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On Bohr compactifications and profinite completions of group extensions

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Abstract

Let $G = N \rtimes H$ be a locally compact group which is a semi-direct product of a closed normal subgroup N and a closed subgroup H. The Bohr compactification Bohr(G) and the profinite completion Prof(G) of G are, respectively, isomorphic to semi-direct products $Q_1 \rtimes$ Bohr(H) and $Q_2 \rtimes$ Prof(H) for appropriate quotients Q_1 of Bohr(N) and Q_2 of Prof(N). We give a precise description of Q_1 and Q_2 in terms of the action of H on appropriate subsets of the dual space of N. In the case where N is abelian, we have Bohr(G) $\cong A \rtimes$ Bohr(H) and Prof(G) $\cong B \rtimes$ Prof(H), where A (respectively B) is the dual group of the group of unitary characters of N with finite H-orbits (respectively with finite image). Necessary and sufficient conditions are deduced for G to be maximally almost periodic or residually finite. We apply the results to the case where $G = \Lambda \wr H$ is a wreath product of discrete groups; we show in particular that, in case H is infinite, Bohr($\Lambda \wr H$) is isomorphic to Bohr($\Lambda^{\mathrm{Ab}} \wr H$) and Prof($\Lambda \wr H$) is isomorphic to Prof($\Lambda^{\mathrm{Ab}} \wr H$), where $\Lambda^{\mathrm{Ab}} = \Lambda/[\Lambda, \Lambda]$ is the abelianisation of Λ. As examples, we compute Bohr(G) and Prof(G) when G is a lamplighter group and when G is the Heisenberg group over a unital commutative ring.

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1. *Introduction*

There are two distinguished compact groups associated to a general topological group G. A **Bohr compactification** (respectively, a **profinite completion**) of G is a pair consisting of a compact (respectively, profinite) group K and a continuous homomorphism $\beta : G \to K$ with dense image satisfying the following universal property: for every compact group (respectively, profinite group) L and every continuous homomorphism $\alpha : G \to L$, there exists a continuous homomorphism $\alpha' : K \to L$ such that the diagram

$$\begin{tikzcd} & K \arrow[d, dashed, "\alpha'"] \\ G \arrow[ur, "\beta"] \arrow[r, "\alpha"] & L \end{tikzcd}$$

commutes. Bohr compactifications and profinite completions (K, β) of G are unique in the following sense: if (K', β') is a pair consisting of a compact (respectively, profinite) group

K' and a continuous homomorphism $\beta' : G \to K'$ with dense image satisfying the same universal property, then there exists an isomorphism $f : K \to K'$ of topological groups such that $\beta' = f \circ \beta$. Concerning existence, we give below (Proposition 4) models of Bohr compactifications and profinite completions. For more on Bohr compactifications, see [**Dix77**, section 16], [**BdlH**, 4·C] or [**Wei40**, chapter VII]; for more details on profinite completions, see [**RZ00**].

We will often denote by $(\mathrm{Bohr}(G), \beta_G)$ and $(\mathrm{Prof}(G), \alpha_G)$ a Bohr compactification and a profinite completion of G. In the sequel, for two topological groups H and L, we write $H \cong L$ if H and L are topologically isomorphic.

The universal property of $\mathrm{Bohr}(G)$ gives rise to a continuous surjective homomorphism $\alpha : \mathrm{Bohr}(G) \to \mathrm{Prof}(G)$ such that $\alpha_G = \alpha \circ \beta_G$. It is easy to see (see [**Bek23**, proposition 7]) that the kernel of α is $\mathrm{Bohr}(G)_0$, the connected component of $\mathrm{Bohr}(G)$, and so

$$\mathrm{Prof}(G) \cong \mathrm{Bohr}(G)/\mathrm{Bohr}(G)_0.$$

Every continuous homomorphism $G_1 \xrightarrow{f} G_2$ of topological groups induces continuous homomorphisms

$$\mathrm{Bohr}(G_1) \xrightarrow{\mathrm{Bohr}(f)} \mathrm{Bohr}(G_2) \qquad \text{and} \qquad \mathrm{Prof}(G_1) \xrightarrow{\mathrm{Prof}(f)} \mathrm{Prof}(G_2)$$

such that $\beta_{G_2} \circ f = \mathrm{Bohr}(f) \circ \beta_{G_1}$ and $\alpha_{G_2} \circ f = \mathrm{Prof}(f) \circ \alpha_{G_1}$.

Consider the category **TGrp** of topological groups, with objects the topological groups and morphisms the continuous homomorphisms between topological groups. The Bohr compactification and the profinite completion are covariant functors

$$\mathrm{Bohr} : \mathbf{TGrp} \to \mathbf{CGrp} \qquad \text{and} \qquad \mathrm{Prof} : \mathbf{TGrp} \to \mathbf{PGrp}$$

from **TGrp** to the subcategory **CGrp** of compact groups and the subcategory **PGrp** of profinite groups.

Assume that we are given an extension

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} G/N \longrightarrow 1 \qquad (*)$$

of topological groups. The functors Bohr and Prof are right exact and so the diagrams

$$\mathrm{Bohr}(N) \xrightarrow{\mathrm{Bohr}(i)} \mathrm{Bohr}(G) \xrightarrow{\mathrm{Bohr}(p)} \mathrm{Bohr}(G/N) \longrightarrow 1$$

and

$$\mathrm{Prof}(N) \xrightarrow{\mathrm{Prof}(i)} \mathrm{Prof}(G) \xrightarrow{\mathrm{Prof}(p)} \mathrm{Prof}(G/N) \longrightarrow 1$$

are exact; this means that

$$\mathrm{Bohr}(p) \text{ and } \mathrm{Prof}(p) \text{ are surjective and}$$
$$\mathrm{Ker}(\mathrm{Bohr}(p)) = \overline{\beta_G(N)} \text{ and } \mathrm{Ker}(\mathrm{Prof}(p)) = \overline{\alpha_G(N)},$$

where $\overline{A}$ denotes the closure of a subset A; these facts are well known and easy to prove (see, e.g., [**HK01**, lemma 2·2] and [**RZ00**, proposition 3·2·5]; see also Proposition 7 below). However, the functors Bohr and Prof are not left exact, that is, $\mathrm{Bohr}(i) : \mathrm{Bohr}(N) \to \mathrm{Bohr}(G)$ and $\mathrm{Prof}(i) : \mathrm{Prof}(N) \to \mathrm{Prof}(G)$ are in general not injective (see e.g. the examples given by Corollaries F and G below).

