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INVARIANT COMPLEMENTS TO CLOSED INVARIANT SUBSPACES

M. P. THOMAS

Introduction. The question under what conditions a closed invariant subspace possesses a closed invariant complement is of major importance in operator theory. In general it remains unanswered. In this paper we drop the requirement that the invariant complement be closed. We show in section 1 that the question is answerable under fairly mild conditions for a quasinilpotent operator (Theorem 1.5). These conditions will cover the case of a quasinilpotent operator with dense range and no point spectrum. In section 2 we discuss the consequences for the Volterra operator V. Since V is unicellular, its proper closed invariant subspaces do not possess closed invariant complements. However, they are all algebraically complemented (Proposition 2.1).

Section 1. Let X be a Banach space and let $B(X)$ denote the algebra of all bounded linear operators on X. If $T \in B(X)$ is fixed, X becomes a module over $\mathbf{C}[x]$, the polynomials with complex coefficients as follows:

$$py \equiv p(T)y, \text{ for } y \in X, p \in \mathbf{C}[x].$$

Definition 1.1. Let $D(T)$ denote the largest algebraic subspace D of X such that $(T - \lambda)D = D$, all $\lambda \in \mathbf{C}$, [**4**, theorem 3]. We call $D(T)$ the *largest T divisible subspace of X.*

If $p \in \mathbf{C}[x]$ and $p \neq 0$, then $p(z) = (z - \lambda_n) \ldots (z - \lambda_1)$ for some $\lambda_1, \lambda_2, \ldots, \lambda_n$ in $\mathbf{C}$, the complex numbers. Hence $pD(T) = D(T)$, and $D(T)$ is also the largest divisible submodule of X. Recall that a divisible module is injective (ref. [**4**, theorem 2]) and, as a consequence, always a direct summand as follows: Let P_1 be the identity map from $D(T)$ to $D(T)$ and extend it to P, a module homomorphism from X to $D(T)$. In terms of diagrams:

$$\begin{tikzcd} 0 \arrow[r] & D(T) \arrow[r] \arrow[d, "P_1"'] & X \arrow[dl, dashed, "P"] \\ & D(T) & \end{tikzcd}$$

Then $PT = TP$ and $P^2 = P$. Letting $R(T) \equiv \{y \in X : Py = 0\}$ one sees that $X \cong D(T) \oplus R(T)$ as $\mathbf{C}[x]$ modules. In general P is not continuous. Also $R(T)$ is not unique but is always isomorphic to the quotient module $X/D(T)$

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and we may write

$$X \cong D(T) \oplus X/D(T),$$

as $\mathbf{C}[x]$ modules. A submodule is called *reduced* if it has no divisible submodules other than (0). Hence $R(T)$ is reduced. Let T be *quasinilpotent*: i.e. $\sigma(T) = \{0\}$. It is clear that

$$D(T) \subseteq \bigcap_{n=1}^{\infty} T^n X.$$

If $\sigma_p(T) = \emptyset$ (i.e. T is one to one) it is easily seen that

$$D(T) = \bigcap_{n=1}^{\infty} T^n X.$$

We will assume the following:

i) $D(T) = \bigcap_{n=1}^{\infty} T^n X.$

ii) $\overline{TX} = X.$

We define a metric topology on $X/D(T)$. Let

$$\{T^k X + D(T) : k = 1, 2, 3 \ldots\}$$

be a local base at $0 + D(T)$. Let a dot over a vector in X indicate the coset in $X/D(T)$. This is a metric topology if one defines $d(\dot{x}, \dot{y}) \equiv 2^{-n}$ provided

$$(x - y) \in T^n X \sim T^{n+1} X.$$

It is easily seen that $d(\dot{x} + \dot{y}, 0) \leqq \max\{d(\dot{x}, 0), d(\dot{y}, 0)\}$, so d is non-archemedian. We will refer to this topology as the *natural p-adic topology on* $X/D(T)$. Of course this metric will never produce a topological vector space since $(\lambda, y) \rightarrow \lambda y$ is not continuous. We have the following rather surprising result which is similar in technique to a lemma by G. Allan [**1**, lemma 2].

PROPOSITION 1.2. *Let X be a Banach space and T a quasinilpotent operator on X. Suppose*

i) $D(T) = \bigcap_{n=1}^{\infty} T^n X.$

ii) $\overline{TX} = X.$

Then the natural p-adic topology on $X/D(T)$ is complete.

