# ON FAMILIES OF CAUCHY TRANSFORMS AND BMOA

## YUSUF ABU MUHANNA and EL-BACHIR YALLAOUI

Communicated by P. C. Fenton

### Abstract

In this paper we prove a number of results on Cauchy transforms of generalized type given by Borel measures supported on the class of analytic functions mapping the unit disc into the unit disk. We also give a BMOA characterization using these families.

## 1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, $\Gamma = \{x : |x| = 1\}$ and let $\mathcal{M}$ denote the sets of complex-valued Borel measures on $\Gamma$.

For $z \in \Delta$ and $\alpha \geq 0$, let $F_\alpha$ denote the family of functions $f$ for which there exists a measure $\mu \in \mathcal{M}$ such that

$$(1.1) \qquad f(z) = \begin{cases} \displaystyle\int_\Gamma \frac{1}{(1 - \bar{x}z)^\alpha} \, d\mu(x) & \text{for} \quad \alpha > 0; \\[2ex] \displaystyle\int_\Gamma \log \frac{1}{1 - \bar{x}z} \, d\mu(x) + f(0) & \text{for} \quad \alpha = 0. \end{cases}$$

These families are known as families of Cauchy transforms and have been well studied in [4, 3, 8, 10].

The class $F_\alpha$ is a Banach space with respect to the norm

$$(1.2) \qquad \|f\|_{F_\alpha} = \begin{cases} \inf \|\mu\| & \text{for} \quad \alpha > 0; \\[1ex] \inf \|\mu\| + |f(0)| & \text{for} \quad \alpha = 0, \end{cases}$$

where $\mu$ varies over all measures in $\mathcal{M}$ for which (1.1) holds and where $\|\mu\|$ denotes the total variation norm of $\mu$.

It is known from [5, 6] that

- $F_\alpha \subset F_\beta$, whenever $0 \leq \alpha < \beta$.
- $F_\alpha$ is Möbius invariant for $\alpha \geq 0$.
- $f \in F_\alpha$ if and only if $f' \in F_{1+\alpha}$.
- $\|f'\|_{F_{1+\alpha}} \leq k\|f\|_{F_\alpha}$ for some $k > 0$.

We now define generalized families of Cauchy transforms by extending $\Gamma$ and $\mathcal{M}$. We extend $\Gamma$ to $B$, the set of analytic selfmaps $\phi$ mapping the open unit disc $\Delta$ into itself and such that $\phi(0) = 0$. We also extend $\mathcal{M}$ to $\mathcal{N}$, the sets of complex-valued Borel measures on $B$. $B$ is equipped with the topology of uniform convergence on compact subsets. Here $\mathcal{M}$ is equivalent to the subset of $\mathcal{N}$ consisting of all those measures supported on the set $\{xz : |x| = 1\}$.

For $z \in \Delta$ and $\alpha \geq 0$, let $A_\alpha$ denote the family of functions $f$ for which there exists a measure $\mu \in \mathcal{N}$ such that

$$(1.3) \qquad f(z) = \begin{cases} \displaystyle\int_B \frac{1}{(1 - \phi(z))^\alpha}\, d\mu(\phi) & \text{for } \alpha > 0; \\ \displaystyle\int_B \log \frac{1}{1 - \phi(z)}\, d\mu(\phi) + f(0) & \text{for } \alpha = 0. \end{cases}$$

The class $A_\alpha$ is a Banach space with respect to the norm

$$(1.4) \qquad \|f\|_{A_\alpha} = \begin{cases} \inf \|\mu\| & \text{for } \alpha > 0; \\ \inf \|\mu\| + |f(0)| & \text{for } \alpha = 0, \end{cases}$$

where $\mu$ varies over all measures in $\mathcal{N}$.

Clearly the classes $F_\alpha$ are subsets of $A_\alpha$ when the measures $\mu$ in (1.3) are in $\mathcal{M}$. Furthermore, for $f \in F_\alpha$, $\|f\|_{F_\alpha} \geq \|f\|_{A_\alpha}$. It is also known from [1] that for $\alpha \geq 1$, $F_\alpha = A_\alpha$.

We show in this paper that for $0 \leq \alpha < \beta$,

$$(1.5) \qquad A_\alpha \subset A_\beta \quad \text{and} \quad \|f\|_{A_\beta} \leq \|f\|_{A_\alpha}.$$

This generalizes similar results for $F_\alpha$ (see [7, 9]).