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For now on, we will deal only with *locally compact* groups. and with *split* extensions. So, we will consider locally compact groups $G = N \rtimes H$ which are a semi-direct product of a normal closed subgroup N and a closed subgroup H. It is easy to see that Bohr(G), respectively Prof(G), is a semi-direct product of $\overline{\beta_G(N)}$ with $\overline{\beta_G(H)}$, respectively of $\overline{\alpha_G(N)}$ with $\overline{\alpha_G(H)}$ (see [**Jun78**, **GZ11**]). Our results give a precise description of the structure of these semi-direct products.

Denote by $\widehat{N}_{\mathrm{fd}}$ the set of equivalence classes (modulo unitary equivalence) of irreducible finite dimensional unitary representations of N. Every such representation $\sigma : N \to U(n)$ gives rise to the unitary representation $\mathrm{Bohr}(\sigma) : \mathrm{Bohr}(N) \to U(n)$ of Bohr(N); here (and elsewhere) we identify Bohr$(U(n))$ with $U(n)$.

Observe that H acts on $\widehat{N}_{\mathrm{fd}}$: for $\sigma \in \widehat{N}_{\mathrm{fd}}$ and $h \in H$, the conjugate representation $\sigma^h \in \widehat{N}_{\mathrm{fd}}$ is defined by $\sigma^h(n) = \sigma(h^{-1}nh)$ for all $n \in N$.

Define $\widehat{N}_{\mathrm{fd}}^{H-\mathrm{per}}$ as the set of $\sigma \in \widehat{N}_{\mathrm{fd}}$ with *finite* H-orbit.

Observe that, due to the universal property of Bohr(N), the group H acts by automorphisms on Bohr(N). However, this action does not extend in general to an action of Bohr(H) on Bohr(N).

Our first result shows that Bohr(G) is a split extension of Bohr(H) by an appropriate quotient of Bohr(N).

THEOREM A. *Let $G = N \rtimes H$ be a semi-direct product of locally compact groups. Let $\varphi_N : \mathrm{Bohr}(N) \to \overline{\beta_G(N)}$ and $\varphi_H : \mathrm{Bohr}(H) \to \overline{\beta_G(H)}$ be the maps such that $\varphi_N \circ \beta_N = \beta_G|_N$ and $\varphi_H \circ \beta_H = \beta_G|_H$ Set*

$$C := \bigcap_{\sigma \in \widehat{N}_{\mathrm{fd}}^{H-\mathrm{per}}} \mathrm{Ker}(\mathrm{Bohr}(\sigma)).$$

(i) *We have* $\mathrm{Ker}\varphi_N = C$ *and so* φ_N *induces a topological isomorphism* $\overline{\varphi_N} : \mathrm{Bohr}(N)/C \to \overline{\beta_G(N)}$.

(ii) $\varphi_H : \mathrm{Bohr}(H) \to \overline{\beta_G(H)}$ *is a topological isomorphism.*

(iii) *The action of H by automorphisms on* Bohr(N) *induces an action of* Bohr(H) *by automorphisms on* Bohr$(N)/C$ *and the maps $\overline{\varphi_N}$ and φ_H give rise to an isomorphism*

$$\mathrm{Bohr}(G) \cong (\mathrm{Bohr}(N)/C) \rtimes \mathrm{Bohr}(H).$$

We turn to the description of Prof(G). Let $\widehat{N}_{\mathrm{finite}}$ be the set of equivalence classes of irreducible unitary representations σ of N with finite image $\sigma(N)$. Observe that the action of H on $\widehat{N}_{\mathrm{fd}}$ preserves $\widehat{N}_{\mathrm{finite}}$. Let $\widehat{N}_{\mathrm{finite}}^{H-\mathrm{per}}$ be the subset of $\widehat{N}_{\mathrm{finite}}$ of representations with finite H-orbit. Every $\sigma \in \widehat{N}_{\mathrm{finite}}$ gives rise to the unitary representation Prof(σ) of Prof(N).

A result completely similar to Theorem A holds for Prof(G).

THEOREM B. *Let $G = N \rtimes H$ be a semi-direct product of locally compact groups. Let $\psi_N : \mathrm{Prof}(N) \to \overline{\alpha_G(N)}$ and $\psi_H : \mathrm{Prof}(H) \to \overline{\alpha_G(H)}$ be the maps such that $\psi_N \circ \alpha_N = \alpha_G|_N$ and $\psi_H \circ \alpha_H = \alpha_G|_H$ Set*

$$D := \bigcap_{\sigma \in \widehat{N}_{\mathrm{finite}}^{H-\mathrm{per}}} \mathrm{Ker}(\mathrm{Prof}(\sigma)).$$

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(i) *We have* $\mathrm{Ker}\psi_N = D$ *and so* ψ_N *induces a topological isomorphism* $\overline{\psi_N} : \mathrm{Prof}(N)/D \to \overline{\alpha_G(N)}$.

(ii) $\psi_H : \mathrm{Prof}(H) \to \overline{\alpha_G(H)}$ *is a topological isomorphism.*

(iii) *The action of* H *by automorphisms on* $\mathrm{Prof}(N)$ *induces an action of* $\mathrm{Prof}(H)$ *by automorphisms on* $\mathrm{Prof}(N)/D$ *and the maps* $\overline{\psi_N}$ *and* ψ_H *give rise to an isomorphism*

$$\mathrm{Prof}(G) \cong (\mathrm{Prof}(N)/D) \rtimes \mathrm{Prof}(H).$$

When N is a finitely generated (discrete) group, we obtain the following well known result (see [**GZ11**, proposition 2·6]).