Proof. We first note that it suffices to show that

$$\sum_{n=0}^{\infty} T^n \dot{x}_n$$

converges for any choice of $\dot{x}_n$ in $X/D(T)$. Define affine maps φ_n from X to X by

$$\varphi_n(y) \equiv Ty + x_{n-1},\ n = 1, 2, 3 \ldots.$$

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The φ_n are continuous, and $\overline{\varphi_n X} = X$ for all n. Thus the "Mittag-Leffler" theorem [**2**, p. 212] implies that the following projective limit is dense in each factor, and, in particular, non-void:

$$X \xleftarrow{\varphi_1} X \xleftarrow{\varphi_2} X \xleftarrow{\varphi_3} \cdots X \xleftarrow{\varphi_n} \cdots.$$

Hence there exists $x \in X$ and $\{y_i\}_{i=1}^{\infty} \subseteq X$ such that

$$x = \varphi_1(y_1) \quad \text{and} \quad y_n = \varphi_{n+1}(y_{n+1}), n = 1, 2, 3 \ldots.$$

We have:

$$x = Ty_1 + x_0$$
$$y_1 = Ty_2 + x_1 \quad \text{so} \quad x = T^2y_2 + Tx_1 + x_0$$
$$y_2 = Ty_3 + x_2 \quad \text{so} \quad x = T^3y_3 + T^2x_2 + Tx_1 + x_0.$$

In general it follows by induction that

$$(*) \quad x - \sum_{k=0}^{n} T^k x_k \in T^{n+1}X, \quad \text{all } n,$$

and thus

$$\dot{x} = \sum_{k=0}^{\infty} T^k \dot{x}_k.$$

We remark that the "Mittag-Leffler" theorem implies that there is a dense set of such x satisfying (*). Hence, only the coset $\dot{x}$ is unique and $D(T)$ is necessarily dense in X.

Algebraically the structure of objects such as $X/D(T)$ has been worked out (ref. [**4**, pps. 42–52]). First, $X/D(T)$ is primary for the prime $z \in \mathbf{C}[x]$: i.e. $p \in \mathbf{C}[x]$ and $(p, z) = 1$ implies that p maps $X/D(T)$ bijectively onto itself. Hence it becomes a module over the discrete valuation ring

$$\mathbf{C}[x]_z \equiv \{p/q \in \mathbf{C}(x) : (q, z) = 1\}.$$

This ring has a p-adic topology itself with local base at 0 consisting of the sets $\{z^n(\mathbf{C}[x]_z) : n = 1, 2, 3 \cdots\}$. Its completion is usually denoted $(\mathbf{C}[x]_z)^*$ and is naturally isomorphic to $\mathbf{C}[[x]]$, the formal power series with complex coefficients. Note $\mathbf{C}[x]_z$ is embeddable in $\mathbf{C}[[x]]$ since if $(q, z) = 1$, $1/q$ can be written as a formal power series. By $(\mathbf{C}[x]_z)^*$ being naturally isomorphic to $\mathbf{C}[[x]]$ we mean by an isomorphism which preserves this embedding. In particular, a polynomial is taken to itself. Since $X/D(T)$ is complete it becomes a module over $\mathbf{C}[[x]]$: i.e.

$$\left(\sum_{n=0}^{\infty} \lambda_n z^n\right)\dot{y} \equiv \sum_{n=0}^{\infty} T^n(\lambda_n \dot{y}), \quad \text{for } \dot{y} \in X/D(T), \left(\sum_{n=0}^{\infty} \lambda_n z^n\right) \in \mathbf{C}[[x]].$$

Hence $X/D(T)$ is a complete module over the complete discrete valuation ring $\mathbf{C}[[x]]$. See [**4**, theorem 22] for a structure theorem.

Definition 1.3. Let K be a closed T invariant subspace of X. Hence K is a closed $\mathbf{C}[x]$ submodule of X. We say that K is *algebraically complemented* provided there is an algebraic subspace M, $TM \subseteq M$, and $X \cong K \oplus M$ as complex vector spaces. We will also say $X \cong K \oplus M$ as $\mathbf{C}[x]$ modules.

