). Fix $\epsilon > 0$, $e \in \mathscr{P}(\mathcal{M})$ with $\tau(e) < \epsilon$ and N an integer. We have that

$$\big\|(e^\perp\, f_n\, e^\perp)_{n>N}\big\|_{L_p[\ell_\infty]} = \big\|(e^\perp\, (f_n - h_n)\, e^\perp)_{n>N}\big\|_{L_p[\ell_\infty]},$$

for every h_n, satisfying that $e^\perp\, h_n\, e^\perp \in L_p[\ell_\infty^N]$. Therefore, we can take an infimum in such identity to obtain that

$$\begin{aligned} \big\|(e^\perp\, f_n\, e^\perp)_{n>N}\big\|_{L_p[\ell_\infty]} &= \inf\Big\{ \big\|(e^\perp\, (f_n - h_n)\, e^\perp)_{n>N}\big\|_{L_p[\ell_\infty]} : e^\perp\, h_n\, e^\perp = 0, \forall n > N \Big\} \\ &\leq \inf\Big\{ \big\|(e^\perp\, (f_n - h_n)\, e^\perp)_{n}\big\|_{L_p[\ell_\infty]} : e^\perp\, h_n\, e^\perp = 0, \forall n > N \Big\} \qquad \text{by (1.6)} \\ &\leq \inf\Big\{ \big\|(f_n - h_n)_{n}\big\|_{L_p[\ell_\infty]} : e^\perp\, h_n\, e^\perp = 0, \forall n > N \Big\} \qquad \text{by (1.5)} \end{aligned}$$

and that concludes the proof. □

Recall that, whenever needed, it is possible to describe a natural operator space structure for the spaces $L_p^{\mathrm{loc}}(\mathcal{M}; \ell_\infty/c_0)$. Observe that the functor $E \mapsto S_p^m[E]$ is projective; therefore, we can identify the following spaces

$$S_p^m\Big[\frac{L_p(\mathcal{M}; \ell_\infty)}{L_p^{\mathrm{loc}}(\mathcal{M}; c_0)}\Big] = \frac{S_p^m\big[L_p(\mathcal{M}; \ell_\infty)\big]}{S_p^m\big[L_p^{\mathrm{loc}}(\mathcal{M}; c_0)\big]} = \frac{L_p(M_m \otimes \mathcal{M}; \ell_\infty)}{S_p^m\big[L_p^{\mathrm{loc}}(\mathcal{M}; c_0)\big]}$$

isometrically. We can represent $L_p^{\mathrm{loc}}(\mathcal{M}; c_0)$ as the closure of

$$\bigcap_{\epsilon>0} \bigcup_{\substack{e \in \mathscr{P}(\mathcal{M}) \\ \tau(e) < \epsilon}} \bigcup_{N \in \mathbf{N}} e^\perp\, L_p(\mathcal{M}; \ell_\infty^N)\, e^\perp.$$

But using the fact that

$$S_p^m[e^\perp\, L_p(\mathcal{M}; \ell_\infty^N)\, e^\perp] = e^\perp\, L_p(M_m \otimes \mathcal{M}; \ell_\infty^N)\, e^\perp,$$

we obtain that

$$S_p^m\big[L_p^{\mathrm{loc}}(\mathcal{M}; \ell_\infty/c_0)\big] = L_p^{\mathrm{loc}}(M_m \otimes \mathcal{M}; \ell_\infty/c_0).$$

The following two propositions link the boundedness of the noncommutative lim sup with the bilateral almost uniform convergence.

Proposition 2.4. *Let $(f_n)_n \subset L_p(\mathcal{M})$ be a sequence such that $f_n \to f$ bilaterally almost uniformly. We have that*

$$\Big\|\limsup_{n\to\infty}{}^{+} f_n\Big\|_p = \|f\|_p. \tag{2.3}$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Proof. The proof is almost trivial. Start by noticing that in the case of finite von Neumann algebras, we have that

$$\|(f_n)_n\|_{L_p[\ell_\infty]} \leq \|(f_n)_n\|_{\ell_\infty \overline{\otimes} \mathcal{M}}.$$

Therefore, if $g_n \to 0$ b.a.u.. we have that

$$\sup_{\epsilon>0} \inf_{\substack{e\in\mathscr{P}(\mathcal{M})\\ \tau(e)<\epsilon}} \inf_{N\in\mathbf{N}} \left\|(e^\perp g_n e^\perp)_{n>N}\right\|_{L_1[\ell_\infty]} \leq \sup_{\epsilon>0} \inf_{\substack{p\in\mathscr{P}(\mathcal{M})\\ \tau(p)<\epsilon}} \inf_{N\in\mathbf{N}} \left\|(p^\perp g_n p^\perp)_{n>N}\right\|_{\ell_\infty \overline{\otimes} \mathcal{M}},$$

and the last expression tends to 0. Taking $g_n = f - f_n$ gives (2.3). □

Remark 2.5. In the previous proposition, we impose the condition that $(\mathcal{M}, \tau)$ must be finite. There are pathologies that can occur in the semifinite case. For instance, the sequence $f_n = \mathbf{1}_{[n,n+1]}$ converges pointwise to zero and is uniformly bounded inside $L_1(\mathbf{R}_+)$. Nevertheless, if $\epsilon > 0$ is a finite number and $E \subset \mathbf{R}_+$ is a measurable set such that $m(E) < \epsilon$, then $(\mathbf{1}_{E^c} f_n)_{N<n}$ is not in $L_1(\mathbf{R}_+; \ell_\infty)$ for any N.

Proposition 2.6. *If $(S_n)_n$ is a bounded family of operators $S_n : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ such that*

(i) *There is a bounded normal map $F : \mathcal{M} \to \mathcal{M}$ such that $\|S_n(f) - F(f)\|_\infty \to 0$ for all $f \in \mathcal{S}$, where $\mathcal{S} \subset L_1(\mathcal{M}) \cap \mathcal{M}$ is dense in the Orlicz space $L_\Phi(\mathcal{M})$ associated with a convex increasing function Φ that fixes 0 and ∞,*

(ii) *For every $f \in L_\Phi(\mathcal{M})$, we have*

$$\left\|\limsup{}^+_{n\to\infty} S_n(f)\right\|_1 \lesssim \|f\|_{L_\Phi}.$$

Then, for every $f \in L_\Phi$, $S_n(f) \to F(f)$ bilaterally almost uniformly.

Proof. Fix $f \in L_\Phi(\mathcal{M})$. We want to prove that, for every $\epsilon > 0$ and $\delta > 0$, there is an integer N and projection $e \in \mathscr{P}(\mathcal{M})$ such that $\tau(e) < \epsilon$ and

$$\left\|e^\perp (S_n(f) - F(f))\, e^\perp\right\|_\infty < \delta,$$

for every $n > N$. Denote $S_n - F$ by T_n and denote by C the optimal quantity satisfying that

$$\left\|\limsup{}^+_{n\to\infty} T_n(f)\right\|_1 \leq C\, \|f\|_\Phi.$$

By the density of $\mathcal{S}$ inside $L_\Phi(\mathcal{M})$, we can choose $g \in \mathcal{S}$ such that $\|f - g\|_\Phi < \delta_0$, where $\delta_0 < \delta\, \epsilon/(8C)$. Now, we have that

$$T_n(f) = T_n(f - g) + T_n(g)$$

and the second terms tends to 0 in the L_∞-norm, therefore there is an integer N_1 such that $\|T_n(g)\|_\infty < \delta$ for every $n > N_1$. For the first term, we have that

$$\left\|\limsup{}^+_{n\to\infty} T_n(f - g)\right\|_1 \leq C\, \delta_0.$$

But by (1.3) we do have that there is a projection $p \in \mathscr{P}(\mathcal{M})$ such that $\tau(p) < \epsilon/2$ and an integer N_0 such that

$$\left\|p^\perp (T_n(f - g))_{n>N_0}\, p^\perp\right\|_{L_1[\ell_\infty]} \leq 2C\delta_0.$$

Therefore, we can factor the term above as $p^\perp (T_n(f - g))\, p^\perp = \alpha\, h_n\, \beta$, where $\|\alpha\|_2 = \|\beta\|_2 = \sqrt{2C\delta_0}$ and $\|h_n\|_{\ell_\infty \otimes \mathcal{M}} = 1$ Fix a $\lambda > 0$, to be determined later, and define the projections

$$r = \mathbf{1}_{[\lambda,\infty)}(\alpha\alpha^*) \quad \text{and} \quad q = \mathbf{1}_{[\lambda,\infty)}(\beta^*\beta).$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

By Chebyshev's inequality, we have that

$$\tau(r) \leq \frac{\|\alpha\|_2^2}{\lambda} \quad \text{and} \quad \tau(q) \leq \frac{\|\beta\|_2^2}{\lambda}.$$

Taking $\lambda = \delta$ gives that $e = r \vee q \vee p$ satisfies that

$$\tau(e) \leq \frac{\epsilon}{2} + \frac{4C\delta_0}{\lambda} = \frac{\epsilon}{2} + \frac{\delta\epsilon}{2\,\delta} = \epsilon$$

but we also have that

$$\begin{aligned} \left\| e^\perp T_n(f-g)\, e^\perp \right\|_\infty &= \left\| e^\perp\, p^\perp T_n(f-g)\, p^\perp\, e^\perp \right\|_\infty \\ &= \left\| e^\perp \alpha\, h_n\, \beta\, e^\perp \right\|_\infty \\ &\leq \left\| e^\perp \alpha \right\|_\infty \left(\sup_n \left\| h_n \right\|_\infty \right) \left\| \beta\, e^\perp \right\|_\infty \quad < \quad \delta^{\frac{1}{2}} \delta^{\frac{1}{2}}. \end{aligned}$$

Taking $N = \max\{N_0, N_1\}$ and $e \in \mathscr{P}(\mathcal{M})$ gives that

$$\left\| e^\perp \left(F(f) - S_n(f) \right) e^\perp \right\|_\infty < \delta + \delta,$$

for every $n > N$ and that ends the proof. □

Throughout the text, we will need to work also with limits in more than one index. We will use the convention that, for scalar sequences $(a_{n,m})_{n,m}$,

$$\limsup_{n,m\to\infty} |a_{n,m}| = \inf_{0\leq N,M} \sup_{n\geq N, m\geq M} |a_{n,m}|. \tag{2.4}$$

This notion of lim sup is chosen to avoid the case in which the pair (n, m) tends to infinite, in the sense of escaping of every finite set, while one of the coordinates n or m does not. For any given pair of sequences $n(j)$ and $m(j)$, both of them tending to infinity, we have that

$$\limsup_{j\to\infty} |a_{n(j),m(j)}| \leq \limsup_{n,m\to\infty} |a_{n,m}|,$$

and the inequality is tight in the sense that we can make those quantities the same by choosing adequate subsequences $n(j)$ and $m(j)$ as above. That lim sup can be represented as a quotient norm is as follows. Let $L_{N,M} \subset \ell_\infty(\mathbf{N}\times\mathbf{N})$ be the following space

$$L_{N,M} = \ell_\infty^N \,\overline{\otimes}\, \ell_\infty + \ell_\infty \,\overline{\otimes}\, \ell_\infty^M.$$

We define $c_\Delta(\mathbf{N}\times\mathbf{N})$ as the norm closure of all $L_{N,M}$

$$c_\Delta(\mathbf{N}\times\mathbf{N}) = \overline{\operatorname{span}}^{\|\cdot\|_\infty} \left\{ L_{N,M} : N, M \right\} \subset \ell_\infty(\mathbf{N}\times\mathbf{N}).$$

We have that

$$\limsup_{n,m\to\infty} |a_{n,m}| = \left\| (a_{n,m})_{n,m} \right\|_{\ell_\infty(\mathbf{N}\times\mathbf{N})/c_\Delta(\mathbf{N}\times\mathbf{N})} \tag{2.5}$$

in a way analogous to what we did before. We are going to define a nonclosed linear subspace $\mathcal{L}_{\text{loc}}^\Delta \subset L_p(\mathcal{M}; \ell_\infty(\mathbf{N}\times\mathbf{N}))$ given those sequences $(f_{n,m})_{n,m}$ such that for every $\epsilon > 0$, there is a projection $p \in \mathscr{P}(\mathcal{M})$ and some integers N, M such that

- $\tau(p) < \epsilon$.
- $p^\perp f_{n,m}\, p^\perp = 0$ whenever $n > N$ and $m > M$.