COROLLARY C. *Assume that* N *is finitely generated. Then* $\mathrm{Prof}(G) \cong \mathrm{Prof}(N) \rtimes \mathrm{Prof}(H)$.

In the case where N is abelian, we can give a more explicit description of the quotients $\mathrm{Bohr}(N)/C$ and $\mathrm{Prof}(N)/D$ appearing in Theorems A and B. Recall that, in this case, the dual group $\widehat{N}$ is the group of continuous homomorphisms from N to the circle group $\mathbf{S}^1$. We will also consider the subgroup $\widehat{N}_{\mathrm{fin}}$ of $\chi \in \widehat{N}$ with finite image $\chi(N)$, that is, with values in the subgroup of m-th roots of unity in $\mathbf{C}$ for some integer $m \geq 1$. Observe also that $\widehat{N}^{H-\mathrm{per}}$ and $\widehat{N}^{H-\mathrm{per}}_{\mathrm{finite}}$ are subgroups of $\widehat{N}$.

COROLLARY D. *Assume that* N *is an abelian locally compact group. Let* $\widehat{N}^{H-\mathrm{per}}$ *and* $\widehat{N}^{H-\mathrm{per}}_{\mathrm{finite}}$ *be equipped with the discrete topology. Let* A *and* B *be their respective dual groups. Then*

$$\mathrm{Bohr}(G) \cong A \rtimes \mathrm{Bohr}(H) \qquad \textit{and} \qquad \mathrm{Prof}(G) \cong B \rtimes \mathrm{Prof}(H).$$

Recall that G is **maximally almost periodic**, or **MAP**, if $\widehat{G}_{\mathrm{fd}}$ separates its points (equivalently, if $\beta_G : G \to \mathrm{Bohr}(G)$ is injective); recall also that G is **residually finite**, or **RF**, if $\widehat{G}_{\mathrm{finite}}$ separates its points (equivalently, if $\alpha_G : G \to \mathrm{Prof}(G)$ is injective).

COROLLARY E. *Let* $G = N \rtimes H$ *be a semi-direct product of locally compact groups.*

(i) G *is MAP if and only if* H *is MAP and* $\widehat{N}^{H-\mathrm{per}}_{\mathrm{fd}}$ *separates the points of* N.

(ii) G *is RF if and only if* H *is RF and* $\widehat{N}^{H-\mathrm{per}}_{\mathrm{finite}}$ *separates the points of* N.

We give an application of our results to wreath products. Let H, Λ be groups, X a non empty set, and $H \curvearrowright X$ an action of H on X. Then H acts on the direct sum $\oplus_{x\in X}\Lambda$, by shifting the indices. The **(permutational) wreath product**, denoted $\Lambda \wr_X H$, is the semidirect product

$$\Lambda \wr_X H := (\oplus_{x\in X} \Lambda) \rtimes H.$$

When the action of H on X is simply transitive, we obtain the standard wreath product denoted $\Lambda \wr H$. Observe that $\Lambda^{\mathrm{Ab}} \wr_X H$ is a quotient of $\Lambda \wr_X H$, where Λ^{Ab} is the abelianization $\Lambda/[\Lambda, \Lambda]$ of Λ.

Initially, we formulated the next two corollaries only for standard wreath products; the extension of these results to more general wreath products was suggested to us by the referee.

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COROLLARY F. *Let H, Λ be groups, and let $H \curvearrowright X$ be a transitive action of H on a set X. Let $\Lambda \wr_X H$ be equipped with the discrete topology.*

(i) *When X is finite, we have*

$$\mathrm{Bohr}(\Lambda \wr_X H) \cong (\oplus_{x\in X}\mathrm{Bohr}(\Lambda)) \rtimes \mathrm{Bohr}(H) \text{ and}$$
$$\mathrm{Prof}(\Lambda \wr_X H) \cong (\oplus_{x\in X}\mathrm{Prof}(\Lambda)) \rtimes \mathrm{Prof}(H).$$

(ii) *When X is infinite, the quotient map $\Lambda \wr_X H \to \Lambda^{\mathrm{Ab}} \wr_X H$ induces isomorphisms*

$$\mathrm{Bohr}(\Lambda \wr_X H) \cong \mathrm{Bohr}(\Lambda^{\mathrm{Ab}} \wr_X H) \text{ and } \mathrm{Prof}(\Lambda \wr_X H) \cong \mathrm{Prof}(\Lambda^{\mathrm{Ab}} \wr_X H)$$

In particular, if Λ is perfect (that is, $\Lambda = [\Lambda, \Lambda]$), the quotient map $\Lambda \wr_X H \to H$ induces isomorphisms

$$\mathrm{Bohr}(\Lambda \wr_X H) \cong \mathrm{Bohr}(H) \text{ and } \mathrm{Prof}(\Lambda \wr_X H) \cong \mathrm{Prof}(H).$$

The following definition was suggested to us by the referee.

Definition 1. An action $H \curvearrowright X$ of a group H on a set X is **residually finite** or **RF**, if, for any pair x_1, x_2 of distinct elements of X, there exists a finite index subgroup L of H such that $Lx_1 \neq Lx_2$.

Observe that $H \curvearrowright X$ is RF if and only if $H \curvearrowright Y$ is RF for every H-orbit $Y \subset X$. Observe also that, when $H \curvearrowright X$ is simply transitive, the action $H \curvearrowright X$ is RF if and only if the group H is RF.

Item (ii) of the following result was proved, with different methods, in [**Gru57**, theorem 3·2] for standard wreath products and in [**Cor14**, proposition 1·7] for permutational wreath products.

COROLLARY G. *Let Λ, H be groups, and let $H \curvearrowright X$ be an action of H on a set X. Let $\Lambda \wr_X H$ be equipped with the discrete topology.*

Assume that Λ has at least two elements.