Alternatively, K is algebraically complemented if there is a linear map P onto K, $P^2 = P$ and $PT = TP$. Then $M = \{y \in X : Py = 0\}$ and is closed if and only if P is continuous. We note also that M is not unique in general.

Definition 1.4. Let $D(T|K)$ denote the largest algebraic subspace $D \subseteq K$ such that $(T - \lambda)D = D$, all $\lambda \in \mathbf{C}$.

We first consider necessary conditions for algebraic complementation. Let X be a Banach space and T a quasinilpotent operator on X. Suppose $\overline{TX} = X$. Let K be a closed T invariant subspace which is algebraically complemented by some subspace M: i.e. $X \cong K \oplus M$ as $\mathbf{C}[x]$ modules. Suppose $T^n y \in K$. Since $y = k + m$, for unique $k \in K$, $m \in M$, this means $T^n m = 0$. Hence $T^n y = T^n k$ and a necessary condition is

i) $T^n X \cap K = T^n K$, all n.

This is essentially a requirement that K be a pure submodule of X (i.e. $pX \cap K = pK$, all $p \in \mathbf{C}[x]$) since any polynomial acts like a power of T times a unit in $B(X)$. Let P be the projection of X onto K along M. Since $\overline{TX} = X$ and $\overline{PX} = PX = K$ it follows by a theorem of A. Sinclair that

$$\overline{T^N K} = \overline{T^{N+1} K}$$

for some N [**5**, theorem 3.5]. In this theorem P is regarded as a map from X to K and $R = T|K$, so $PT = RP$. We then have a second necessary condition

ii) $\overline{T^N K} = \overline{T^{N+1} K}$, for some N.

The surprising fact is that with some minor restrictions on T, these necessary conditions are also sufficient.

THEOREM 1.5. *Let X be a Banach space and T a quasinilpotent operator on X. Suppose that*

$$\overline{TX} = X \quad \textit{and} \quad D(T) = \bigcap_{n=1}^{\infty} T^n X.$$

Let K be a closed T invariant subspace of X with $D(T|K) = \bigcap_n T^n K$. Then K is algebraically complemented if and only if

i) $T^n X \cap K = T^n K, \quad n = 1, 2, 3 \ldots.$

ii) $\overline{T^N K} = \overline{T^{N+1} K}$ *for some N.*

Proof. We have already shown necessity. Hence, let i) and ii) hold. Let

$$C = \overline{T^N K}.$$

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Then $\overline{TC} = C$ by ii) and $\bigcap_n T^nC = \bigcap_n T^nK$ is T divisible. Thus $D(T|C) = D(T|K) = \bigcap_n T^nC$. Let D_1 denote $D(T|C)$. Then T is a quasinilpotent operator on the Banach space C, and C/D_1 is complete in the p-adic topology with local base $\{T^k(C/D_1)\}$ by Proposition 1.2. Let R_1 be a reduced submodule of K such that $K \cong D_1 \oplus R_1$. Let D_2 be a divisible submodule of X so that $D(T) \cong D_1 \oplus D_2$, [**4**, theorem 2]. We claim $D(T) \cap R_1 \equiv (0)$. Let $\mathbf{y} \in D(T) \cap R_1$. Then $\mathbf{y} \in R_1 \subseteq K$ and

$$\mathbf{y} \in (\bigcap_n T^nX) \cap K = \bigcap_n T^nK = D_1,$$

by i). Thus $\mathbf{y} \in D_1 \cap R_1 \equiv (0)$. Hence $D(T) \cap R_1 \equiv (0)$ and we also have the direct sum: $D_1 \oplus R_1 \oplus D_2$. Let Q be the projection of $D_1 \oplus R_1 \oplus D_2$ onto $D_1 \oplus D_2$ along R_1. Extend Q to all of X by injectivity, since $D_1 \oplus D_2$ is divisible. Let $R_2 = \{y \in X : Qy = 0\}$. It is clear that $R_1 \subseteq R_2$ and $X \cong D_1 \oplus D_2 + R_2$ as $\mathbf{C}[x]$ modules. To prove the theorem it suffices to show that R_1 is a $\mathbf{C}[x]$ direct summand of R_2. Now $R_2 \cong X/(D_1 \oplus D_2) \cong X/D(T)$ which is a complete module over $\mathbf{C}[[x]]$ by Proposition 1.2. Also $R_1 \cong K/D_1 \cong (K + D(T))/D(T)$ since $(D(T) \cap K) = (\bigcap_n T^nX) \cap K = \bigcap_n T^nK = D_1$ by i). So $K + D(T)/D(T)$ sits as a $\mathbf{C}[x]$ submodule of $X/D(T)$. We first show that $K + D(T)/D(T)$ is actually a $\mathbf{C}[[x]]$ submodule of $X/D(T)$. Let $f(x) = \sum_n^\infty c_n z^n \in \mathbf{C}[[x]]$ and let $k \in K$. Then