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Observe that the second condition can be expressed as saying that $p^{\perp} f_{n,m} p^{\perp} \in L_p(\mathcal{M}; L_{N,M})$. We define the space $L_p^{\mathrm{loc}}(\mathcal{M}; c_\Delta(\mathbf{N}\times\mathbf{N}))$ as the norm closure of $\mathcal{L}_{\mathrm{loc}}^{\Delta}$ inside $L_p(\mathcal{M}; \ell_\infty)$. We will omit the dependency on the iterated products of the integers $\mathbf{N}\times\mathbf{N}$, when they can be understood from the context. We define

$$L_p^{\mathrm{loc}}(\mathcal{M}; \ell_\infty/c_\Delta) = \frac{L_p(\mathcal{M}; \ell_\infty)}{L_p^{\mathrm{loc}}(\mathcal{M}; c_\Delta)}.$$

We will denote the seminorm induced in $L_p(\mathcal{M}; \ell_\infty)$ by the above quotient as

$$\left\| \limsup_{n,m\to\infty}{}^{+} f_{n,m} \right\|_p := \left\| (f_{n,m})_{n,m} \right\|_{L_p^{\mathrm{loc}}[\ell_\infty/c_\Delta]}.$$

Similarly, we can define multi-indexed versions of the above lim sup and express them as a quotient. The operator space structures of the spaces $L_p^{\mathrm{loc}}(\mathcal{M}; \ell_\infty/c_\Delta)$ are defined in the same way. The analogues of Proposition 2.3 also hold true, as well as the connections with the bilateral almost uniform convergence in more than one variable in Propositions 2.4 and 2.6.

In the next section, we will use the following lemma, whose proof is immediate and left to the reader.

Lemma 2.7.

(i) *Let $(n(j))_j$ and $(m(j))_j$ be sequences of integers converging to infinity; then*

$$\left\| \limsup_{j\to\infty}{}^{+} f_{n(j),m(j)} \right\|_p \le \left\| \limsup_{n,m\to\infty}{}^{+} f_{n,m} \right\|_p. \tag{2.6}$$

(ii) *When (f_n) only depends on one of the indices, we have that*

$$\left\| \limsup_{n,m\to\infty}{}^{+} f_n \right\|_p = \left\| \limsup_{n\to\infty}{}^{+} f_n \right\|_p. \tag{2.7}$$

Remark 2.8. In the same way in which Propositions 2.4 and 2.6 connect the boundedness of the $\limsup^{+}$ with the bilateral almost uniform convergence, it is possible to introduce asymmetric versions of the space $L_p^{\mathrm{loc}}(\mathcal{M}; \ell_\infty)$ that are connected to the column and row almost uniform convergence. Start by recalling the definition of the asymmetric $L_p[\ell_\infty]$-spaces from [28, 27] (see also [7, 3.1.2]), given by

$$L_p(\mathcal{M}; \ell_\infty^c) = \left(\mathcal{M} \,\overline{\otimes}\, \ell_\infty\right) L_p(\mathcal{M}),$$
$$L_p(\mathcal{M}; \ell_\infty^r) = L_p(\mathcal{M}) \left(\mathcal{M} \,\overline{\otimes}\, \ell_\infty\right),$$

where, as before, the norms are given by taking the infima over all decompositions of (f_n) as $f_n = h_n\,\alpha$ or $f_n = \alpha\, h_n$ of $\|\alpha\|_p \sup_n \|h_n\|_\infty$. Those spaces have been studied for their connection with the column/row almost uniform convergence. Let us focus our attention on the column case since the other one follows symmetrically. We can define a linear non-closed subspace $\mathcal{L}_c \subset L_1(\mathcal{M}; \ell_\infty^c)$ given by all sequences $(f_n)_n$ such that for every $\epsilon > 0$, there is a projection $e \in \mathcal{P}(\mathcal{M})$ and an integer N such that

- $\tau(e) < \epsilon$,
- $f_n\, e^{\perp} = 0$, for every $n > N$.

The closure of $\mathcal{L}_c$ will be denoted by $L_p^{\mathrm{loc}}(\mathcal{M}; c_0^c) \subset L_p(\mathcal{M}; \ell_\infty^c)$. We define the quasi-Banach space

$$L_p^{\mathrm{loc}}(\mathcal{M}; \ell_\infty^c/c_0^c) = \frac{L_p(\mathcal{M}; \ell_\infty^c)}{L_p(\mathcal{M}; c_0^c)}.$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

The analogues of Propositions 2.4 and 2.6 hold in this case. Indeed, if $(\mathcal{M}, \tau)$ is a finite von Neumann algebra, it holds that

$$f_n \longrightarrow f \text{ column almost uniformly} \quad \Longrightarrow \quad \left\|(f_n)_n\right\|_{L_p^{\mathrm{loc}}[\ell_\infty^c/c_0^c]} \leq \|f\|_p.$$

Similarly, if in Proposition 2.6, we change the third assumption by

$$\left\|(T_n)_n : L_\Phi(\mathcal{M}) \to L_p^{\mathrm{loc}}(\mathcal{M}; \ell_\infty^c/c_0^c)\right\| < \infty.$$

Then the consequence is that $T_n(f) \to F(f)$ column almost uniformly.

3. The Jessen-Marcinkiewicz-Zygmund inequality

In this section, we will prove one of the main results of the article, Theorem A1. Despite following relatively easily from the theory that we have already developed, that theorem is the main ingredient of Corollary A2. We will also explore applications to multiparametric ergodic theory and asymmetric—that is, column/row—versions of those results.

Proof (of Theorem A1.) Since all the operators involved B_m and A_n are assumed to be positivity preserving, it is enough to prove the result within the positive cone. Fix $f \in L_\Phi(\mathcal{M})$ with $f \geq 0$. By Proposition 2.3, we have that the norm of $L_1^{\mathrm{loc}}(\mathcal{M}; \ell_\infty/c_0)$ is monotone with respect to the natural order inherited by $L_1(\mathcal{M}; \ell_\infty)$. Using assumption (i) as well as characterization (1.3), we have that for every $\epsilon > 0$, there is a $h \in L_1(\mathcal{M})$ such that

$$B_m(f) \leq h, \text{ for every } m$$

and

$$\|h\|_1 \leq (1+\epsilon)\left\|(B_m)_m : L_\Phi(\mathcal{M}) \to L_1(\mathcal{M}; \ell_\infty)\right\| \|f\|_\Phi.$$

The monotonicity and Lemma 2.7 give that

$$\left\|\limsup_{n,m\to\infty}{}^{+} A_n \circ B_m(f)\right\|_1 \leq \left\|\limsup_{n,m\to\infty}{}^{+} A_n(h)\right\|_1 \leq \left\|\limsup_{n\to\infty}{}^{+} A_n(h)\right\|_1.$$

But since $A_n(h) \to F(h)$ bilaterally almost uniformly by (ii), we can apply Proposition 2.4 to conclude that

$$\begin{aligned}
&\left\|\limsup_{n,m\to\infty}{}^{+} A_n \circ B_m(f)\right\|_1 \\
&\leq \left\|F : L_1(\mathcal{M}) \to L_1(\mathcal{M})\right\| \|h\|_1 \\
&\leq (1+\epsilon)\left\|F : L_1(\mathcal{M}) \to L_1(\mathcal{M})\right\| \left\|(B_m)_m : L_\Phi(\mathcal{M}) \to L_1(\mathcal{M}; \ell_\infty)\right\| \|f\|_\Phi.
\end{aligned}$$

And since ϵ is arbitrary, we can conclude. □

Now we have all the ingredients to obtain Corollary A2.

Proof (of Corollary A2.) Since $B_m(f) = \mathbf{E}_m^{[2]}(f)$ is a martingale sequence, we can apply Theorem 1.10 to obtain that the maximal operator associated with B_m maps $L\log^2 L(\mathcal{M})$ into $L_1(\mathcal{M}; \ell_\infty)$. Similarly, by Cuculescu's theorem, we obtain that $\mathbf{E}_n^{[1]}(f)$ converge b.a.u. for every $f \in L_1(\mathcal{M})$. An application Theorem A1 gives the result. □

The following corollary gives a similar result for multiparametric ergodic means.

Corollary 3.1. *Let $(\mathcal{M}^{[i]}, \tau_i)$ for $i \in \{1, 2\}$ be two finite von Neumann algebras and $T_i : \mathcal{M}^{[i]} \to \mathcal{M}^{[i]}$ be Markovian operators and F_i the projections onto their fixed subspaces* $\ker(\mathrm{id} - T_i)$.

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

(i) *Let* $(\mathcal{M},\tau) = (\mathcal{M}^{[1]} \overline{\otimes} \mathcal{M}^{[2]}, \tau_1 \otimes \tau_2)$*; it holds that*

$$\left\| \limsup_{n,m\to\infty}{}^{+} \big(M_n(T_1) \otimes M_m(T_2)\big)(f) \right\|_1 \lesssim \|f\|_{L\log^2 L(\mathcal{M})}.$$

As a consequence $(M_n(T_1)\otimes M_m(T_2))(f) \to (F_1\otimes F_2)(f)$ *bilaterally almost uniformly as* $n,m\to\infty$*, for every* $f \in L\log^2 L(\mathcal{M})$.

(ii) *Let* $(\mathcal{M},\tau) = (\mathcal{M}^{[1]} * \mathcal{M}^{[2]}, \tau_1 * \tau_2)$*; it holds that*

$$\left\| \limsup_{n,m\to\infty}{}^{+} \big(M_n(T_1) * M_m(T_2)\big)(f) \right\|_1 \lesssim \|f\|_{L\log^2 L(\mathcal{M})}.$$

As a consequence $(M_n(T_1) * M_m(T_2)) \to (F_1 * F_2)(f)$ *bilaterally almost uniformly as* $n,m\to\infty$*, for every* $f \in L\log^2 L(\mathcal{M})$.

(iii) *Furthermore, if* T_2 *is symmetric, we can change* $M_m(T_2)$ *by* T_2^m *in points (i) and (ii) above.*

Proof. Again, the proof is an application of Theorem A1. We will only prove points (i) and (iii) since (ii) follows like point (i).

Notice that, by Theorem 1.7, we have that the maximal associated to $\mathrm{id}\otimes M_m(T_2)$ maps $L\log^2 L(\mathcal{M})$ into $L_1(\mathcal{M};\ell_\infty)$. It is a consequence of Yeadon's results that $M_n(T_1)\otimes\mathrm{id}$ converge b.a.u. Gathering both results allows one to apply Theorem A1. For (iii), we use that, when T_2 is symmetric, so is $T_2\otimes\mathrm{id}$; and therefore, by 1.7(iv), the maximal operator given by powers of $\mathrm{id}\otimes T_2$ is maximally bounded from $L\log^2 L(\mathcal{M})$ into $L_1(\mathcal{M};\ell_\infty)$. □

There are two natural generalization of the results above. One is in the case of asymmetric spaces, the other relates to higher iterations: that is, working with $\mathbf{E}_{n_1}^{[1]} \otimes \mathbf{E}_{n_2}^{[2]} \ldots \otimes \mathbf{E}_{n_r}^{[r]}$ instead of just $\mathbf{E}_{n_1}^{[1]} \otimes \mathbf{E}_{n_2}^{[2]}$. We will briefly cover both directions here.

The asymmetric result can be seen as a generalization of [31] (see also [19, 18]). We will need the following proposition, whose proof is immediate (and thus we omit it).

Proposition 3.2.

(i) $\big\|f^* f\big\|_{L_1\log^s L}^{\frac12} \sim_{(s)} \|f\|_{L_2\log^s L} \sim_{(s)} \big\|f\, f^*\big\|_{L_1\log^s L}^{\frac12}$.