(i) *The group $\Lambda \wr_X H$ is MAP if and only if Λ and H are MAP, and either*
   - *Λ is abelian and $H \curvearrowright X$ is RF, or*
   - *X is finite.*

(ii) ([**Gru57**], [**Cor14**]) *The group $\Lambda \wr_X H$ is RF if and only if Λ and H are RF, and either*
   - *Λ is abelian and $H \curvearrowright X$ is RF, or*
   - *X is finite.*

Remark 2.

(i) The Bohr compactification of an abelian locally compact group A is easy to describe: $\mathrm{Bohr}(A)$ can be identified with $\widehat{\Gamma}$, where $\Gamma = \widehat{A}$ is viewed as discrete group; in case A is finitely generated, a more precise description of $\mathrm{Bohr}(A)$ is available (see [**Bek23**, proposition 11]).

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(ii) Provided Bohr(H) and Prof(H) are known, Corollary F together with Corollary D give, in view of (i), a complete description of the Bohr compactification and the profinite completion of *any* wreath product $\Lambda \wr_X H$ in case X is infinite.

(iii) Bohr compactifications of group and semigroup extensions have been studied by several authors, in a more abstract and less explicit setting ([**DL83**, **JL81**, **Jun78**, **JM02**, **Lan72**, **Mil83**]); profinite completions of group extensions appear at numerous places in the literature ([**GZ11**, **RZ00**]).

This paper is organised as follows. Section 2 contains some general facts about Bohr compactifications and profinite completions as well as some reminders on projective representations. In Section 3, we give the proof of Theorems A and B. Section 4 contains the proof of the corollaries. Section 5 is devoted to the explicit computation of the Bohr compactification and profinite completions for two groups: the lamplighter group $(\mathbf{Z}/n\mathbf{Z}) \wr \mathbf{Z}$ and the Heisenberg group $H(R)$ over an arbitrary commutative ring R.

2. *Preliminaries*

2·1. *Models for Bohr compactifications and profinite completions*

Let G be a topological group. We give well known models for Bohr(G) and Prof(G). For this, we use finite dimensional unitary representations of G, that is, continuous homomorphisms $\pi : G \to U(n)$ for some integer $n \geq 1$. We denote by $\widehat{G}_{\mathrm{fd}}$ the set of equivalence classes of irreducible finite dimensional unitary representations of G. Let $\widehat{G}_{\mathrm{finite}}$ be the subset of $\widehat{G}_{\mathrm{fd}}$ consisting of representations π with finite image $\pi(G)$.

For a compact (respectively, profinite) group K, the set $\widehat{K}_{\mathrm{fd}}$ (respectively, $\widehat{K}_{\mathrm{finite}}$) coincides with the dual space $\widehat{K}$, that is, the set of equivalence classes of unitary representations of K.

A useful tool for the identification of Bohr(G) or Prof(G) is given by the following proposition; for the easy proof, see [**Bek23**, propositions 5 and 6].

PROPOSITION 3

(i) *Let K be a compact group and $\beta : G \to K$ a continuous homomorphism with dense image; then (K, β) is a Bohr compactification of G if and only if the map $\widehat{\beta} : \widehat{K} \to \widehat{G}_{\mathrm{fd}}$, given by $\widehat{\beta}(\pi) = \pi \circ \beta$, is surjective.*

(ii) *Let L a be profinite group and $\alpha : G \to L$ a continuous homomorphism with dense image; then (L, α) is a profinite completion of G if and only if the map $\widehat{\beta} : \widehat{L} \to \widehat{G}_{\mathrm{finite}}$, given by $\widehat{\beta}(\pi) = \pi \circ \beta$, is surjective.*

The following proposition is an immediate consequence of Proposition 3.

PROPOSITION 4. *Choose families*

$$(\pi_i : G \to U(n_i))_{i \in I} \quad \textit{and} \quad (\sigma_j : G \to U(n_j))_{j \in J}$$

of representatives for the sets $\widehat{G}_{\mathrm{fd}}$ and $\widehat{G}_{\mathrm{finite}}$, respectively.

(i) *Let $\beta : G \to \prod_{i \in I} U(n_i)$ be given by $\beta(g) = \bigoplus_{i \in I} \pi_i(g)$ and let K be the closure of $\beta(G)$. Then (K, β) is a Bohr compactification of G.*

(ii) *Let $\alpha : G \to \prod_{j \in J} U(n_j)$ be given by $\alpha(g) = \bigoplus_{j \in J} \sigma_j(g)$ and let L be the closure of $\alpha(G)$. Then (L, α) is a profinite completion of G.*

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We observe that a more common model for the profinite completion of G is the projective limit $\varprojlim G/H$, where H runs over the family of the normal subgroups of finite index of G, together with the natural homomorphism $G \to \varprojlim G/H$ (see e.g. [**RZ00**, 2·1·6])

2·2. *Extension of representations*

We will also use the notion of a projective representation. Let G be a locally compact group. A map $\pi : G \to U(n)$ is a **projective representation** of G if the following holds:

$\pi(e) = I$,
for all $g_1, g_2 \in G$, there exists $c(g_1, g_2) \in \mathbf{S}^1$ such that

$$\pi(g_1 g_2) = c(g_1, g_2)\pi(g_1)\pi(g_2),$$

π is Borel measurable.

The map $c : G \times G \to \mathbf{S}^1$ is a 2-cocycle with values in the unit circle $\mathbf{S}^1$. The conjugate representation $\overline{\pi} : G \to U(n)$ is another projective representation defined by $\overline{\pi}(g) = J\pi(g)J$, where $J : \mathbf{C}^n \to \mathbf{C}^n$ is the anti-linear map given by conjugation of the coordinates,

The proof of the following lemma is straightforward.