$$f(k + D(T)) = \sum_{n=0}^{N} c_n T^n(k + D(T)) + \sum_{p=1}^{\infty} c_{N+p} T^p(u + D(T)),$$

where $u = T^N k \in C$. Furthermore $C + D(T)/D(T)$ is isomorphic to C/D_1 since $C \cap D(T) = D_1$. Since C/D_1 is complete in the p-adic topology as noted above it follows that

$$\sum_{p=1}^{\infty} c_{N+p} T^p(u + D(T)) \in C + D(T).$$

Hence $f(k + D(T)) \in K + D(T)$. We next show that $K + D(T)/D(T)$ is pure in $X/D(T)$, i.e.:

$$f(X/D(T)) \cap (K + \mathrm{D}(T)/D(T)) = f(K + D(T)/D(T))$$

for all $f \in \mathbf{C}[[x]]$. Let $f \in \mathbf{C}[[x]]$ and $y \in X$ with $f(y + D(T)) = (k + D(T))$ for some $k \in K$. Note that $f = z^i g$ where g is a unit in $\mathbf{C}[[x]]$. Hence $z^i(y + D(T)) = (k_1 + D(T))$, where $k_1 \in K$ and $(k_1 + D(T)) = g^{-1}(k + D(T))$. We wish to show there is some $k_2 \in K$ such that $z^i(k_2 + D(T)) = (k_1 + D(T))$. This will show purity upon application of g to both sides. But there is $d \in D(T)$ such that $T^i y + d = k_1$ and $d = T^i e$ for some $e \in D(T)$. Thus $T^i(y + e) = k_1 \in K$. Using i), there is $k_2 \in K$ such that $T^i k_2 = k_1$, hence

$$z^i(k_2 + D(T)) = (k_1 + D(T)),$$

and purity of $K + D(T)/D(T)$ in $X/D(T)$ follows. Finally we claim that

$K + D(T)/D(T)$ is closed and complete in $X/D(T)$ for the natural p-adic topology. So suppose $k_n \in K$ and $y \in X$ such that

$$(y - k_n) + D(T) \in T^n(X/D(T)), n = 1, 2, 3 \ldots.$$

This implies there is $y_n \in X$ such that $k_{n+1} - k_n = T^n y_n$, all n. Again i) implies there is $z_n \in K$ such that $k_{n+1} - k_n = T^n z_n$, all n. Let $T^N z_n = u_n \in C$. Thus

$$k_{N+p+1} - k_{N+p} = T^p u_n, \quad \text{all } n.$$

So

$$\sum_{p=1}^{\infty} (k_{N+p+1} - k_{N+p}) + D(T) = u + D(T)$$

for some $u \in C$ since $C + D(T)/D(T)$ is complete by our previous remarks. Thus

$$y + D(T) = u + k_1 + \sum_{n=1}^{N} (k_{n+1} - k_n) + D(T),$$

and hence $y + D(T) \in K + D(T)/D(T)$. We have then shown that $K + D(T)/D(T)$ is a pure, complete $\mathbf{C}[[x]]$ submodule of $X/D(T)$. Invoking [**4**, theorem 23] we have that $K + D(T)/D(T)$ is a $\mathbf{C}[[x]]$ and hence a $\mathbf{C}[x]$ direct summand of $X/D(T)$. Hence $R_2 \cong R_1 \oplus R_3$ for some reduced submodule R_3. Letting $M \equiv D_2 \oplus R_3$, since $K = D_1 \oplus R_1$, we have shown that K is algebraically complemented: i.e. $X \cong K \oplus M$ as $\mathbf{C}[x]$ modules. This proves the theorem.