(ii) $\big\|(f_n^* f_n)_n\big\|_{L_1[\ell_\infty]}^{\frac12} = \big\|(f_n)_n\big\|_{L_2[\ell_\infty^c]}$ *and* $\big\|(f_n\, f_n^*)_n\big\|_{L_1[\ell_\infty]}^{\frac12} = \big\|(f_n)_n\big\|_{L_2[\ell_\infty^r]}$.

(iii) $\big\|(f_n^* f_n)_n\big\|_{L_1^{\mathrm{loc}}[\ell_\infty/c_0]}^{\frac12} = \|(f_n)_n\|_{L_2^{\mathrm{loc}}[\ell_\infty^c/c_0^c]}$ *and* $\big\|(f_n\, f_n^*)\big\|_{L_1^{\mathrm{loc}}[\ell_\infty/c_0]}^{\frac12} = \big\|(f_n)_n\big\|_{L_2^{\mathrm{loc}}[\ell_\infty^r/c_0^r]}$.

Now we will formulate the asymmetric analogue of Corollary 3.1.

Corollary 3.3. *Let* $(\mathcal{M}^{[i]},\tau_i)$ *for* $i\in\{1,2\}$ *be two finite von Neumann algebras and* $(\mathcal{M}_m^{[i]})_m$ *and* $\mathbf{E}_m^{[i]} : \mathcal{M}\to\mathcal{M}_m^{[i]}$ *be as before. Let* (M,τ) *be* $(\mathcal{M}^{[1]}\overline{\otimes}\mathcal{M}^{[2]},\tau_1\otimes\tau_2)$*. It holds that*

$$\Big\|\big(\mathbf{E}_n^{[1]}\otimes\mathbf{E}_m^{[2]}\big)_{n,m} : L_2\log^2 L(\mathcal{M}) \to L_2^{\mathrm{loc}}(\mathcal{M};\ell_\infty^\dagger/c_0^\dagger)\Big\| < \infty,$$

where $\dagger\in\{r,c\}$*. As a consequence,* $(\mathbf{E}_n^{[1]}\otimes\mathbf{E}_m^{[2]})(f) \to f$ *row and column almost uniformly as* $n,m\to\infty$*, for every* $f\in L_2\log^2 L(\mathcal{M})$.

The same result follows with trivial changes for free products and in the case of ergodic means.

Proof. We are only going to prove the column case since both are identical. For every $f\in L_2\log^2 L(\mathcal{M})$, we have that $g = f^* f \in L\log^2 L(\mathcal{M})$ by Proposition 3.2 (i). Thus, applying $\mathbf{E}_n^{[1]}\otimes\mathbf{E}_m^{[2]}$ to g and using Corollary A2 (i) gives that

$$\Big\|\big(\mathbf{E}_n^{[1]}\otimes\mathbf{E}_m^{[2]}\big)(g)\Big\|_{L_1^{\mathrm{loc}}[\ell_\infty/c_0]}^{\frac12} \lesssim \|g\|_{L\log^2 L}^{\frac12},$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

but the right-hand side is equivalent to the $L_2 \log^2 L$-norm of f, while, by Kadison's inequality, the left-hand side satisfies that

$$\big(\mathbf{E}_n^{[1]} \otimes \mathbf{E}_m^{[2]}\big)(g) = \big(\mathbf{E}_n^{[1]} \otimes \mathbf{E}_m^{[2]}\big)(f^* f) \geq \big(\mathbf{E}_n^{[1]} \otimes \mathbf{E}_m^{[2]}\big)(f)^* \big(\mathbf{E}_n^{[1]} \otimes \mathbf{E}_m^{[2]}\big)(f).$$

Applying Proposition 3.2 (iii) gives the desired result. □

To extend Corollaries 3.1 and A2 to the case of larger index sets, we need the following proposition, which is a straightforward generalization of [19, page 1121]. Observe also the the proposition below is a generalization of [20, Theorem 2.3] or Theorem 1.5.

Proposition 3.4. *Let $(S_m)_m$ be a family of Markovian operators such that*

$$\big\|(S_m)_m : L_p(\mathcal{M}) \to L_p(\mathcal{M}; \ell_\infty)\big\| \lesssim \max\left\{1, \left(\frac{1}{p-1}\right)^{\alpha}\right\},$$

for some $\alpha \geq 0$. Then for every $\beta \geq 0$, it holds that

$$\big\|(S_m)_m : L \log^{\beta+\alpha} L(\mathcal{M}) \to L \log^{\beta} L(\mathcal{M}; \ell_\infty)\big\| \lesssim \infty.$$

The proof uses as a key ingredient the atomic decomposition of noncommutative $L \log^s L$ spaces obtained in [19, Appendix] as direct generalization of [46]. Indeed, every element of the unit ball of $L \log^s L$ can be expressed as a combination of *atoms*

$$f = \sum_{k=0}^{\infty} \lambda_k \, a_k,$$

where $\lambda_k \geq 0$ satisfy that $\sum_{k=1} \lambda_k \leq 1$. An element $a \in \mathcal{M}$ is an atom for $L \log^s L$ iff it is supported in projection $e \in \mathscr{P}(\mathcal{M})$ with $\tau(e) = 2^k$ for some $k \geq 0$ and

$$\|a\|_\infty \leq \frac{1}{\tau(e)} \left(1 + \log_+ \left(\frac{1}{\tau(e)}\right)\right)^{-s}.$$

The atomic decomposition also gives an equivalence in norm

$$\|f\|_{L \log^s L} \sim \inf\left\{\sum_{k=0}^{\infty} \lambda_k : f = \sum_{k=0}^{\infty} \lambda_k a_k \quad \text{for } a_k \text{ atom}\right\}.$$

Proof. Apart from the atomic decomposition, the other ingredient is the fact that, for finite algebras $\mathcal{M}$, we have

$$\|f\|_{L \log^s L} \lesssim_{(s)} \frac{1}{(p-1)^s} \|f\|_p,$$

for every $1 < p \leq \infty$ (see [2]). Start by noticing that, by Minkowski's inequality, it is enough to prove the result for atoms. Let us fix an atom $a \in L \log^{\alpha+\beta} L(\mathcal{M})$. We have that

$$\begin{aligned}
\big\|\big(S_m(a)\big)_m\big\|_{L \log^\beta L[\ell_\infty]} &\lesssim_{(s)} \frac{1}{(p-1)^\beta} \big\|\big(S_m(a)\big)_m\big\|_{L_p[\ell_\infty]} \\
&\lesssim \frac{1}{(p-1)^{\alpha+\beta}} \big\|a\big\|_p \qquad (3.1) \\
&\leq \frac{1}{(p-1)^{\alpha+\beta}} \tau(e)^{\frac{1}{p}} \frac{1}{\tau(e)} \Big(1 + \log_+ \frac{1}{\tau(e)}\Big)^{-(\alpha+\beta)}, \qquad (3.2)
\end{aligned}$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

for every $1 < p \leq \infty$; we have used the hypothesis on (3.1) and the conditions on the atom in (3.2). We can now fix p, depending on the atom, to be

$$p - 1 = \left(1 + \log_+\left(\frac{1}{\tau(e)}\right)\right)^{-1}.$$

That gives

$$\left\|(S_m(a))\right\|_{L\log^{\beta} L[\ell_\infty]} \leq \tau(e)^{(1-p)/p},$$

and we conclude, noticing that the last expression is uniformly bounded by the choice of p relative to $\tau(e)$. □

As a corollary, we obtain the following.

Corollary 3.5. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra, T a Markovian operator and $\mathbf{E}_n : \mathcal{M} \to \mathcal{M}_m \subset \mathcal{M}$ the conditional expectations associated with a filtration $(\mathcal{M}_m)$. We have that, for every $\beta \geq 0$,*

(i) $\left\|(\mathbf{E}_n)_n : L\log^{\beta+2} L(\mathcal{M}) \to L\log^{\beta} L(\mathcal{M}; \ell_\infty)\right\| < \infty$.
(ii) $\left\|(M_n(T))_n : L\log^{\beta+2} L(\mathcal{M}) \to L\log^{\beta} L(\mathcal{M}; \ell_\infty)\right\| < \infty$.
(iii) $\left\|(T^n)_n : L\log^{\beta+2} L(\mathcal{M}) \to L\log^{\beta} L(\mathcal{M}; \ell_\infty)\right\| < \infty$, *whenever T is symmetric.*

Now we have all the ingredients to formulate our result in the multiparametric case. In particular, we obtain the following.

Corollary 3.6. *Let $\mathcal{M}^{[i]}$ be a collection of von Neumann algebras, for $i \in \{1, 2, ..., d\}$, and assume that there are filtrations $\mathcal{M}_m^{[i]} \subset \mathcal{M}^{[i]}$ approximating each algebram, and denote by $\mathbf{E}_m^{[i]}$ their conditional expectations. Let $(\mathcal{M}, \tau) = (\mathcal{M}^{[1]} \overline{\otimes} \mathcal{M}^{[2]} \overline{\otimes} ... \overline{\otimes} \mathcal{M}^{[d]}, \tau_1 \otimes \tau_2 ... \otimes \tau_d)$. It holds that*

$$\left\| \limsup_{n_1 ... n_d \to \infty}{}^{+} \left(\mathbf{E}_{n_1}^{[1]} \otimes \mathbf{E}_{n_2}^{[2]} ... \otimes \mathbf{E}_{n_d}^{[d]}\right)(f) \right\|_1 \lesssim \|f\|_{L\log^{2(d-1)} L(\mathcal{M})}.$$

As a consequence, $\mathbf{E}_{n_1}^{[1]} \otimes \mathbf{E}_{n_2}^{[2]} ... \otimes \mathbf{E}_{n_d}^{[d]}(f) \to f$ bilaterally almost uniformly as $n_1, n_2, ...n_d \to \infty$, for every $f \in L\log^{2(d-1)} L(\mathcal{M})$. The same result holds after changing the tensor products by free products as in Corollary 3.1.

Proof. The proof is just an iteration of Theorem A1 using the positivity of the conditional expectations. Indeed, starting with an element $f \in L\log^{2(d-1)} L$ gives, by Proposition 3.5 (i), an element $f_1 \in L\log^{2(d-1)} L$ such that

$$\mathrm{id} \otimes \mathbf{E}_{n_d}^{[d]}(f) \leq f_1,$$

and its $L\log^{2(d-2)} L$ norm is controlled by the $L\log^{2(d-2)} L$ norm of f. Applying this process $d - 1$ times gives an element $f_{d-1} \in L_1$, and we can apply the bilateral almost uniform convergence of $\mathbf{E}_{n_1}^{[1]} \otimes \mathrm{id}(f)$ for every $f \in L_1$ to obtain a bound of the $\limsup^+$ after Proposition 2.4. □

The same result holds for ergodic means, giving the following.

Corollary 3.7. *Let $\mathcal{M}^{[i]}$ be a collection of finite von Neumann algebras, for $i \in \{1, 2, ..., d\}$, and $T_i : \mathcal{M}^{[i]} \to \mathcal{M}^{[i]}$ be Markovian operators. Fix $(M, \tau) = (\mathcal{M}^{[1]} \overline{\otimes} \mathcal{M}^{[2]} \overline{\otimes} \cdots \mathcal{M}^{[d]}, \tau_1 \otimes \tau_2 ... \tau_d)$. It holds that*

$$\left\| \limsup_{n_1 ... n_d \to \infty}{}^{+} \left(M_{n_1}(T_1) \otimes M_{n_2}(T_2) ... \otimes M_{n_d}(T_d)\right)(f) \right\|_1 \lesssim \|f\|_{L\log^{2(d-1)} L(\mathcal{M})}.$$

As a consequence, $M_{n_1}(T_1) \otimes M_{n_2}(T_2) ... \otimes M_{n_d}(T_d)(f) \to \left(F_1 \otimes F_2 ... \otimes F_d\right) f$.

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Furthermore, if T_2, T_3, ..., T_d are symmetric, we have that

$$\Big\| \limsup_{n_1 \dots n_d \to \infty}{}^{+} \big(M_{n_1}(T_1) \otimes T_2^{n_2} \dots \otimes T_d^{n_d}\big)(f) \Big\|_1 \lesssim \|f\|_{L\log^{2(d-1)} L(\mathcal{M})}.$$

The same result holds after changing the tensor products by free products.