LEMMA 5. *Let $\pi : G \to U(n)$ be a projective representation of G, with associated cocycle $c : G \times G \to \mathbf{S}^1$. Let $\pi' : G \to U(m)$ be another projective representation of G with associated cocycle 2-cocycle $c' : G \times G \to \mathbf{S}^1$.*

(i) *$\overline{\pi} : G \to U(n)$ is a projective representation of G with $\overline{c}$ as associated cocycle.*

(ii) *The tensor product*

$$\pi \otimes \pi' : G \to U(nm), \qquad g \mapsto \pi(g) \otimes \pi'(g)$$

is a projective representation of G with cc' as associated cocycle.

Let N be a closed normal subgroup of G. Recall that the stabiliser G_π in G of an irreducible unitary representation π of N is the set of $g \in G$ such that π^g is equivalent to π. Observe that G_π contains N.

The following proposition is a well known fact from the Clifford–Mackey theory of unitary representations of group extensions (see [**CR62**, chapter 1, section 11] and [**Mac58**]).

PROPOSITION 6. *Let $G = N \rtimes H$ be the semi-direct product of the locally compact groups H and N. Let $\pi : N \to U(m)$ be an irreducible unitary representation of N and assume that $G = G_\pi$. There exists a projective representation $\widetilde{\pi} : G \to U(m)$ with the following properties*:

(i) *$\widetilde{\pi}$ extends π, that is, $\widetilde{\pi}(n) = \pi(n)$ for every $n \in N$*;

(ii) *the 2-cocycle $\widetilde{c} : G \times G \to \mathbf{S}^1$ associated to $\widetilde{\pi}$ has the form $\widetilde{c} = c \circ (p \times p)$, for a map $c : H \times H \to \mathbf{S}^1$, where $p : G \to H$ is the canonical homomorphism.*

Proof. Let $S \subset U(m)$ be a Borel transversal for the quotient space $PU(m) = U(m)/\mathbf{S}^1$ with $I_m \in S$. Let $h \in H$. Since $G = G_\pi$ and since π is irreducible, there exists a unique matrix

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$\widetilde{\pi}(h) \in S$ such that

$$\pi(hnh^{-1}) = \widetilde{\pi}(h)\pi(n)\widetilde{\pi}(h)^{-1} \qquad \text{for all} \quad n \in N.$$

Define $\widetilde{\pi} : G \to U(n)$ by

$$\widetilde{\pi}(nh) = \pi(n)\widetilde{\pi}(h) \qquad \text{for all} \quad n \in N, h \in H.$$

It is clear that $\widetilde{\pi}|_N = \pi$ and that

$$\pi(gng^{-1}) = \widetilde{\pi}(g)\pi(n)\widetilde{\pi}(g)^{-1} \qquad \text{for all} \quad g \in G, n \in N.$$

It can be shown (see [**Mac58**, proof of theorem 8·2]) that $\widetilde{\pi}$ is a measurable map.

Let $g_1, g_2 \in G$. For every $n \in N$, we have, on the one hand,

$$\pi(g_1g_2ng_2^{-1}g_1) = \widetilde{\pi}(g_1g_2)\pi(n)\widetilde{\pi}(g_1g_2)^{-1}$$

and on the other hand

$$\begin{aligned}\pi(g_1g_2ng_2^{-1}g_1) &= \widetilde{\pi}(g_1)\pi(g_2ng_2^{-1})\widetilde{\pi}(g_1)^{-1}\\ &= \widetilde{\pi}(g_1)\widetilde{\pi}(g_2)\pi(n)\widetilde{\pi}(g_1)^{-1}\widetilde{\pi}(g_2)^{-1}.\end{aligned}$$

Since π is irreducible, it follows that

$$\widetilde{\pi}(g_1g_2) = \widetilde{c}(g_1, g_2)\widetilde{\pi}(g_1)\widetilde{\pi}(g_2)$$

for some scalar $\widetilde{c}(g_1, g_2) \in \mathbf{S}^1$.

Moreover, for $g_1 = n_1h_1, g_2 = n_2h_2$, we have, on the one hand,

$$\begin{aligned}\widetilde{\pi}(g_1g_2) &= \widetilde{c}(g_1, g_2)\widetilde{\pi}(g_1)\widetilde{\pi}(g_2)\\ &= \widetilde{c}(n_1h_1, n_2h_2)\pi(n_1)\widetilde{\pi}(h_1)\pi(n_2)\widetilde{\pi}(h_2)\end{aligned}$$

and, on the other hand,

$$\begin{aligned}\widetilde{\pi}(g_1g_2) &= \widetilde{\pi}(n_1(h_1n_2h_1^{-1})h_1h_2)\\ &= \pi(n_1(h_1n_2h_1^{-1}))\widetilde{\pi}(h_1h_2)\\ &= \pi(n_1)\pi(h_1n_2h_1^{-1})\widetilde{\pi}(h_1h_2)\\ &= \pi(n_1)\widetilde{\pi}(h_1)\pi(n_2)\widetilde{\pi}(h_1)^{-1}\widetilde{\pi}(h_1h_2)\\ &= \widetilde{c}(h_1, h_2)\pi(n_1)\widetilde{\pi}(h_1)\pi(n_2)\widetilde{\pi}(h_1)^{-1}\widetilde{\pi}(h_1)\widetilde{\pi}(h_2)\\ &= \widetilde{c}(h_1, h_2)\pi(n_1)\widetilde{\pi}(h_1)\pi(n_2)\widetilde{\pi}(h_2);\end{aligned}$$

this shows that $\widetilde{c}(n_1h_1, n_2h_2) = \widetilde{c}(h_1, h_2)$.

2·3. *Bohr compactification and profinite completion of quotients*

Let G be a topological group and N a closed normal subgroup of G. Let $(\mathrm{Bohr}(G), \beta_G)$ and $(\mathrm{Prof}(G), \alpha_G)$ be a Bohr compactification and a profinite completion of G. Let $\mathrm{Bohr}(p) : \mathrm{Bohr}(G) \to \mathrm{Bohr}(G/N)$ and $\mathrm{Prof}(p) : \mathrm{Bohr}(G) \to \mathrm{Bohr}(G/N)$ be the morphisms induced by the canonical epimorphism $p : G \to G/N$. The following proposition is well known (see [**HK01**, lemma 2·2] or [**Bek23**, proposition 10] for (i) and [**RZ00**, proposition 3·2·5] for (ii)). For the convenience of the reader, we give for (ii) a proof which is different from the one in [**RZ00**]

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PROPOSITION 7

(i) $\mathrm{Bohr}(p)$ *is surjective and its kernel is* $\overline{\beta_G(N)}$.