We also show that $A_0 = \text{BMOA}$ and that the norm $\|\cdot\|_{A_0}$ is equivalent to well known BMO norms. Furthermore, we show that, for all $n \geq 1$ and $\alpha \geq 0$

$$\|z^n\|_{A_\alpha} \leq k$$

where the constant $k$ is independent of $n$ and $\alpha$.

## 2. The classes $A_\alpha$

In this section we establish for $0 \le \alpha < \beta$ the relationship between $A_\alpha$ and $A_\beta$ as well as their respective norms.

THEOREM 2.1. *If* $0 \le \alpha < \beta$, *then* $A_\alpha \subset A_\beta$ *and* $\|f\|_{A_\beta} \le \|f\|_{A_\alpha}$.

PROOF. Note that since $A_\alpha = F_\alpha$ for $\alpha \ge 1$, [1], and for $0 < \alpha < \beta$, $F_\alpha \subset F_\beta$ and $\|f\|_{F_\beta} \le \|f\|_{F_\alpha}$ [7], then all we have to prove is the case $0 \le \alpha < \beta < 1$.

(i) Let $f \in A_\alpha$, where $0 < \alpha < \beta$, then we can write

$$(2.1) \qquad f(z) = \int_B \frac{1}{(1 - \psi(z))^\alpha} \, d\mu(\psi).$$

Since $1/(1 - z)^\alpha \in F_\alpha \subset F_\beta$, we can write

$$(2.2) \qquad \frac{1}{(1 - z)^\alpha} = \int_\Gamma \frac{1}{(1 - \bar{x}z)^\beta} \, d\nu(x)$$

and

$$(2.3) \qquad \left\| \frac{1}{(1 - z)^\alpha} \right\|_{F_\beta} \le \left\| \frac{1}{(1 - z)^\alpha} \right\|_{F_\alpha} = 1.$$

Now by replacing $z$ in (2.2) by $\psi(z)$ and putting the result in (2.1) we get

$$(2.4) \qquad f(z) = \int_B \int_\Gamma \frac{1}{(1 - \bar{x}\psi(z))^\beta} \, d\nu(x) \, d\mu(\psi).$$

Suppose without loss of generality that $\nu$ is a positive measure and let

$$g_n(z) = \sum_{k=1}^n \frac{\nu_k}{(1 - \bar{x}_k z)^\beta}.$$

Then by (2.1)

$$\int_B g_n(\psi) \, d\mu(\psi)$$

converges locally uniformly to

$$f(z) = \int_B \int_\Gamma \frac{1}{(1 - \bar{x}\psi(z))^\beta} \, d\nu(x) \, d\mu(\psi).$$

Let $\eta_n(\psi) = \sum_{k=1}^n \nu_k \mu(\psi)$, then

$$\int_B g_n(\psi) \, d\mu(\psi) = \int_B \frac{1}{(1 - \psi)^\beta} \, d\eta_n(\psi),$$

where $\|\eta_n\| \le \|\nu\|\|\mu\|$ for all $n$. Hence by compactness, there exists a measure $\sigma$ such that

$$(2.5) \qquad f(z) = \int_B \frac{1}{(1 - \psi(z))^\beta} \, d\sigma(\psi),$$

which shows that $f \in A_\beta$. Furthermore, $\|\sigma\| \le \|\nu\|\|\mu\|$. Consequently,

$$\|f\|_{A_\beta} \le \|\sigma\| \le \|\mu\|\|\nu\| \quad \text{for all } \mu.$$

However, since $\mu$ and $\nu$ are arbitrary measures that give (2.1) and (2.2), then

$$\|f\|_{A_\beta} \le \inf\{\|\mu\|\}\inf\{\|\nu\|\}.$$

Hence by (2.3)

$$\|f\|_{A_\beta} \le \|f\|_{A_\alpha}.$$

(ii) Now let $f \in A_0$. We want to show that $f \in A_\alpha$ for any $\alpha > 0$. By the definition,