COROLLARY 1.6. *Let X be a Banach space and T a quasinilpotent operator on X. Let $\mathcal{N}(T) \equiv \{y \in X : Ty = 0\}$. Suppose that $\overline{TX} = X$ and $\mathcal{N}(T) \subseteq \bigcap_n T^n X$. Let K be a closed T invariant subspace of X. Then K is algebraically complemented if and only if*

i) $T^n X \cap K = T^n K, \quad n = 1, 2, 3 \ldots.$

ii) $\overline{T^N K} = \overline{T^{N+1} K}$, *for some N.*

Proof. Necessity is clear from our previous remarks. If $y = T^n y_n$, $n = 1, 2, 3 \ldots$, then $y_1 - T^{n-1} y_n \in \mathcal{N}(T)$, $n = 2, 3, 4 \ldots$. Thus, since $\mathcal{N}(T) \subseteq \bigcap_n T^n X$, there is $z_n \in X$ such that $y_1 - T^{n-1} y_n = T^{n-1} z_n$. This shows $y_1 \in \bigcap_n T^n X$ and hence

$$T\left(\bigcap_{n=1}^{\infty} T^n X\right) = \bigcap_{n=1}^{\infty} T^n X,$$

since $y = Ty_1$. This fact has been noted elsewhere (see, for example, [**3**, theorem 1]). Thus $D(T) = \bigcap_n T^n X$. If $k = T^n k_n$, $n = 1, 2, 3, \ldots$, where $k \in K$, $k_n \in K$, all n. Then $k_1 - T^{n-1} k_n \in \mathcal{N}(T)$, so $k_1 - T^{n-1} k_n = T^{n-1} z_n$ for some $z_n \in X$. But by i), $T^{n-1} z_n = T^{n-1} c_n$ for some $c_n \in K$. Thus

$k_1 = T^{n-1}(k_n + c_n)$, all n. Again this shows:

$$T\left(\bigcap_{n=1}^{\infty} T^n K\right) = \bigcap_{n=1}^{\infty} T^n K,$$

since $k = Tk_1$. Thus $D(T|K) = \bigcap_n T^n K$. Hence the Corollary follows from Theorem 1.5.

We note that the condition $\mathcal{N}(T) \subseteq \bigcap_n T^n X$ if often easy to verify, as for example in the case of a weighted left shift on l_2. The case where $\sigma_p(T) = \emptyset$ is handled by the following corollary.

COROLLARY 1.7. *Let X be a Banach space and T a quasinilpotent operator on X. Suppose $\sigma_p(T) = \emptyset$ and $\overline{TX} = X$. Let K be a closed T invariant subspace of X. Then K is algebraically complemented if and only if*

i) *$Tx \in K$ implies $x \in K$.*

ii) $\overline{T^N K} = \overline{T^{N+1}K}$, for some N.

Proof. If $T^n X \cap K = T^n K$ and $\sigma_p(T) = \emptyset$ then $T^n x \in K$ implies $x \in K$. But induction and i) imply $T^n x \in K$ only if $x \in K$. Hence under the condition of $\sigma_p(T) = \emptyset$, $T^n X \cap K = T^n K$ and i) are equivalent conditions. Thus, necessity follows from our previous remarks. If $\sigma_p(T) = \emptyset$ it is trivial that

$$D(T) = \bigcap_{n=1}^{\infty} T^n X \quad \text{and} \quad D(T|K) = \bigcap_{n=1}^{\infty} T^n K.$$

Hence applying Theorem 1.5, we obtain the result.

In the next section we will consider the consequences of Corollary 1.7 for the Volterra operator.

Section 2. We discuss an application of the results in Section 1. Let $X = L^2[0, 1]$ and $T = V$, the Volterra operator, i.e.:

$$Vf(t) \equiv \int_0^t f(x)dx,$$

all $f \in L^2[0, 1]$. It is elementary that $||v^n|| \leqq 1/(n-1)!$, so V is quasinilpotent. Also $\sigma_p(V) = \emptyset$ and $\overline{VX} = X$. For each $a \in [0, 1]$, let