(ii) $\mathrm{Prof}(p)$ *is surjective and its kernel is* $\overline{\alpha_G(N)}$.

Proof. To show (ii), set $K := \overline{\alpha_G(N)}$. Let $(\mathrm{Prof}(G/N), \overline{\alpha})$ be a profinite completion of G/N. We have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "p"] \arrow[d, "\alpha_G"] & G/N \arrow[d, "\overline{\alpha}"] \\ \mathrm{Prof}(G) \arrow[r, "\mathrm{Prof}(p)"] & \mathrm{Prof}(G/N). \end{tikzcd}$$

It follows that $\alpha_G(N)$ and hence K is contained in $\mathrm{Ker}(\mathrm{Prof}(p))$. So, we have induced homomorphisms $\beta : G/N \to \mathrm{Prof}(G)/K$ and $\beta' : \mathrm{Prof}(G)/K \to \mathrm{Prof}(G/N)$, giving rise to a commutative diagram

$$\begin{tikzcd} & G/N \arrow[dl, "\beta"'] \arrow[d, "\overline{\alpha}"] \\ \mathrm{Prof}(G)/K \arrow[r, "\beta'"] & \mathrm{Prof}(G/N). \end{tikzcd}$$

It follows that $(\mathrm{Prof}(G)/K, \beta)$ has the same universal property for G/N as $(\mathrm{Prof}(G/N), \overline{\alpha})$; it is therefore a profinite completion of G/N.

3. *Proof of Theorems* A *and* B

3·1. *Proof of Theorem* A

Set $K := \overline{\beta_G(N)}$, where β_G is the canonical map from the locally compact group $G = N \rtimes H$ to $\mathrm{Bohr}(G)$.

(i) *First step.* We claim that

$$\left\{\widehat{\sigma} \circ (\beta_G|_N) : \widehat{\sigma} \in \widehat{K}\right\} \subset \widehat{N}_{\mathrm{fd}}^{H-\mathrm{per}}.$$

Indeed, let $\widehat{\sigma} \in \widehat{K}$. Then $\sigma := \widehat{\sigma} \circ (\beta_G|_N) \in \widehat{N}_{\mathrm{fd}}$. Let $\widehat{\rho} \in \widehat{\mathrm{Bohr}(G)}$ be an irreducible subrepresentation of the induced representation $\mathrm{Ind}_K^{\mathrm{Bohr}(G)} \widehat{\sigma}$. Then, by Frobenius reciprocity, $\widehat{\sigma}$ is equivalent to a subrepresentation of $\widehat{\rho}|_K$. Hence, σ is equivalent to a subrepresentation of $(\widehat{\rho} \circ \beta_G)|N$. The decomposition of the finite dimensional representation $(\widehat{\rho} \circ \beta_G)|_N$ into isotypical components shows that σ has a finite H-orbit (see [**Bek23**, proposition 12]).

(ii) *Second step.* We claim that

$$\widehat{N}_{\mathrm{fd}}^{H-\mathrm{per}} \subset \left\{\widehat{\sigma} \circ (\beta_G|_N) : \widehat{\sigma} \in \widehat{K}\right\}.$$

Indeed, let $\sigma : N \to U(m)$ be a representation of N with finite H-orbit. By Proposition 6, there exists a *projective* representation $\widetilde{\sigma}$ of $G_\sigma = NH_\sigma$ which extends σ and the associated cocycle $c : G_\sigma \times G_\sigma \to \mathbf{S}^1$, factorises through $H_\sigma \times H_\sigma$.

Define a projective representation $\tau : G_\sigma \to U(m)$ of G_σ by

$$\tau(nh) = \overline{\widetilde{\sigma}}(h) \qquad \text{for all} \quad nh \in NH_\sigma.$$

Observe that τ is trivial on N and that its associated cocycle is $\overline{c}$. Consider the tensor product representation $\widetilde{\sigma} \otimes \tau$ of G_σ. Lemma 5 shows that $\widetilde{\sigma} \otimes \tau$ is a projective representation for the cocyle $c\overline{c} = 1$. So, $\widetilde{\sigma} \otimes \tau$ is a measurable homomorphism from G_σ to $U(m)$. This implies that $\widetilde{\sigma} \otimes \tau$ is continuous (see [**BHV08**, lemma A·6·2]) and so $\widetilde{\sigma} \otimes \tau$ is an *ordinary* representation of G_σ.

It is clear that $\widetilde{\sigma} \otimes \tau$ is finite dimensional. Observe that the restriction $(\widetilde{\sigma} \otimes \tau)|_N$ of $\widetilde{\sigma} \otimes \tau$ to N is a multiple of σ. Let

$$\rho := \mathrm{Ind}_{G_\sigma}^G(\widetilde{\sigma} \otimes \tau).$$

Then ρ is finite dimensional, since $\widetilde{\sigma} \otimes \tau$ is finite dimensional and G_σ has finite index in G. As G_σ is open in G, $\widetilde{\sigma} \otimes \tau$ is equivalent to a subrepresentation of the restriction $\rho|_{G_\sigma}$ of ρ to G_σ (see e.g. [**BdlH**, 1·F]); consequently, σ is equivalent to a subrepresentation of $\rho|_N$. Since ρ is a finite dimensional unitary representation of G, there exists a unitary representation $\widehat{\rho}$ of $\mathrm{Bohr}(G)$ such that $\widehat{\rho} \circ \beta_G = \rho$. So, σ is equivalent to a subrepresentation of $(\widehat{\rho} \circ \beta_G)|_N$, that is, there exists a subspace V of the space of $\widehat{\rho}$ which is invariant under $\beta_G(N)$ and defining a representation of N which is equivalent to σ. Then V is invariant under $K = \overline{\beta_G(N)}$ and defines therefore an irreducible representation $\widehat{\sigma}$ of K for which $\widehat{\sigma} \circ (\beta_G|_N) = \sigma$ holds.