4. Almost uniform convergence in $\mathbf{F}_2$

We will also denote by $\mathbf{C}[\mathbf{F}_2] \subset \mathcal{L}\mathbf{F}_2$ the free group algebra given by finite combinations of elements λ_ω in $\mathcal{L}\mathbf{F}_2$. Clearly $\mathbf{C}[\mathbf{F}_2]$ is weak-$*$ dense in $\mathcal{L}\mathbf{F}_2$ and norm dense in every $L_p(\mathcal{L}\mathbf{F}_2)$ for $1 \leq p < \infty$ as well as in the Orlicz spaces $L\log^\alpha L(\mathcal{L}\mathbf{F}_2)$. Again, it is immediate that $\|S_t(f) - f\|_\infty \to 0$ as $t \to 0$ for every $f \in \mathbf{C}[\mathbf{F}_2]$. As stated in the introduction (see Problem C1, the boundedness of the maximal operator associated with $(S_t)_t$ will give bilateral almost uniform convergence to the initial data as $t \to 0$. Sadly, that result is still unknown in L_1. Here, we will improve the best-known result by giving a class of operators $\mathcal{C}$ between $L\log^2 L(\mathcal{L}\mathbf{F}_2)$ and $L_1(\mathcal{L}\mathbf{F}_2)$ for which bilateral almost uniform convergence holds.

Let $\langle\langle a\rangle\rangle$ and $\langle\langle b\rangle\rangle$ be the normal subgroups generated by a and b, respectively. Denote its associated quotient maps by $n_a : \mathbf{F}_2 \to \mathbf{F}_2/\langle\langle b\rangle\rangle \cong \mathbf{Z}$ and $n_b : \mathbf{F}_2 \to \mathbf{F}_2/\langle\langle a\rangle\rangle \cong \mathbf{Z}$. For some reduced word $\omega = s_1^{n_1} s_2^{n_2} \dots s_\ell^{n_\ell}$, where each $s_i \in \{a, b\}$, the quotient maps are given by

$$n_a\big(s_1^{n_1} s_2^{n_2} \dots s_\ell^{n_\ell}\big) = \sum_{i\,|\,s_i = a} n_i,$$
$$n_b\big(s_1^{n_1} s_2^{n_2} \dots s_\ell^{n_\ell}\big) = \sum_{i\,|\,s_i = b} n_i.$$

Intuitively, n_a and n_b count the number of a's and the number of b's with multiplicity. Let $\Sigma \subset \mathbf{F}_2$ be the subset of reduced words satisfying that there are no sign changes in a or b. That is, for every word ω, all of the a's have positive (respectively, negative) exponent, and the same holds for the b's. The key observation that will allow us to apply transference techniques is that, for every $\omega \in \Sigma$, we have that

$$|\omega| = |n_a(\omega)| + |n_b(\omega)|.$$

Let $\Theta : \mathbf{C}[\mathbf{F}_2] \to \mathbf{C}[\mathbf{F}_2] \otimes_{\text{alg}} \mathbf{C}[\mathbf{Z}] \otimes_{\text{alg}} \mathbf{C}[\mathbf{Z}]$ be the map given by linear extension of $\lambda_\omega \mapsto \lambda_\omega \otimes \lambda_{n_a(\omega)} \otimes \lambda_{n_b(\omega)}$. We have the following.

Proposition 4.1. *The map Θ extends to a normal and faithful and trace preserving $*$-homomorphism*

$$\Theta : \mathcal{L}\mathbf{F}_2 \longrightarrow \mathcal{L}\mathbf{F}_2 \,\overline{\otimes}\, \mathcal{L}(\mathbf{Z}^2) \cong \mathcal{L}\mathbf{F}_2 \,\overline{\otimes}\, L_\infty(\mathbf{T}^2).$$

The proof is a routine application of Fell's absorption principle. Indeed, after identifying $\ell_2(\mathbf{Z}^2)$ with $\ell_2(\mathbf{Z}) \otimes_2 \ell_2(\mathbf{Z})$, the map $V : \ell_2(\mathbf{F}_2) \otimes_2 \ell_2(\mathbf{Z}^2) \to \ell_2(\mathbf{F}_2) \otimes_2 \ell_2(\mathbf{Z}^2)$ given by linear extension of

$$\delta_\omega \otimes \delta_{k_1} \otimes \delta_{k_2} \longmapsto \delta_\omega \otimes \delta_{k_1 - n_a(\omega)} \otimes \delta_{k_2 - n_b(\omega)}$$

is unitary, and the following diagram commutes.

$$\begin{array}{ccc} \ell_2(\mathbf{F}_2) \otimes_2 \ell_2(\mathbf{Z}^2) & \xrightarrow{\quad V \quad} & \ell_2(\mathbf{F}_2) \otimes_2 \ell_2(\mathbf{Z}^2) \\ \Big\downarrow{\scriptstyle \lambda_\omega \otimes \lambda_{n_a(\omega)} \otimes \lambda_{n_a(\omega)}} & & \Big\downarrow{\scriptstyle \lambda_\omega \otimes \mathbf{1} \otimes \mathbf{1}} \\ \ell_2(\mathbf{F}_2) \otimes_2 \ell_2(\mathbf{Z}^2) & \xrightarrow{\quad V \quad} & \ell_2(\mathbf{F}_2) \otimes_2 \ell_2(\mathbf{Z}^2) \end{array}$$

The result follows after identifying $\mathcal{L}\mathbf{Z}^2$ with $L_\infty(\mathbf{T}^2)$. The fact that Θ is trace preserving, where the natural trace of $\mathcal{L}\mathbf{F}_2 \otimes L_\infty(\mathbf{T}^2)$ is given by the tensor product of τ and integration against the Haar

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

measure of $\mathbf{T}^2$, is trivial. Therefore, Θ is faithful, and its image is isomorphic to $\mathcal{L}\mathbf{F}_2$. Given any $\bar{\eta} \in \mathbf{T}^2$, we will denote by $\Theta_{\bar{\eta}}$ the trace-preserving automorphism of $\mathcal{L}\mathbf{F}_2$ given by

$$\Theta_{\bar{\eta}}(f) = \mathrm{ev}_{\bar{\eta}} \circ \Theta(f),$$

where $\mathrm{ev}_{\bar{\eta}}$ denotes the evaluation in $\bar{\eta} \in \mathbf{T}^2$. Alternatively, one can see $\Theta_{\bar{\eta}} : \mathcal{L}\mathbf{F}_2 \to \mathcal{L}\mathbf{F}_2$ as the map given by

$$\Theta_{\bar{\eta}}\Big(\sum_{\omega \in \mathbf{F}_2} \widehat{f}(\omega)\lambda_\omega \Big) = \sum_{\omega \in \mathbf{F}_2} e^{2\pi i \left(n_a(\omega)\eta_1 + n_b(\omega)\eta_2\right)} \widehat{f}(\omega)\lambda_\omega.$$

We shall denote by $\mathcal{L}\mathbf{F}_2|_\Sigma \subset \mathcal{L}\mathbf{F}_2$ the subset of elements supported in Σ: that is,

$$\mathcal{L}\mathbf{F}_2|_\Sigma = \Big\{ f \in \mathcal{L}\mathbf{F}_2 : f = \sum_{\omega \in \Sigma \subset \mathbf{F}_2} \widehat{f}(\omega)\, \lambda_\omega \Big\}.$$

Similarly, we will denote by $L_p(\mathcal{L}\mathbf{F}_2)|_\Sigma$ or $L\log^\alpha L(\mathcal{L}\mathbf{F}_2)|_\Sigma$ the subsets of elements $f \in L_p(\mathcal{L}\mathbf{F}_2)$ or $L\log^\alpha L(\mathcal{L}\mathbf{F}_2)$ such that $\tau(f\lambda_g^*) = 0$ for every $g \notin \Sigma$. Let $P_s : L_\infty(\mathbf{T}^d) \to L_\infty(\mathbf{T}^d)$ be the Poisson semigroup of the torus, given by

$$P_s(f) = P_s\Big(\sum_{k \in \mathbf{Z}^d} \widehat{f}(k) e^{2\pi i k\theta} \Big) = \sum_{k \in \mathbf{Z}^d} \widehat{f}(k) e^{-s|k|} e^{2\pi i k\theta}.$$

We have that the following diagram commutes.

$$\begin{array}{ccccc}
\mathcal{L}\mathbf{F}_2|_\Sigma & \xrightarrow{\ \subset\ } & \mathcal{L}\mathbf{F}_2 & \xrightarrow{\ \Theta\ } & \mathcal{L}\mathbf{F}_2 \,\overline{\otimes}\, L_\infty(\mathbf{T}^2) \\
\big\downarrow{\scriptstyle S_t} & & & & \big\downarrow{\scriptstyle \mathrm{id}\otimes P_t} \\
\mathcal{L}\mathbf{F}_2|_\Sigma & \xrightarrow{\ \subset\ } & \mathcal{L}\mathbf{F}_2 & \xrightarrow{\ \Theta\ } & \mathcal{L}\mathbf{F}_2 \,\overline{\otimes}\, L_\infty(\mathbf{T}^2)
\end{array}$$

To extend the commutativity of the diagram above to the L_1, we need the following proposition.

Proposition 4.2. *Let $\Theta : \mathcal{L}\mathbf{F}_2 \longrightarrow \mathcal{L}\mathbf{F}_2 \,\overline{\otimes}\, L_\infty(\mathbf{T}^2)$ be as before. We have that*

(i) $\Theta : L_1(\mathcal{L}\mathbf{F}_2) \longrightarrow L_1(\mathcal{L}\mathbf{F}_2 \otimes L_\infty(\mathbf{T}^2))$ *extends to a positivity-preserving isometry.*
(ii) $\Theta : L_1^{\mathrm{loc}}(\mathcal{L}\mathbf{F}_2; \ell_\infty/c_0) \longrightarrow L_1(\mathcal{L}\mathbf{F}_2 \otimes L_\infty(\mathbf{T}^2); \ell_\infty/c_0)$ *extends to a order-preserving quasi-isometry of the positive cones.*

Proof. The first point is a trivial consequence of the fact that Θ is trace-preserving. To prove (ii), we have to show that the following two inequalities hold for every $f \in L_1(\mathcal{M})_+$:

$$\big\|(\Theta(f_n))_n\big\|_{L_1^{\mathrm{loc}}[\ell_\infty/c_0]} \leq \big\|(f_n)_n\big\|_{L_1^{\mathrm{loc}}[\ell_\infty/c_0]}. \tag{4.1}$$

$$\big\|(f_n)_n\big\|_{L_1^{\mathrm{loc}}[\ell_\infty/c_0]} \leq 2\big\|(\Theta(f_n))_n\big\|_{L_1^{\mathrm{loc}}[\ell_\infty/c_0]}. \tag{4.2}$$

For (4.1), start by fixing $\epsilon > 0$. We have that there is a $p \in \mathscr{P}(\mathcal{L}\mathbf{F}_2)$, N an integer and $g \in L_1(\mathcal{L}\mathbf{F}_2)$ such that $p^\perp f_n\, p^\perp \leq g$ for every $N < n$ and $\|g\|_1 \leq (1+\delta)\|(f_n)_n\|_{L_1^{\mathrm{loc}}[\ell_\infty/c_0]}$ for every δ. This implies that for $\epsilon > 0$, there is a projection $q = \Theta(p)$ with $\tau(q) < \epsilon$ and $h \in L_1(\mathcal{L}\mathbf{F}_2 \,\overline{\otimes}\, L_\infty(\mathbf{T}^2))$ given by $h = \Theta(g)$ satisfying that $q^\perp \Theta(f_n)\, q^\perp \leq h$ for every $N < n$. By point (ii), we have that $\|h\|_1 \leq (1+\delta)\|(f_n)_n\|_{L_1^{\mathrm{loc}}[\ell_\infty/c_0]}$. Since δ is arbitrarily small, we can conclude. For (4.2), we are going to prove that

$$\big\|(\Theta(f_n))_n\big\|_{L_1^{\mathrm{loc}}[\ell_\infty/c_0]} \leq 1 \quad \text{implies that} \quad \big\|(f_n)_n\big\|_{L_1^{\mathrm{loc}}[\ell_\infty/c_0]} \leq 2.$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