$$(2.6) \qquad f(z) = \int_B \log \frac{1}{1 - \phi(z)} \, d\mu(\phi) + f(0).$$

Since $\log(1 - z)^{-1} \in F_0 \subset F_\alpha$ see [5], then

$$\log \frac{1}{1 - z} = \int_\Gamma \frac{1}{(1 - \overline{x}z)^\alpha} \, d\nu(x),$$

where $\nu$ depends only on $\alpha$. Hence (2.6) becomes

$$(2.7) \qquad f(z) = \int_B \int_\Gamma \frac{1}{(1 - \overline{x}\phi(z))^\alpha} \, d\nu(x) \, d\mu(\phi) + f(0),$$

where the integral in (2.7) looks exactly like the one in (2.3) with $\alpha$ replacing $\beta$. Hence using an argument similar to the one in (i) we get that

$$(2.8) \qquad f(z) = \int_B \int_\Gamma \frac{1}{(1 - \psi(z))^\alpha} \, d\sigma(\psi) + f(0)$$

which shows that $f \in A_\alpha$. Furthermore, $\|\sigma\| \le \|\nu\|\|\mu\|$ and hence

$$(2.9) \qquad \|f\|_{A_\alpha} \le \inf\{\|\mu\|\} + |f(0)| = \|f\|_{A_0}. \qquad \square$$

## 3. Characterization of $A_0$

It is known from [2, page 248] that $\phi \in$ BMO if and only if there exist functions $\phi_1$ and $\phi_2$ in $L^\infty$ such that

$$\phi = \phi_1 + \tilde{\phi}_2 + \alpha,$$

where both $\|\phi_1\|_\infty$ and $\|\phi_2\|_\infty$ are less than $C\|\phi\|_*$, $C$ is a constant and $\| \ \|_*$ is the classical BMO norm.

Consequently $f \in$ BMOA if and only if there are analytic functions $f_1$ and $f_2$ such that

(3.1)                           $$f = f_1 + f_2 + \alpha$$

where $\|\operatorname{Re} f_1\|_\infty \leq C$ and $\|\operatorname{Im} f_2\|_\infty \leq C$. If we define on BMOA the norm

(3.2)            $$\|f\|^* = \inf\{\|\operatorname{Re} f_1\|_\infty + \|\operatorname{Im} f_2\|_\infty : f = f_1 + f_2 + \alpha\},$$

then by [2], the norms $\|f\|^*$ and $\|f\|_*$ are equivalent.

Now we have the following proposition which establishes a set equality between $A_0$ and BMOA and thus gives a BMOA characterization using generalized families of Cauchy transforms.

THEOREM 3.1. $A_0 =$ BMOA.

PROOF. Suppose that $f \in A_0$, then according to (1.3) there exists a measure $\mu \in \mathcal{N}$ such that

(3.3)                 $$f(z) = \int_B \log \frac{1}{1 - \phi(z)} \, d\mu(\phi) + f(0).$$

Assume without loss of generality that $\mu$ is a probability measure. Then $f$ is subordinate to $\log(1 - z)^{-1} + f(0)$ and consequently by (3.1) $f \in$ BMOA. The proof of the other inclusion follows from (3.1) and subordination.                                □

THEOREM 3.2. The norms $\| \cdot \|_*$ and $\| \cdot \|_{A_0}$ are equivalent, that is, there exist positive constants $c_1$ and $c_2$ such that

(3.4)                           $$c_1\|f\|_* \leq \|f\|_{A_0} \leq c_2\|f\|_*.$$

PROOF. Suppose $f \in$ BMOA, then $f$ can be decomposed as in (3.1). Let $d_1$ denote $\|\operatorname{Re} f_1\|_\infty$ and $d_2$ denote $\|\operatorname{Im} f_2\|_\infty$. Then

(3.5)                           $$\left| \frac{\pi}{2d_1} \operatorname{Im} i f_1(z) \right| \leq \frac{\pi}{2}$$

and

(3.6)
$$\left| \frac{\pi}{2d_2} \operatorname{Im} f_2(z) \right| \le \frac{\pi}{2}$$

for all $z \in \Delta$. Consequently, by subordination

(3.7)
$$if_1(z) = \frac{2d_1}{\pi} \left( \log \frac{1}{1 - \phi(z)} - \log \frac{1}{1 + \phi(z)} \right) + if_1(0)$$

(3.8)
$$f_2(z) = \frac{2d_2}{\pi} \left( \log \frac{1}{1 - \psi(z)} - \log \frac{1}{1 + \psi(z)} \right) + f_2(0)$$

for all $z \in \Delta$ and $\phi, \psi \in B$. Therefore,

(3.9)
$$\| f \|_{A_0} \le \frac{4}{\pi} (d_1 + d_2),$$

and hence

(3.10)
$$\| f \|_{A_0} \le \frac{4}{\pi} \| f \|^* \le c_2 \| f \|_*$$

which gives the right inequality in (3.4).