Let $\varphi_N : \mathrm{Bohr}(N) \to K = \overline{\beta_G(N)}$ be the homomorphism such that $\varphi_N \circ \beta_N = \beta_G|_N$.

(iii) *Third step.* We claim that

$$\mathrm{Ker}\varphi_N = \bigcap_{\sigma \in \widehat{N}_{\mathrm{fd}}^{H-\mathrm{per}}} \mathrm{Ker}(\mathrm{Bohr}(\sigma)),$$

where $\mathrm{Bohr}(\sigma)$ is the representation of $\mathrm{Bohr}(N)$ such that $\mathrm{Bohr}(\sigma) \circ \beta_N = \sigma$.

Indeed, by the first and second steps, we have

$$\widehat{N}_{\mathrm{fd}}^{H-\mathrm{per}} = \left\{\widehat{\sigma} \circ (\beta_G|_N) : \widehat{\sigma} \in \widehat{K}\right\} = \left\{(\widehat{\sigma} \circ \varphi_N) \circ \beta_N : \widehat{\sigma} \in \widehat{K}\right\};$$

since obviously $\widehat{\sigma} \circ \varphi_N = \mathrm{Bohr}(\sigma)$ for $\sigma = (\widehat{\sigma} \circ \varphi_N) \circ \beta_N$, it follows that

$$\bigcap_{\sigma \in \widehat{N}_{\mathrm{fd}}^{H-\mathrm{per}}} \mathrm{Ker}(\mathrm{Bohr}(\sigma)) = \bigcap_{\widehat{\sigma} \in \widehat{K}} \mathrm{Ker}(\widehat{\sigma} \circ \varphi_N).$$

As $\varphi_N(\mathrm{Bohr}(N)) = K$ and $\widehat{K}$ separates the points of K, we have $\bigcap_{\widehat{\sigma} \in \widehat{K}} \mathrm{Ker}(\widehat{\sigma} \circ \varphi_N) = \mathrm{Ker}\varphi_N$ and the claim is proved.

Set $L := \overline{\beta_G(H)}$.

(iv) *Fourth step.* We claim that the map $\varphi_H : \mathrm{Bohr}(H) \to L$, defined by the relation $\varphi_H \circ \beta_H = \beta_G|_H$, is an isomorphism. Indeed, the canonical isomorphism $H \to G/N$ induces an isomorphism $\mathrm{Bohr}(H) \to \mathrm{Bohr}(G/N)$. Using Proposition 7 (i), we obtain a continuous epimorphism

$$f : L \to \mathrm{Bohr}(H)$$

such that $f(\beta_G(h)) = \beta_H(h)$ for all $h \in H$. Then $\varphi_H \circ f$ is the identity on $\beta_G(H)$ and hence on L, by density. This implies that φ_H is an isomorphism.

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Observe that, by the universal property of Bohr(N), every element $h \in H$ defines a continuous automorphism $\theta_b(h)$ of Bohr(N) such that

$$\theta_b(h)(n) = \beta_N(hnh^{-1}) \qquad \text{for all} \quad n \in N.$$

The corresponding homomorphism $\theta_b : H \to \mathrm{Aut}(\mathrm{Bohr}(N))$ defines an action of H on the compact group Bohr(N). By duality, we have an action, still denoted by θ_b, of H on $\widehat{\mathrm{Bohr}(N)}$ and we have

$$\mathrm{Bohr}(\sigma^h) = \theta_b(h)(\mathrm{Bohr}(\sigma)) \qquad \text{for all} \quad \sigma \in \widehat{N}_{\mathrm{fd}}, h \in H.$$

This implies that the normal subgroup

$$\mathrm{Ker}\varphi_N = \bigcap_{\sigma \in \widehat{N}_{\mathrm{fd}}^{H-\mathrm{per}}} \mathrm{Ker}(\mathrm{Bohr}(\sigma))$$

of Bohr(N) is H-invariant. We have therefore an induced action $\overline{\theta_b}$ of H on $\mathrm{Bohr}(N)/\mathrm{Ker}\varphi_N$. Observe that the isomorphism

$$\mathrm{Bohr}(N)/\mathrm{Ker}\varphi_N \to K$$

induced by φ_N is H-equivariant for $\overline{\theta_b}$ and the action of H on K given by conjugation with $\beta_G(h)$ for $h \in H$.

(v) *Fifth step*. We claim that the action $\overline{\theta_b}$ induces an action of Bohr(H) by automorphisms on $\mathrm{Bohr}(N)/\mathrm{Ker}\varphi_N$ and that the map

$$(\mathrm{Bohr}(N)/\mathrm{Ker}\varphi_N) \rtimes \mathrm{Bohr}(H) \to \mathrm{Bohr}(G), (x\mathrm{Ker}\varphi_N, y) \mapsto \varphi_N(x)\varphi_H(y)$$

is an isomorphism.

Indeed, $\overline{\beta_G(N)}$ is a normal subgroup of Bohr(G) and so $\overline{\beta_G(H)}$ acts by conjugation on K. By the third and the fourth step, the maps

$$\overline{\varphi_N} : \mathrm{Bohr}(N)/\mathrm{Ker}\varphi_N \to K, \qquad x\mathrm{Ker}\varphi_N \mapsto \varphi_N(x)$$

and

$$\varphi_H : \mathrm{Bohr}(H) \to L$$

are isomorphisms. We define an action

$$\widehat{\theta} : \mathrm{Bohr}(H) \to \mathrm{Aut}(\mathrm{Bohr}(N)/\mathrm{Ker}\varphi_N)$$

by

$$\widehat{\theta}(y)(x\mathrm{Ker}\varphi_N) = (\overline{\varphi_N})^{-1}\left(\varphi_H(y)\varphi_N(x)\varphi_H(y)^{-1}\right)$$

for $x \in \mathrm{Bohr}(N)$ and $y \in \mathrm{Bohr}(H)$. The claim follows.