$$K(a) \equiv \{f \in L^2[0, 1] : f \equiv 0 \text{ a.e. on } [0, a]\}.$$

Then it can be shown that the $K(a)$ are closed V invariant subspaces of X, and that all closed V invariant subspaces are of this form. Hence V is unicellular (i.e. the closed invariant subspaces are totally ordered). Also $Vf \in K(a)$ implies $f \in K(a)$ and $\overline{VK(a)} = K(a)$, all $a \in [0, 1]$. By Corollary 1.7 there is an algebraic V invariant subspace $M(a)$, for each a, such that $X \cong K(a) \oplus M(a)$, as $\mathbf{C}[x]$ modules. Note each $M(a)$ is dense, if $a \neq 0$, since $\overline{M(a)}$ is a closed V invariant subspace and $K(a) + \overline{M(a)} \supseteq X$. This can happen only if $\overline{M(a)} = X$. In general if $a \neq 0$ or 1, $M(a)$ is not unique. However one can choose $M(a)$ with nicer properties as follows: Let $D_0 \equiv D(V) \cap K(a)^\perp$. Since $D(V) = \bigcap_n V^n X$ it easily follows that

$$D(V) = \{f : f \in C^\infty[0, 1], f^{(n)}(0) = 0 \text{ all } n\}.$$

Of course, when we say $f \in C^\infty[0, 1]$, we mean f is equal almost everywhere to a $C^\infty[0, 1]$ function. Hence

$$D_0 = \{f : f \in C^\infty[0, 1], f^{(n)}(0) = 0 \text{ all } n, \text{ and } f([a, 1]) \equiv (0)\}.$$

If $Ve \in D_0$ it is easily seen that $e \in D_0$ also. Hence $V^{-1}D_0 \subseteq D_0$. Let E_0 be the $\mathbf{C}[x]$ module generated by D_0, i.e.:

$$E_0 = \left\{\sum_{k=1}^{n} p_k d_k : p_k \in C[x], d_k \in D_0\right\}.$$

Since D_0 is an algebraic subspace, it follows that

$$E_0 = \left\{\sum_{k=0}^{m} V^k d_k : d_k \in D_0\right\}.$$

We claim $E_0 \cap K(a) \equiv (0)$. So suppose that there are $d_0, d_1, d_2, \ldots, d_m$ in D_0 such that

$$\left\langle \sum_{k=0}^{m} V^k d_k, l \right\rangle = 0$$

for all $l \in K(a)^\perp$. Then

$$\left\langle V^m \left(\sum_{k=0}^{m} e_k \right), l \right\rangle = 0$$

for all $l \in K(a)^\perp$, where $e_k \in D_0$ and $V^{m-k}e_k = d_k$, $k = 0, 1, 2, \ldots, m$. Hence

$$\left\langle \sum_{k=0}^{m} e_k, (V^*)^m l \right\rangle = 0$$

for all $l \in K(a)^\perp$. But

$$\overline{V^* K(a)^\perp} = K(a)^\perp, \quad \text{thus}$$

$$\left\langle \sum_{k=0}^{m} e_k, l \right\rangle = 0$$

for all $l \in K(a)^\perp$. Hence

$$\sum_{k=0}^{m} e_k \in K(a),$$

which forces

$$\sum_{k=0}^{m} e_k = 0$$

as $\sum^m e_k \in D_0$. Hence $\sum^m V^k d_k = V^m \sum^m e_k = 0$ and it follows that $E_0 \cap K(a) \equiv (0)$. Now examining the proof of Theorem 1.5, one sees that D_2 can be chosen to contain E_0 as follows: We have the direct sum $D_1 \oplus E_0$ as a consequence of $E_0 \cap K(a) \equiv (0)$. Let Q_0 be the projection of $D_1 \oplus E_0$ onto D_1 along E_0 and extend Q_0 to all of $D(T)$ by injectivity ($E_0 \subseteq D(T)$). Let $D_2 \equiv \{d \in D(T) : Q_0 d = 0\} \supseteq E_0$. Proceed with the proof as before. Since $M \cong D_2 \oplus R_3$ we see that $M \supseteq E_0 \supseteq D_0$. We have thus proved:

PROPOSITION 2.1. *Let V be the Volterra operator on $L^2[0, 1]$. Let $a \in (0, 1]$. Then the closed T invariant subspace $K(a)$ of V is algebraically complemented by a V invariant subspace $M(a)$ satisfying:*

i) $\overline{M(a)} = L^2[0, 1]$.

ii) $g \in C^\infty[0, 1]$, $g([a, 1]) \equiv (0)$ *and* $g^{(n)}(0) = 0$, *all* n, *implies* $g \in M(a)$.

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The University of Texas at Austin,
Austin, Texas