By the first assumption and the definition of the $L_1^{\mathrm{loc}}[\ell_\infty/c_0]$-norm, we have that for every ϵ, there is a projection $q \in \mathscr{P}(\mathcal{L}\mathbf{F}_2 \overline{\otimes} L_\infty(\mathbf{T}^2))$ with $\varphi(q) < \epsilon$, an integer N and $g \in L_1(\mathcal{L}\mathbf{F}_2 \overline{\otimes} L_\infty(\mathbf{T}^2))$ with $\|g\|_1 \leq 1+\delta$ such that $q^\perp \Theta(f_n)\, q^\perp \leq g$ for nN. To reduce the identity to $\mathcal{L}\mathbf{F}_2$, notice that for almost every $\bar{\eta} \in \mathbf{T}^2$, we have that

$$q_{\bar{\eta}}^\perp \Theta_{\bar{\eta}}(f)\, q_{\bar{\eta}}^\perp \leq g_{\bar{\eta}}$$

holds, where $q_{\bar{\eta}} = (\mathrm{ev}_{\bar{\eta}} \otimes \mathrm{id})(q)$ and the same for g. Observe that we have the following inequalities as a consequence of Chebyshev's inequality:

$$\left|\left\{\bar{\theta} \in \mathbf{T}^2 : \tau(p_{\bar{\theta}}) < 2\epsilon\right\}\right| \geq \frac{1}{2},$$
$$\left|\left\{\bar{\theta} \in \mathbf{T}^2 : \|g_{\bar{\theta}}\|_1 < (2+\delta)\|g\|_1\right\}\right| > \frac{1}{2}.$$

Therefore, there is a $\bar{\theta}$ that lies in the intersection of both sets. Then $\bar{\theta}$ satisfies that

- $\tau(q_{\bar{\theta}}) < 2\epsilon$.
- $q_{\bar{\theta}}^\perp \Theta_{\bar{\theta}}(f_n)\, q_{\bar{\theta}}^\perp \leq g_{\bar{\theta}}$ for every $N < n$.
- $\|g_{\bar{\theta}}\|_1 \leq (2+\delta)$.

Applying $\Theta_{-\bar{\theta}}$ to $q_{\bar{\theta}}$, $g_{\bar{\theta}}$, and using the fact that the sup in the formula in Proposition 2.3 is monotone and thus can be exchanged by a limit $\epsilon \to 0$, gives that the $L_1^{\mathrm{loc}}[\ell_\infty/c_0]$-norm is smaller than $2+\delta$. But since δ can be taken arbitrarily small, we can conclude. □

Now we can proceed to prove the main theorem of this section.

Proof (of Theorem C2) Fix t_n. Observe that we have the following commutative diagram.

$$\begin{array}{ccccc}
L_1(\mathcal{L}\mathbf{F}_2)|_\Sigma & \xrightarrow{\subset} & L_1(\mathcal{L}\mathbf{F}_2) & \xrightarrow{\Theta} & L_1(\mathcal{L}\mathbf{F}_2 \overline{\otimes} L_\infty(\mathbf{T}^2)) \\
\downarrow{\scriptstyle (S_{t_n})_n} & & & & \downarrow{\scriptstyle (\mathrm{id}\otimes P_{t_n})_n} \\
L_1^{\mathrm{loc}}(\mathcal{L}\mathbf{F}_2; \ell_\infty/c_0) & \xrightarrow{=} & L_1^{\mathrm{loc}}(\mathcal{L}\mathbf{F}_2; \ell_\infty/c_0) & \xrightarrow{\Theta} & L_1^{\mathrm{loc}}(\mathcal{L}\mathbf{F}_2 \overline{\otimes} L_\infty(\mathbf{T}^2); \ell_\infty/c_0)
\end{array}$$

We have that $(\mathrm{id} \otimes P_s)$ is a subordinated semigroup. Therefore, by Proposition 1.9, $(Id \otimes P_s)(f) \to f$ bilaterally almost uniformly as $s \to 0^+$. By Theorem 2.4, we have that

$$\left\|(\mathrm{id} \otimes P_{s_n}) : L_1(\mathcal{L}\mathbf{F}_2 \overline{\otimes} L_\infty(\mathbf{T}^2)) \longrightarrow L_1^{\mathrm{loc}}(\mathcal{L}\mathbf{F}_2 \overline{\otimes} L_\infty(\mathbf{T}^2); \ell_\infty/c_0)\right\| \leq 1.$$

But now, by Proposition 4.1, we have that the horizontal arrows in the commutative diagram are quasi-isometric. That implies (0.1) holds. The result follows. □

The two-parametric problem in the free group

There is a natural two-parametric generalization of Problem C1 for the free group $\mathbf{F}_2$. That is, to determine the largest class $\mathcal{D} \subset L_1(\mathcal{L}\mathbf{F}_2)$ for which the two-parametric symmetric Markovian semigroup given by $(s,t) \mapsto P_s * P_t$ satisfies that

$$\big(P_s * P_t\big)(f) \to f \quad \text{bilaterally almost uniformly, as } s,t \to 0^+,$$

for every $f \in \mathcal{D}$. A straightforward adaptation of Corollary 3.1 gives the following.

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Corollary 4.3. *For every $f \in L\log^2 L(\mathcal{L}\mathbf{F}_2)$ and sequences $s_n, t_n \to 0$, we have that*

$$\Big\| \limsup_{n,m\to\infty}{}^{+} P_{s_n} * P_{t_m}(f) \Big\|_1 \lesssim \|f\|_{L\log^2 L}.$$

As a consequence, $P_s * P_t(f) \to f$ bilaterally almost uniformly as $s, t \to 0$, for every $f \in L\log^2 L(\mathcal{L}\mathbf{F}_2)$.

It is natural to conjecture that, following the same techniques employed to prove Theorem C2, there would be two-parametric convergence for $P_s * P_t$ whenever f lies in the class $\mathcal{D}$:

$$\mathcal{D} = L\log^2 L(\mathcal{L}\mathbf{F}_2) + L\log L(\mathcal{L}\mathbf{F}_2)|_\Sigma. \tag{4.3}$$

Nevertheless, this result is conditioned by the problem of determining whether the following holds:

Problem 4.4. *Let $(P_s)_{s\geq 0}$ be the usual Poisson semigroup on $\mathbf{T}^d$. Does it hold that*

$$\big\|(P_s)_s : L_p(\mathbf{T}^d) \to L_p(\mathbf{T}^d; \ell_\infty)\big\|_{\mathrm{cb}} \lesssim \max\Big\{1, \frac{1}{p-1}\Big\}?$$

Recall that, by (1.1) and (1.4), we have that the complete norm above is given by

$$\begin{aligned}
&\big\|(P_s)_s : L_p(\mathbf{T}^d) \to L_p(\mathbf{T}^d; \ell_\infty)\big\|_{\mathrm{cb}} \\
&\quad = \sup_{m\geq 1} \Big\{ \big\|(\mathrm{id}\otimes P_s)_s : S_p^m[L_p(\mathbf{T}^d)] \to S_p^m[L_p(\mathbf{T}^d; \ell_\infty)\big\| \Big\} \\
&\quad = \sup_{m\geq 1} \Big\{ \big\|(\mathrm{id}\otimes P_s)_s : L_p(M_m \otimes L_\infty(\mathbf{T}^d)) \to L_p(M_m \otimes L_\infty(\mathbf{T}^d); \ell_\infty)\big\| \Big\}.
\end{aligned}$$

It is also easily obtained that the complete norm in L_p of the maximal operator associated with the Poisson semigroup on $\mathbf{T}^d$ has a norm that is bounded by $(p-1)^{-2}$. But, as far as the knowledge of the authors goes, it is not known whether this bound can be lowered to $(p-1)^{-1}$, which is the classical non-complete norm. If Problem 4.4 were to have a positive solution, that would imply, by a routine application of Theorem A1, that the rightmost vertical arrow in the commutative diagram below is bounded.

$$\begin{array}{ccccc}
L\log L(\mathcal{L}\mathbf{F}_2)|_\Sigma & \xrightarrow{\subset} & L\log L(\mathcal{L}\mathbf{F}_2) & \xrightarrow{\Theta} & L\log L(\mathcal{L}\mathbf{F}_2 \,\overline{\otimes}\, L_\infty(\mathbf{T}^2)) \\
\downarrow{\scriptstyle (P_{t_n} * P_{s_m})_{n,m}} & & & & \downarrow{\scriptstyle (\mathrm{id}\otimes P_{t_n}\otimes P_{s_m})_{n,m}} \\
L_1^{\mathrm{loc}}(\mathcal{L}\mathbf{F}_2; \ell_\infty/c_\Delta) & \xrightarrow{=} & L_1^{\mathrm{loc}}(\mathcal{L}\mathbf{F}_2; \ell_\infty/c_\Delta) & \xrightarrow{\Theta} & L_1^{\mathrm{loc}}(\mathcal{L}\mathbf{F}_2 \,\overline{\otimes}\, L_\infty(\mathbf{T}^2); \ell_\infty/c_\Delta)
\end{array}$$

Using that the horizontal arrows are isometric would give that $(\mathrm{id}\otimes P_s)$ induces a maximally bounded map from $L\log L(\mathcal{L}\mathbf{F}_2)|_\Sigma$ to $L_1^{\mathrm{loc}}(\mathcal{L}\mathbf{F}_2; \ell_\infty/c_\Delta)$. Therefore, there would be bilateral almost uniform convergence for functions in the class $\mathcal{D}$ of (4.3) for the two-parametric semigroup $P_s * P_t$.

5. The strong maximal inequality

In this section, we will work under the assumption that $(\mathcal{M}, \tau) = (\mathcal{M}^{[1]} \,\overline{\otimes}\, \mathcal{M}^{[2]}, \tau_1 \otimes \tau_2)$, where $(\mathcal{M}^{[i]}, \tau_i)$ is a noncommutative probability space, $i \in \{1, 2\}$. We also fix filtrations $(\mathcal{M}_m^{[i]})$ of $\mathcal{M}_i$ with associated conditional expectations $\mathbf{E}_m^{[i]} : \mathcal{M}^{[i]} \to \mathcal{M}_m^{[i]}$. We will denote $\mathbf{E}_n^{[1]} \otimes \mathbf{E}_m^{[2]}(f)$ by $f_{n,m}$ at times. Our goal will be to prove Theorem B2, which gives an ε-perturbation of the weak type (Φ, Φ) for the strong maximal in two variables, where $\Phi(t) = t\,(1+\log t)^2$. We also suspect that a similar inequality holds for multiparametric ergodic means using Yeadon's inequality as a tool instead of Cuculescu's inequality.

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

An improvement of Cuculescu's inequality

Let $(\mathcal{N},\tau)$ be a finite von Neumann algebra. Given a positive $f\in L_1(\mathcal{N})$, Cuculescu's inequality gives an explicit combinatorial description of the noncommutative maximal weak type $(1,1)$ inequality for $f\mapsto(\mathbf{E}_n(f))_n$.

Theorem 5.1 (Cuculescu's projections [6]). *Let $(\mathcal{N}_n)_n$ be a filtration over $\mathcal{N}$ and $(\mathbf{E}_n)_n$ the associated conditional expectations. Given $f\in L_1(\mathcal{N})_+$ and $\lambda>0$, the family of decreasing projections $q_1(\lambda)\geq q_2(\lambda)\geq\dots q(\lambda)$ given by the recursive formula*

$$\begin{aligned} q_1(\lambda) &= \mathbf{1},\\ q_n(\lambda) &= q_{n-1}(\lambda)\,\mathbf{1}_{[0,\lambda]}\big(q_{n-1}(\lambda)\,f_n\,q_{n-1}(\lambda)\big)\\ &= \mathbf{1}_{[0,\lambda]}\big(q_{n-1}(\lambda)\,f_n\,q_{n-1}(\lambda)\big)\,q_{n-1}(\lambda),\\ q(\lambda) &= \bigwedge_{n\geq 1} q_n(\lambda), \end{aligned}$$

satisfies

(i) $q_n(\lambda)\in\mathcal{N}_n$.
(ii) $q_n(\lambda)\,f_n\,q_n(\lambda)\leq\lambda q_n(\lambda)$.
(iii) $q_n(\lambda)$ *commutes with* $q_{n-1}(\lambda)\,f_n\,q_{n-1}(\lambda)$.
(iv) *Furthermore,*

$$\tau\big(q(\lambda)^\perp\big)\leq\frac{1}{\lambda}\|f\|_1. \tag{5.1}$$

The following lemma gives a refinement of inequality (5.1) in which the right-hand side is further restricted to the spectral region over which f is large with respect to λ.