Next, we show that the inequality (3.14) holds. Let us write $f$ as in (1.2) and assume without loss of generality that $\mu$ is a positive measure. Then

(3.11)
$$| \operatorname{Im} f(z) | \le c \| \mu \|,$$

where $c > 1$. Thus

(3.12)
$$| \operatorname{Im} f(z) | \le c \| f \|_{A_0}$$

and since

(3.13)
$$\| \operatorname{Im} f \|_\infty \ge \| \operatorname{Im} f \|_*$$

we have

(3.14)
$$\| f \|_* \le k_1 \| \operatorname{Im} f \|_* \le k_2 \| f \|_{A_0} = \frac{1}{c_1} \| f \|_{A_0},$$

where $c_1 = 1/k_2$ and the left inequality in (3.14) follows from [2, page 235]. □

THEOREM 3.3. $\| z^n \|_{A_\alpha} \le k$ for $n \ge 1$.

PROOF. It is enough to show that $\| z^n \|_{A_0} \le k$ for $n \ge 1$. Since we have shown that $\| \cdot \|_*$ and $\| \cdot \|_{A_0}$ are equivalent, let us approximate $\| z^n \|_*$. It is known (see [2, page 240]) that

(3.15)
$$\| g \|_*^2 \approx \sup_\psi \iint_\Delta | \nabla g |^2 (1 - |z|^2) | \psi'(z) | \, dA,$$

where $\psi(z) = (z + z_0)/(1 + \overline{z_0}z)$ is a Möbius transformation. Replace $g$ in (3.15) by $z^n$ and $|z|$ by $r$ to get

$$
\begin{aligned}
I &= \iint_\Delta |\nabla(z^n)|^2 (1 - r^2)|\psi'(z)|\, dA \\
&\leq \iint_\Delta n^2 r^{2n-2}(1 - r^2)|\psi'(z)|\, dA \\
&\leq \int_0^{2\pi} \int_0^1 n^2 r^{2n-2}(1 - r^2)|\psi'(z)|\, dr\, d\theta \\
&\leq 2\pi n^2 \int_0^1 r^{2n-2}(1 - r)\, dr, \qquad \text{because } \int_0^{2\pi} |\psi'(z)|\, d\theta \leq 2\pi
\end{aligned}
$$

$$
(3.16) \qquad \leq 2\pi n^2 \int_0^1 (r^{2n-2} - r^{2n})\, dr = \frac{4\pi n^2}{(4n^2 - 1)} \leq \frac{4\pi}{3}
$$

which gives us the desired result and completes the proof.                          $\square$

The following corollary is a direct consequence of the previous theorem.

COROLLARY 3.4. *If $f(z) = \sum_{n=0}^\infty a_n z^n$ is analytic and if $\sum_{n=1}^\infty |a_n| < \infty$, then $f \in A_\alpha$ for all $\alpha \geq 0$.*

PROOF. It is sufficient to prove that $f \in A_0$ since $A_0 \subset A_\alpha$. To show that $f \in A_0$ all we have to show is that the norm $\|f(z)\|_{A_0}$ is bounded.

$$
\|f(z)\|_{A_0} = \left\| \sum_{n=0}^\infty a_n z^n \right\|_{A_0} \leq \sum_{n=0}^\infty |a_n| \, \|z^n\|_{A_0} \leq \frac{4\pi}{3} \sum_{n=0}^\infty |a_n| < \infty.
$$

$\square$

## Acknowledgement

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Department of Mathematics
American University of Sharjha
PO Box 2666
Sharjha
United Arab Emirates
e-mail: ymuhanna@aus.ac.ae

Department of Mathematics and Statistics
Sultan Qaboos University
PO Box 36
Al-Khod P.C. 123
Muscat
Sultanate of Oman
e-mail: bachiry@squ.edu.om