A Proof of the Addition Theorem for the Legendre Functions.

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§ 1. *A Theorem of Dougall's.* In Vol. XVIII. (p. 78) of these Proceedings Dr Dougall has established the theorem that, if m is a positive integer,

$$\frac{\Pi(n-m)}{\Pi(n+m)} P_n^m(x) P_n^m(x') \frac{\pi}{\sin n\pi}$$

$$= \sum_{p=m}^{\infty} (-1)^p \left\{ \frac{1}{n-p} - \frac{1}{n+p+1} \right\} \frac{\Pi(p-m)}{\Pi(p+m)} P_p^m(x) P_p^m(x'), \quad \ldots (1)$$

where $x = \cos\theta$, $x' = \cos\theta'$. This can be proved by taking the integral

$$\frac{1}{2\pi i} \int \frac{\Pi(\zeta - m)}{\Pi(\zeta + m)} \frac{P_\zeta^m(x) P_\zeta^m(x')}{\zeta - n} \frac{\pi}{\sin \zeta\pi} d\zeta$$

round a circle $|\zeta| = p + \frac{1}{2}$ and making $p \to \infty$. From the formula

$$P_\zeta^m(z) = \frac{(z^2-1)^{\frac{1}{2}m}}{2^m \Pi(m)} \frac{\Pi(\zeta+m)}{\Pi(\zeta-m)} F\left(m-\zeta,\ m+\zeta+1,\ m+1,\ \frac{1-z}{2}\right)$$

it is clear that the integrand is holomorphic except at the zeros of $(\zeta - n)\sin(\zeta\pi)$, and from the expression (10) of the previous paper in which the series are convergent or asymptotic in n for $0 < \theta < \pi$, it can be deduced that the integral round the circle $\to 0$ as $p \to \infty$, provided that $0 < \theta < \pi$, $0 < \theta' < \pi$, and $\theta + \theta' < \pi$.

Similarly from the integral

$$\frac{1}{2\pi i} \int \frac{\Pi(\zeta - m)}{\Pi(\zeta + m)} \frac{P_\zeta^m(x)}{\zeta - n} \frac{\pi}{\sin \zeta\pi} d\zeta$$

it can be deduced that, for $-1 < x < 1$,

$$\frac{\Pi(n-m)}{\Pi(n+m)} P_n^m(x) \frac{\pi}{\sin n\pi}$$

$$= \sum_{p=m}^{\infty} (-1)^p \left\{ \frac{1}{n-p} - \frac{1}{n+p+1} \right\} \frac{\Pi(p-m)}{\Pi(p+m)} P_p^m(x), \quad \ldots\ldots\ldots (2)$$

and, in particular, that, when $m = 0$,

$$P_n(x) = \frac{\sin n\pi}{\pi} \sum_{p=0}^{\infty} (-1)^p \left\{ \frac{1}{n-p} - \frac{1}{n+p+1} \right\} P_p(x). \quad \ldots\ldots (3)$$

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These formulae may be employed to deduce various formulae for the Legendre Functions from the corresponding formulae for the Legendre Polynomials.

§ 2. *The Addition Theorem.* For example, the Addition Theorem for $P_n(z)$ when n is not an integer may be obtained in this way from the Addition Formula for the Legendre Polynomials.

Let $z = xx' - \sqrt{(x^2-1)}\sqrt{(x'^2-1)}\cos\phi$, where $-1 < z < 1$; then, from (3)

$$P_n(z) = \frac{\sin n\pi}{\pi}\sum_{p=0}^{\infty}(-1)^p\left\{\frac{1}{n-p} - \frac{1}{n+p+1}\right\}P_p(z).$$

Now substitute for $P_p(z)$ its expansion (the Addition Formula)

$$P_p(z) = P_p(x)P_p(x') + 2\sum_{m=1}^{p}(-1)^m\frac{(p-m)!}{(p+m)!}\cos m\phi P_p^m(x)P_p^m(x'),$$

and rearrange the resulting double series; thus

$$P_n(z) = \frac{\sin n\pi}{\pi}\sum_{p=0}^{\infty}(-1)^p\left\{\frac{1}{n-p} - \frac{1}{n+p+1}\right\}P_p(x)P_p(x')$$

$$+\frac{2\sin n\pi}{\pi}\sum_{m=1}^{\infty}(-1)^m\cos m\phi\sum_{p=m}^{\infty}(-1)^p\frac{(p-m)!}{(p+m)!}\left\{\frac{1}{n-p} - \frac{1}{n+p+1}\right\}P_p^m(x)P_p^m(x')$$

$$= P_n(x)P_n(x') + 2\sum_{m=1}^{\infty}(-1)^m\cos m\phi\,\frac{\Pi(n-m)}{\Pi(n+m)}P_n^m(x)P_n^m(x') \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

by (1). This is the Addition Theorem; it is valid for all values of x and x' for which the series (4) is convergent.