Lemma 5.2. *Let $f\in L_1(\mathcal{N})_+$, $\lambda\geq 0$ and $q(\lambda)$ be as in Theorem 5.1. We also have that*

$$\tau\big(q(\lambda)^\perp\big)\;\leq\;\frac{2}{\lambda}\tau\big(f\,\mathbf{1}_{(\frac{\lambda}{2},\infty)}(f)\big).$$

Proof. Let $p_n(\lambda)=q_{n-1}(\lambda)-q_n(\lambda)$ for $n>1$, and observe that

$$p_n(\lambda)\,\mathbf{E}_n\big(f\mathbf{1}_{(\frac{\lambda}{2},\infty)}(f)\big)\,p_n(\lambda)\;\geq\;\frac{\lambda}{2}p_n(\lambda). \tag{5.2}$$

Indeed, $p_n(\lambda)f_np_n(\lambda)=p_n(\lambda)q_{n-1}(\lambda)f_nq_{n-1}(\lambda)p_n(\lambda)\geq\lambda p_n(\lambda)$ and

$$\begin{aligned} &p_n(\lambda)\,\mathbf{E}_n\big(f\mathbf{1}_{(\frac{\lambda}{2},\infty)}(f)\big)\,p_n(\lambda)\\ &\qquad= p_n(\lambda)\,f_n\,p_n(\lambda)-p_n(\lambda)\,\mathbf{E}_n\big(f\,\mathbf{1}_{[0,\frac{\lambda}{2}]}(f)\big)\,p_n(\lambda)\;\geq\;\frac{\lambda}{2}\,p_n(\lambda). \end{aligned}$$

Next, we use inequality (5.2) to obtain

$$\begin{aligned} \tau\big(\mathbf{1}-q(\lambda)\big) &= \sum_{n>1}\tau\big(p_n(\lambda)\big)\\ &\leq\frac{2}{\lambda}\sum_{n>1}\tau\Big(p_n(\lambda)\,\mathbf{E}_n\big(f\,\mathbf{1}_{(\frac{\lambda}{2},\infty)}(f)\big)\,p_n(\lambda)\Big)\;\leq\;\frac{2}{\lambda}\tau\big(f\mathbf{1}_{(\frac{\lambda}{2},\infty)}(f)\big), \end{aligned}$$

which proves the claimed assertion. □

Remark 5.3. In the commutative case, the formula in Lemma 5.2 is very well known. For instance, a two-sided version of the inequality was used in [42] to prove that the L_1-norm of the Hardy-Littlewood

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

maximal function associated with f is comparable to the $L\log L$-norm of f. This can be obtained immediately, integrating the formula in Lemma 5.2 with respect to λ and applying Fubini's theorem.

That computation gives an interesting consequence in the noncommutative case. Recall that it was shown in [20] that the noncommutative maximal function with respect to a martingale sequence $(\mathbf{E}_n)_n$ is a bounded map $L\log^2 L(\mathcal{N}) \to L_1(\mathcal{N};\ell_\infty)$, and that the exponent 2 on the log is optimal. By copying the computation of [42], we have that

$$\inf\left\{\int_0^\infty \tau\big(q(\lambda)^\perp\big)\,d\lambda \;:\; \begin{array}{l} q(\lambda)\in\mathscr{P}(\mathcal{N}) \text{ with} \\ q(\lambda)\,f_n\,q(\lambda)\le\lambda\,q(\lambda), \forall n\ge 1\end{array}\right\} \lesssim \|f\|_{L\log L(\mathcal{N})}. \tag{5.3}$$

This leads to the following interesting observation: given a sequence $(f_n)_n \in L_1(\mathcal{N};\ell_\infty)$, the quantity on the left-hand side of (5.3) is not comparable to the $L_1[\ell_\infty]$-norm. Indeed, if it were so, one would get that the $L_1[\ell_\infty]$-norm of $(\mathbf{E}_n(f))_m$ would be bounded by the $L\log L$-norm of f. But that is not possible by [20, Section 3].

This gives an extra source of intuition regarding why it is difficult to obtain maximal L_p inequalities from weak type ones (like those given by Cuculescu's and Yeadon's inequalities). It is not possible to reconstruct the $L_p[\ell_\infty]$-norm from the projections obtained in the weak type inequalities.

Proof of the strong maximal theorem

Now let us prove assertions (i) and (ii) of Theorem B2. If $\lambda \le (2e^2)^{1/\varepsilon}$. Both inequalities trivially hold with $q(\lambda)=0$. When $\lambda > (2e^2)^{1/\varepsilon}$, we may assume that $\lambda = e^r$ for some positive integer $r\in\mathbf{Z}_+$. Let us first construct the projection $q(\lambda)$. Let $(f_n)_{n\ge1}$ denote the positive martingale $(\mathbf{E}_n^{[1]}\otimes\mathrm{id})(f)$, and consider Cuculescu's projections $q_n^{[1]}(e^\ell)$ associated with $(f_n)_n$ at height e^ℓ:

$$q^{[1]}(e^\ell) = \bigwedge_{n\ge1} q_n^{[1]}(e^\ell).$$

Define the family of disjoint projections

$$\pi_j^{[1]} = \bigwedge_{\ell\ge j} q^{[1]}(e^\ell) - \bigwedge_{\ell\ge j-1} q^{[1]}(e^\ell) \quad \text{for} \quad j\in\mathbf{Z}.$$

Let $\rho_\lambda = \sum_{j\ge r}\pi_j^{[1]}$, where r comes from the identity $\lambda = e^r$, and consider the (unbounded) operators

$$\Pi = \sum_{j\ge1} e^j\, j^{1+\varepsilon}\pi_j^{[1]} \quad \text{and} \quad \Pi_\lambda = \rho_\lambda\Pi = \Pi\rho_\lambda. \tag{5.4}$$

Next, construct the sequence of Cuculescu's projections $q_m^{[2]}(e^\ell)$ associated with the positive martingale $(\mathrm{id}\otimes\mathbf{E}_m^{[2]})(\Pi_\lambda)$ at height e^ℓ. As we did above, define the corresponding family of disjoint projections $\pi_j^{[2]}$ accordingly. We finally define

$$q(\lambda) := \mathbf{1} - \sum_{j\ge r}\pi_j^{[2]}.$$

Proof of inequality (i). We start by noticing that

$$\begin{aligned} q_\lambda\, f_{n,m}\, q_\lambda &= q_\lambda\mathbf{E}_m^{[2]}\big(\rho_\lambda\, f_n\,\rho_\lambda + \rho_\lambda\, f_n\,\rho_\lambda^\perp + \rho_\lambda^\perp f_n\rho_\lambda + \rho_\lambda^\perp\, f_n\,\rho_\lambda^\perp\big)q_\lambda \\ &\le 2q_\lambda\mathbf{E}_m^{[2]}\big(\rho_\lambda\, f_n\,\rho_\lambda + \rho_\lambda^\perp f_n\rho_\lambda^\perp\big)\,q_\lambda \;\le\; 2q_\lambda\mathbf{E}_m^{[2]}\big(\rho_\lambda f_n\rho_\lambda\big)q_\lambda + 2e^r\,q_\lambda. \end{aligned}$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Indeed, to justify the last inequality, recall that

$$\rho_\lambda^\perp = \bigwedge_{\ell\in\mathbf{Z}} q^{[1]}(e^\ell) + \sum_{j<r} \pi_j^{[1]}, \tag{5.5}$$
$$\Big\| \bigwedge_{\ell\in\mathbf{Z}} q^{[1]}(e^\ell) f_n^{\frac12} \Big\| = \Big\| f_n^{\frac12} \bigwedge_{\ell\in\mathbf{Z}} q^{[1]}(e^\ell) \Big\| \le \lim_{\ell\to-\infty} e^{\frac{\ell}{2}} q^{[1]}(e^\ell) = 0.$$

In particular, we may estimate $\rho_\lambda^\perp f_n \rho_\lambda^\perp$ as follows

$$\begin{aligned}\big\|\rho_\lambda^\perp f_n \rho_\lambda^\perp\big\|_{\mathcal{M}} &= \sup_{\|\alpha\|_{L_2(\mathcal{M})}\le 1} \sum_{j,k<r} \big\langle \pi_j^{[1]} f_n \pi_k^{[1]}\alpha, \alpha\big\rangle \\ &\le \sup_{\|\alpha\|_{L_2(\mathcal{M})}\le 1} \sum_{j,k<r} \big\|\pi_j^{[1]} f_n \pi_k^{[1]}\big\|_{\mathcal{M}} \big\|\pi_j^{[1]}\alpha\big\|_2 \big\|\pi_k^{[1]}\alpha\big\|_2 \\ &\le \sup_{\|\alpha\|_{L_2(\mathcal{M})}\le 1} \sum_{j,k<r} e^{(j+k)/2} \big\|\pi_j^{[1]}\alpha\big\|_2 \big\|\pi_k^{[1]}\alpha\big\|_2 \;<\; e^r\end{aligned}$$

by Hölder's inequality. Next, to estimate $\rho_\lambda f_n \rho_\lambda = \Pi_\lambda^{\frac12}(\Pi_\lambda^{-\frac12} f_n \Pi_\lambda^{-\frac12})\Pi_\lambda^{\frac12}$, we note that

$$\begin{aligned}\big\|\Pi_\lambda^{-\frac12} f_n \Pi_\lambda^{-\frac12}\big\|_{\mathcal{M}} &= \sup_{\|\alpha\|_{L_2(\mathcal{M})}\le 1} \sum_{j,k\ge r} \frac{e^{-(j+k)/2}}{(jk)^{(1+\varepsilon)/2}} \big\langle \pi_j^{[1]} f_n \pi_k^{[1]}\alpha, \alpha\big\rangle \\ &\le \sup_{\|\alpha\|_{L_2(\mathcal{M})}\le 1} \Big(\sum_{j\ge r} \frac{\|\pi_j^{[1]}\alpha\|_2}{j^{\frac{1+\varepsilon}{2}}} \Big)^2 \;\le\; \sum_{j\ge 1} \frac{1}{j^{1+\varepsilon}} \;=:\; A_\varepsilon.\end{aligned}$$

This implies that $\rho_\lambda f_n \rho_\lambda \le A_\varepsilon \Pi_\lambda$, and therefore

$$q(\lambda)\, \mathbf{E}_m^{[2]}\big(\rho_\lambda f_n \rho_\lambda\big)\, q(\lambda) \le A_\varepsilon\, q(\lambda)\, \mathbf{E}_m^{[2]}(\Pi_\lambda)\, q(\lambda) \le A_\varepsilon \lambda\, q(\lambda).$$

The last inequality follows from the definition of $q(\lambda)$, arguing as in (5.5) and the estimate after it. Altogether, we have proved that $q(\lambda)\, f_{n,m}\, q(\lambda) \le C_\varepsilon \lambda q(\lambda)$ for some constant C_ε independent of λ, which diverges as $\varepsilon \to 0^+$. □

Proof of inequality (ii) We have

$$\begin{aligned}\tau(\mathbf{1} - q_\lambda) &= \sum_{j>r-1} \tau(\pi_j^{[2]}) \;=\; \tau\Big(\mathbf{1} - \bigwedge_{\ell\ge r-1} q_2(2^\ell)\Big) \\ &\le \sum_{\ell\ge r-1} \tau\big(\mathbf{1} - q^{[2]}(e^\ell)\big) \;\le\; \sum_{\ell\ge r-1} e^{-\ell}\tau(\Pi_\lambda) \quad \lesssim \frac{1}{\lambda} \quad \tau\big(\Pi_\lambda\big).\end{aligned} \tag{5.6}$$

We now expand the trace of Π_λ/λ as follows:

$$\tau(\Pi_\lambda/\lambda) = \int_{\mathbf{R}_+} \tau\big(\mathbf{1}_{(\lambda s,\infty)}(\Pi_\lambda)\big)\, ds \le \tau(\rho_\lambda) + \int_1^\infty \tau\big(\mathbf{1}_{(\lambda s,\infty)}(\Pi_\lambda)\big)\, ds.$$

Arguing as above, we see that $\tau(\rho_\lambda) \lesssim \lambda^{-1}\tau(f)$, which gives the first term in the sum defined by the right-hand side of inequality (ii) in Theorem B2. It remains to prove the inequality

$$\int_1^\infty \tau\big(\mathbf{1}_{(\lambda s,\infty)}(\Pi_\lambda)\big)\, ds \;\le\; C_\varepsilon\, \tau\left\{ \frac{f}{\lambda}\Big(1 + \log_+\Big[\frac{f}{\lambda}\Big]\Big)\Big(\log_+\Big[\frac{f}{\lambda^{1-\epsilon}}\Big]\Big)^{1+\varepsilon} \right\}. \tag{5.7}$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Since $\Pi_\lambda = \sum_{j\geq r} e^j j^{1+\varepsilon}\pi_j^{[1]}$ and $\lambda = e^r$, we get

$$\mathbf{1}_{(\lambda s,\infty)}(\Pi_\lambda) \;=\; \sum_{\substack{j\geq r\\ e^j j^{1+\varepsilon}>\lambda s}} \pi_j^{[1]} \;=\; \begin{cases} \displaystyle\sum_{j\geq r} \pi_j^{[1]} & \text{if } s < r^{1+\varepsilon},\\ \displaystyle\sum_{e^j j^{1+\varepsilon}>\lambda s} \pi_j^{[1]} & \text{if } s > r^{1+\varepsilon}. \end{cases}$$

Let $j_0(s)$ be the biggest integer j satisfying $e^j j^{1+\varepsilon} \leq \lambda s$. Arguing as in (5.6) and using Lemma 5.2 gives

$$\begin{aligned}\int_1^\infty \tau\big(\mathbf{1}_{(\lambda s,\infty)}(\Pi_\lambda)\big)\,ds &\leq r^{1+\varepsilon}\tau(\rho_\lambda) + \int_{r^{1+\varepsilon}}^\infty \sum_{j>j_0(s)} \tau(\pi_j^{[1]})\,ds\\ &\leq \big(\log\lambda\big)^{1+\varepsilon} \sum_{\ell\geq r} e^{-\ell}\tau\big(f\mathbf{1}_{(e^\ell/2,\infty)}(f)\big)\\ &\quad + \int_{r^{1+\varepsilon}}^{e^r} \sum_{\ell\geq j_0(s)} e^{-\ell}\tau\big(f\mathbf{1}_{(e^\ell/2,\infty)}(f)\big)\,ds\\ &\quad + \int_{e^r}^\infty \sum_{\ell\geq j_0(s)} e^{-\ell}\tau\big(f\mathbf{1}_{(e^\ell/2,\infty)}(f)\big)\,ds \;=:\; \mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_3.\end{aligned}$$

Notice that, for every $\varepsilon > 0$, we have that

$$\mathbf{1}_{(\frac{\lambda}{2},\infty)}(f) = \mathbf{1}_{(\lambda,\infty)}\left(\left(\frac{2f}{\lambda^{1+\varepsilon}}\right)^{\frac{1}{\varepsilon}}\right)$$

and therefore, we trivially obtain that

$$\begin{aligned}\mathrm{A}_1 &\lesssim \big(1+\log_+\lambda\big)^{1+\varepsilon}\tau\Big(\frac{f}{\lambda}\mathbf{1}_{(\frac{\lambda}{2},\infty)}(f)\Big)\\ &\lesssim \tau\Big\{\frac{f}{\lambda}\big(1+\log_+\lambda\big)^{1+\varepsilon}\mathbf{1}_{(\frac{\lambda}{2},\infty)}(f)\Big\}\\ &\lesssim \Big(1+\frac{1}{\varepsilon}\Big)\tau\Big\{\frac{f}{\lambda}\Big(1+\log_+\frac{f}{\lambda^{1-\varepsilon}}\Big)^{1+\varepsilon}\Big\}.\end{aligned}$$

Next, we use $\lambda > e^2$ to get $j_0(s) \geq 2$ so that

$$e^{-j_0(s)} \;=\; \frac{e(j_0(s)+1)^{1+\varepsilon}}{e^{j_0(s)+1}(j_0(s)+1)^{1+\varepsilon}} \lesssim \frac{(\log(\lambda s))^{1+\varepsilon}}{\lambda s}$$

for $s > r^{1+\varepsilon}$. Since $\log(\lambda s) \leq 2\max\{\log\lambda, \log s\}$, we get

$$\sum_{\ell\geq j_0(s)} e^{-\ell}\tau\big(f\mathbf{1}_{(e^\ell/2,\infty)}(f)\big) \;\lesssim\; \begin{cases} \dfrac{\log^{1+\varepsilon}(\lambda)}{\lambda s}\tau\Big(f\mathbf{1}_{(\frac{\lambda s}{\log^{1+\varepsilon}(\lambda)},\infty)}(10f)\Big) & \text{if } \lambda \geq s,\\ \dfrac{\log^{1+\varepsilon}(s)}{\lambda s}\tau\Big(f\mathbf{1}_{(\frac{\lambda s}{\log^{1+\varepsilon}(s)},\infty)}(10f)\Big) & \text{if } s > \lambda. \end{cases}.$$

This gives the following estimate for A_2, arguing as for A_1:

$$\mathrm{A}_2 \;\leq\; \frac{\log^{1+\varepsilon}(\lambda)}{\lambda}\int_{\log^{1+\varepsilon}(\lambda)}^{\lambda} \tau\Big\{f\mathbf{1}_{(\frac{\lambda s}{\log^{1+\varepsilon}(\lambda)},\infty)}(10f)\Big\}\,\frac{ds}{s}$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

$$\leq \frac{\log^{1+\varepsilon}(\lambda)}{\lambda}\,\tau\Big\{f\log\Big[\frac{4ef}{\lambda}\log^{1+\varepsilon}(\lambda)\Big]\mathbf{1}_{(\frac{\lambda s}{\log^{1+\varepsilon}(\lambda)},\infty)}(10f)\Big\}$$
$$\lesssim_{\varepsilon} \tau\Big\{\frac{f}{\lambda}\Big(1+\log_+\Big(\frac{f}{\lambda}\Big)\Big)\Big(1+\log_+\Big(\frac{f}{\lambda^{1-\varepsilon}}\Big)\Big)^{1+\varepsilon}\Big\}.$$

Similarly, we get

$$\mathrm{A}_3 \leq \frac{1}{\lambda}\int_{\lambda}^{\infty}\tau\Big(f\mathbf{1}_{(\frac{\lambda s}{\log^{1+\varepsilon}(s)},\infty)}(10f)\Big)\,\frac{\log^{1+\varepsilon}(s)}{s}ds.$$

Next, given $s > \lambda > e^{16}$ and $\varepsilon < 1$, observe that

$$\frac{s}{\log^{1+\varepsilon}(s)} < A \;\Rightarrow\; s < A\log^{1+\varepsilon}(s) < 4A\log^{1+\varepsilon}(A).$$

Therefore, approximating the operator $10f/\lambda$ by a positive combination of pairwise disjoint projections $\sum_j a_j P_j$, we may rewrite the above integral as follows and apply Fubini to get the following estimate:

$$\mathrm{A}_3 \leq \tau\left(\frac{f}{\lambda}\int_{\lambda}^{\infty}\sum_{a_j>\frac{s}{\log^{1+\varepsilon}(s)}}P_j\,\frac{\log^{1+\varepsilon}(s)}{s}ds\right)$$
$$\leq \tau\left(\frac{f}{\lambda}\sum_{a_j>\frac{\lambda}{\log^{1+\varepsilon}(\lambda)}}\Big[\int_{\lambda}^{4a_j\log^{1+\varepsilon}(a_j)}\frac{\log^{1+\varepsilon}(s)}{s}ds\Big]P_j\right)$$
$$\leq \tau\left(\frac{f}{\lambda}\sum_{a_j>\frac{\lambda}{\log^{1+\varepsilon}(\lambda)}}\log^{2+\varepsilon}\left(4a_j\log^{1+\varepsilon}(a_j)\right)P_j\right)$$
$$\leq \tau\left(\frac{f}{\lambda}\mathbf{1}_{\left(\frac{\lambda}{\log^{1+\varepsilon}(\lambda)},\infty\right)}\left(\frac{10f}{\lambda}\right)\log^{2+\varepsilon}\left[\frac{40f}{\lambda}\log^{1+\varepsilon}\left(\frac{10f}{\lambda}\right)\right]\right)$$
$$\lesssim \tau\left(\frac{f}{\lambda}\Big(\log_+\Big[\frac{f}{\lambda}\Big]\Big)^{2+\varepsilon}\right).$$

□

Several remarks are in order.

Remark 5.4.

R.1 The proof above can be understood as a noncommutative generalization of a result of de Guzmán. Indeed, in [14], it was proven that if two bases of Borel sets $\mathfrak{B}_i \subset \mathcal{B}(\mathbf{R}^{d_i})$ with finite measure satisfy that their associated maximal operators over $\mathbf{R}^{d_i}$,

$$(M_i f)(x) = \sup_{E\in\mathfrak{B}_i}\fint_E |f(x)|\,dx$$

are of weak type (Φ_i, Φ_i): that is,

$$\big|\{x\in\mathbf{R}^{d_i} \,:\, (M_i f)(x) > \lambda\}\big| \lesssim \int_{\mathbf{R}^{d_i}}\Phi_i\left(\frac{|f(x)|}{\lambda}\right)dx.$$

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

Then the tensor product of both maximal operators $M_1 \otimes M_2$ is of weak type (Ψ, Ψ) over $\mathbf{R}^{d_1+d_2}$, where Ψ is given by a combination of Φ_1 and Φ_2.

When specialized to the commutative case, our proof above gives what is perhaps the simplest version of de Guzmán's and Cordoba/Fefferman inequality CFG in the martingale case. Our proof also highlights the different behavior of the Cuculescu projections compared with the level sets of the classical maximal. Indeed, in the classical case, we have that

$$\Pi \;=\; \sum_{j\geq 1} e^j \pi_j^{[1]} \;=\; \sum_{j\geq 1} e^j \, \mathbf{1}_{\{e^j < M_1(f) \leq e^{j+1}\}}$$

is comparable to the maximal function $(M_1 \otimes \mathrm{id})(f)$, while (see Remark 5.3) the same cannot hold in the noncommutative case. Nevertheless, the modified Π of (5.4) with the extra term $j^{1+\varepsilon}$ can be used to control the behavior of the maximal.

R.2 The proof above can be modified by introducing other terms of summable inverse in the definition of Π, thus giving inequalities like (ii) for more complicated Orlicz spaces like

$$L \log^2 L \log\log^{1+\varepsilon} L(\mathcal{M}).$$

We have chosen not to perform our calculation for that particular space because our inequality recovers what we believe is the optimal case (see Conjecture B1) when $\varepsilon \to 0^+$, while the space above would not.

R.3 A similar result to Theorem B2 has been obtained in the literature (see [19, Theorem 3.5]). Nevertheless, there are key differences between the two approaches. Although the results [19] were formulated in the context of multiparametric ergodic means, they can be easily transferred to the context of martingales. Indeed, following their scheme of proof, we can use an atomic decomposition of the $L \log^\alpha L$ spaces, due originally to Tao [46] in the commutative setting, to obtain that, for every $s \geq 0$,

$$\big\| (\mathbf{E}_n)_n : L \log^{2+s} L(\mathcal{M}) \to L \log^s(\mathcal{M}) \big\| < \infty.$$

Then, starting with an element in $L \log^2 L(\mathcal{M})$, we can bound $\mathbf{E}_n^{[1]} \otimes \mathbf{E}_m^{[2]}$ by iterating the technique above, applying it first to $(\mathrm{id} \otimes \mathbf{E}_m^{[2]})(f)$ for positive f to obtain an element

$$(\mathrm{id} \otimes \mathbf{E}_m^{[2]})(f) \leq g \quad \text{with} \quad \|g\|_1 \leq \|f\|_{L \log^2 L}.$$

Since g is in L_1, we can apply the maximal weak type $(1,1)$ for the family $(\mathbf{E}_n^{[1]} \otimes \mathrm{id})_n$. As a result, we obtain that for every $f \in L \log^2 L(\mathcal{M})_+$ and $\lambda > 0$, there is a projection $q(\lambda) \in \mathscr{P}(\mathcal{M})$ satisfying that

- $q(\lambda)\, (\mathbf{E}_n^{[1]} \otimes \mathbf{E}_m^{[2]})(f)\, q(\lambda) \;\leq\; \lambda q(\lambda).$
- $\tau\big(q(\lambda)^\perp\big) \;\leq\; \dfrac{\|f\|_{L \log^2 L}}{\lambda}.$

This result generalizes to iterated products by starting with an element in $L \log^{2(d-1)} L$. This result can be understood as a bound

$$(\mathbf{E}_n^{[1]} \otimes \mathbf{E}_m^{[2]})_{n,m} : L \log^2 L(\mathcal{M})_+ \to L_{1,\infty}(\mathcal{M}; \ell_\infty)_+.$$

Nevertheless, it is important to notice that neither of the results is stronger than the other. In one direction, we cannot deduce the bound above without an extra ε in the exponent. In the other, our result gives projections with a decay in λ that is strictly faster that $O(\lambda^{-1})$. Since the proof of [19, Theorem 3.5] requires going through $L_1[\ell_\infty]$, by Remark 5.3 one must lose some factor with respect to the optimal size of the projections $q(\lambda)$. We also recall that having noncommutative

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

maximal weak type (Φ, Φ) estimates, with $\Phi(r) = r(1 + \log_+^{2+\varepsilon}(r))$, may be important for the purpose of extending real interpolation results between the weak Orlicz types.

Acknowledgements. The authors are thankful to Simeng Wang for pointing out the existence of [19] and for providing a reference for the asymmetric convergence results in [31]. Those fruitful discussions took place during ICMAT's School I of the Thematic Research Program: Operator Algebras, Groups and Applications to Quantum Information in March 2019. The authors are also indebted to Guixiang Hong for the comments provided at Harbin Institute of Technology, China. Adrián M. González-Pérez was supported by the European Research Council. Consolidator Grant 614195 - RIGIDITY. Javier Parcet was partially supported by CSIC Grant PIE-201650E030 and ICMAT Severo Ochoa Grant SEV-2015-0554.

Conflict of Interest: None.

References

[1] T. Bekjan, Z. Chen and A. Osekowski, 'Noncommutative maximal inequalities associated with convex functions', *Trans. Amer. Math. Soc.* **369**(1) (2017), 409–427.
[2] C. Bennett and R.C. Sharpley, *Interpolation of Operators* (Academic Press, 1988).
[3] D.P. Blecher and C.L. Merdy, *Operator Algebras and Their Modules: An Operator Space Approach*, London Mathematical Society Monographs (Clarendon, 2004).
[4] M. Choda, 'Reduced free products of completely positive maps and entropy for free product of automorphisms', *Publ. Res. Inst. Math. Sci.* **32**(2) (1996), 371–382.
[5] A. Cordoba and R. Fefferman, 'A geometric proof of the strong maximal theorem', *Ann. of Math. (2)* **102**(1) (1975), 95–100.
[6] I. Cuculescu, 'Martingales on von Neumann algebras', *J. Multivariate Analysis* **1**(1), 1971, 17–27.
[7] A. Defant, *Classical Summation in Commutative and Noncommutative* L_p*-Spaces*, vol. **2021** of *Lecture Notes in Mathematics* (Springer Berlin Heidelberg, Berlin, Heidelberg, 2011).
[8] S. Dirksen, 'Weak-type interpolation for noncommutative maximal operators', *J. Operat. Theor.* **73**(2) (2015), 515–532.
[9] J. L. Doob, *Stochastic Processes* vol. **101** (New York , Wiley, 1953).
[10] N. Dunford and J. T. Schwartz, 'Convergence almost everywhere of operator averages', *J. Ration. Mech. Anal* **5**(1) (1956), 129–178.
[11] E.G. Effros and Z.-J. Ruan, *Operator Spaces, vol.* **23** *of London Mathematical Society Monographs New Series* (The Clarendon Press, Oxford University Press, New York, 2000).
[12] G.B. Folland, *A Course in Abstract Harmonic Analysis, Studies in Advanced Mathematics* (CRC Press, Boca Raton, FL, 1995).
[13] G.B. Folland, *Real analysis, Modern Techniques and Their applications, Pure and Applied Mathematics*, second edn. (John Wiley & Sons, Inc., New York, 1999).
[14] M. de Guzmán, 'An inequality for the Hardy-Littlewood maximal operator with respect to the product of differentiation bases', *Studia Math* **49** (1972), 265–286.
[15] U. Haagerup, 'An example of a non nuclear C^*-algebra, which has the metric approximation property', *Invent. Math.* **50**(3) (1978), 279–293.
[16] U. Haagerup, 'Operator-valued weights in von Neumann algebras. I', *J. Funct. Anal.* **32**(2) (1979), 175–206.
[17] G.H. Hardy and J.E. Littlewood, 'A maximal theorem with function-theoretic applications', *Acta Math.* **54** (1930), 81–116.
[18] G. Hong, M. Junge and J. Parcet, 'Algebraic Davis decomposition and asymmetric Doob inequalities', *Commun. Math. Phys* **346**(3), (September 2016), 995–1019.
[19] G. Hong and M. Sun, 'Noncommutative multi-parameter Wiener-Wintner type ergodic theorem', *J. Funct. Anal.* **275**(5) (2018), 1100–1137.
[20] Y. Hu, 'Noncommutative extrapolation theorems and applications', *Ill. J. Math.* **53**(2) (2009), 463–482.
[21] R. Jajte, *Strong Limit Theorems in Non-Commutative Probability* (Lecture Notes in Mathematics (Springer-Verlag, Berlin Heidelberg, 1985).
[22] R. Jajte, *Strong Limit Theorems in Noncommutative* L^2*-Spaces* (Lecture Notes in Mathematics (Springer-Verlag, Berlin Heidelberg, 1991).
[23] B. Jessen, J. Marcinkiewicz and A. Zygmund, 'Note on the differentiability of multiple integrals', *Fundam. Math.* **25**(1) (1935), 217–234.
[24] M. Junge, 'Doob's inequality for non-commutative martingales', *J. Reine Angew. Math.* **549** (2002), 149–190.
[25] M. Junge, 'Fubini's theorem for ultraproducts of noncommutative L_p-spaces', *Can. J. Math.* **56**(5) (2004), 983–1021.
[26] M. Junge, T. Mei and J. Parcet, 'Noncommutative Riesz transforms—dimension free bounds and Fourier multipliers', *J. Eur. Math. Soc.* **20**(3) (2018).
[27] M. Junge and J. Parcet, 'Mixed-norm inequalities and operator space L_p-embedding theory', *Mem. Amer. Math. Soc.* 203(**953**) (2010.
[28] M. Junge and Q. Xu, 'Noncommutative maximal ergodic theorems', *J. Am. Math. Soc.* **20**(2) (2007), 385–439.
[29] W. Kunze, 'Noncommutative Orlicz Spaces and Generalized Arens Algebras', *Math. Nachr.* **147**(1) (1990), 123–138.

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press

[30] E.C. Lance, 'Ergodic theorems for convex sets and operator algebras', *Invent. Math.* **37**(3) (1976), 201–214.
[31] S. Litvinov, 'Uniform equicontinuity of sequences of measurable operators and non-commutative ergodic theorems', *Proc. Amer. Math. Soc.* **140**(7) (2012), 2401–2409.
[32] T. Mei and É. Ricard, 'Free Hilbert transforms', *Duke Math. J.* **166**(11) (2017), 2153–2182.
[33] T. Mei and Q. Xu, 'Free Fourier multipliers associated with the first segment' (September 2019), arXiv 1909.06879.
[34] D. Ornstein, 'On the pointwise behavior of iterates of a self-adjoint operator', *J. Appl. Math. Mech.* **18**(5), 1968, 473–477.
[35] G. K. Pedersen, *C^*-algebras and their automorphism groups*, vol. **14** of *London Mathematical Society Monographs* (Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979).
[36] G. Pisier, 'Non-commutative vector valued L_p-spaces and completely p-summing maps', *Astérisque* (**247**) (1998).
[37] G. Pisier, *Introduction to Operator Space Theory,* vol. **294** of *London Mathematical Society Lecture Note Series* (Cambridge University Press, Cambridge, 2003).
[38] G. Pisier and Q. Xu, 'Non-commutative L^p-spaces', in: *Handbook of the Geometry of Banach Spaces* vol. **2** (North-Holland, Amsterdam, 2003), 1459–1517.
[39] G. Pisier, *Martingales in Banach Spaces* vol. **155** (Cambridge University Press, 2016).
[40] G. Pisier and Q. Xu, 'Non-commutative martingale inequalities', *Comm. Math. Phys.* **189**(3) (1997), 667–698.
[41] R.A. Ryan, *Introduction to Tensor Products of Banach Spaces, Springer Monographs in Mathematics* (Springer London, 2013).
[42] E.M. Stein, 'A note on the class $LlogL$', *Studia Math.* **3**(32) (1969), 305–310.
[43] E.M. Stein, *Topics in Harmonic Analysis Related to the Littlewood-Paley theory, Annals of Mathematics Studies* no. **63** (Princeton University Press, Princeton, NJ, 1970).
[44] M. Takesaki, *Theory of Operator Algebras I,* vol. **124** of *Encyclopaedia of Mathematical Sciences* (Springer-Verlag, Berlin, 2002).
[45] M. Takesaki, *Theory of Operator Algebras II,* vol. **125** of *Encyclopaedia of Mathematical Sciences* (Springer-Verlag, Berlin, 2003).
[46] T. Tao, 'A converse extrapolation theorem for translation-invariant operators', *J. Funct. Anal.* **180**(1) (2001), 1–10.
[47] T. Tao, 'Failure of the L^1pointwise and maximal ergodic theorems for the free group', in: *Forum Math. Sigma* vol. **3** (Cambridge University Press, 2015).
[48] M. Terp, 'L^p spaces associated with von Neumann algebras', *Notes, Math. Institute, Copenhagen Univ* **3** (1981).
[49] D. V. Voiculescu, K. J. Dykema and A. Nica, *Free Random Variables* (American Mathematical Society, 1992).
[50] S. Yano, 'Notes on Fourier analysis (XXIX): An extrapolation theorem', *J. Math. Soc. Jpn.* **3**(2) (1951), 296–305.
[51] F. J. Yeadon, 'Ergodic theorems for semifinite von Neumann algebras I', *J. London Math. Soc.* **16**(2), 1977, 326–332.
[52] F. J. Yeadon, 'Ergodic theorems for semifinite von Neumann algebras: II', *Math. Proc. Camb. Philos. Soc.* **88** (1980), 135–147.
[53] K. Yosida and S. Kakutani, 'Birkhoff's Ergodic Theorem and the Maximal Ergodic Theorem', *Proc. Imp. Academy* **15**(6) (1939), 165–168.
[54] K. Yosida and S. Kakutani, 'Operator-Theoretical Treatment of Markoff's Process and Mean Ergodic Theorem', *Ann. Math.* **42**(1)(1941), 188–228.

https://doi.org/10.1017/fms.2020.37 Published online by Cambridge University Press