3·2. *Proof of Theorem* B

The proof is similar to the proof of Theorem A. The role of $\widehat{N}_{\mathrm{fd}}$ is now played by the space $\widehat{N}_{\mathrm{finite}}$ of finite dimensional irreducible representations of N with finite image. We will go

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quickly through the steps of the proof of Theorem A; at some places (especially the second step) there will be a few crucial changes and new arguments which we will emphasise.

Set $L := \overline{\alpha_G(N)}$, where $\alpha_G : G \to \mathrm{Prof}(G)$ is the canonical map. Observe that L is profinite.

(i) *First step*. We claim that $\left\{\widehat{\sigma} \circ (\alpha_G|_N) : \widehat{\sigma} \in \widehat{L}\right\} \subset \widehat{N}^{H-\mathrm{per}}_{\mathrm{finite}}$. Indeed, let $\widehat{\sigma} \in \widehat{L}$. Then $\sigma := \widehat{\sigma} \circ (\alpha_G|_N) \in \widehat{N}_{\mathrm{finite}}$, since L is profinite. Let $\widehat{\rho}$ be an irreducible subrepresentation of $\mathrm{Ind}_L^{\mathrm{Prof}(G)}\widehat{\sigma}$. Since $\mathrm{Prof}(G)$ is compact, $\widehat{\rho}$ is finite dimensional. Since σ is equivalent to a subrepresentation of $\widehat{\rho} \circ (\alpha_G)|N)$, it has therefore a finite H-orbit.

(ii) *Second step*. We claim that $\widehat{N}^{H-\mathrm{per}}_{\mathrm{finite}} \subset \left\{\widehat{\sigma} \circ (\alpha_G|_N) : \widehat{\sigma} \in \widehat{L}\right\}$. Indeed, let $\sigma : N \to U(m)$ be an irreducible representation with finite image. By Proposition 6, there exists a projective representation $\widetilde{\sigma}$ of $G_\sigma = NH_\sigma$ which extends σ and the associated cocycle $c : G_\sigma \times G_\sigma \to \mathbf{S}^1$, factorises through $H_\sigma \times H_\sigma$. We need to show that we can choose $\widetilde{\sigma}$ so that $\widetilde{\sigma}(G_\sigma)$ is finite.

Choose a projective representation $\widetilde{\sigma} : G_\sigma \to U(m)$ as above and modify $\widetilde{\sigma}$ as follows: define

$$\widetilde{\sigma}_1(nh) = \frac{1}{(\det \widetilde{\sigma}(h))^{1/m}}\widetilde{\sigma}(h)\sigma(n) \qquad \text{for all} \quad n \in N,\ h \in H_\sigma.$$

Then $\widetilde{\sigma}_1$ is again a projective representation of $G_\sigma = NH_\sigma$ which extends σ and the associated cocycle $c : G_\sigma \times G_\sigma \to \mathbf{S}^1$ factorises through $H_\sigma \times H_\sigma$; moreover, $\widetilde{\sigma}_1(h) \in SU(m)$ for every $h \in H_\sigma$.

Every $h \in H_\sigma$ induces a bijection φ_h of $\sigma(N)$ given by

$$\varphi_h : \sigma(n) \mapsto \widetilde{\sigma}_1(h)\sigma(n)\widetilde{\sigma}_1(h)^{-1} = \sigma(hnh^{-1}) \qquad \text{for all} \quad n \in N.$$

So, we have a map

$$\varphi : \widetilde{\sigma}_1(H_\sigma) \to \mathrm{Sym}(\sigma(N)), \quad \widetilde{\sigma}_1(h) \mapsto \varphi_h,$$

where $\mathrm{Sym}(\sigma(N))$ is the set of bijections of $\sigma(N)$. For $h_1, h_2 \in H_\sigma$, we have $\varphi_{h_1} = \varphi_{h_2}$ if and only if $\widetilde{\sigma}_1(h_2) = \lambda\widetilde{\sigma}_1(h_1)$ for some scalar $\lambda \in \mathbf{S}^1$, by irreducibility of σ. Since $\det(\widetilde{\sigma}_1(h_1)) = 1$ and $\det(\widetilde{\sigma}_1(h_2)) = 1$, it follows that λ is a mth root of unity. This shows that the fibers of the map φ are finite. Since $\sigma(N)$ is finite, $\mathrm{Sym}(\sigma(N))$ and hence $\widetilde{\sigma}_1(H_\sigma)$ is finite. It follows that $\widetilde{\sigma}_1(G_\sigma) = \widetilde{\sigma}_1(H_\sigma)\sigma(N)$ is finite.

Let $\tau : G_\sigma \to U(m)$ be the projective representation of G_σ given by

$$\tau(nh) = \overline{\widetilde{\sigma}_1}(h) \qquad \text{for all} \quad nh \in NH_\sigma.$$

Then $\widetilde{\sigma}_1 \otimes \tau$ is a ordinary representation of G_σ and has finite image. The induced representation $\rho := \mathrm{Ind}_{G_\sigma}^G(\widetilde{\sigma}_1 \otimes \tau)$ has finite image, since G_σ has finite index in G. As $\widetilde{\sigma}_1 \otimes \tau$ is equivalent to a subrepresentation of the restriction $\rho|_{G_\sigma}$ of ρ to G_σ, the representation σ is equivalent to a subrepresentation of $\rho|_N$. Since $\rho(G)$ has finite image, there exists a unitary representation $\widehat{\rho}$ of $\mathrm{Prof}(G)$ such that $\widehat{\rho} \circ \alpha_G = \rho$. So, there exists a subspace V of the space of $\widehat{\rho}$ which is invariant under $\alpha_G(N)$ and defining a representation of N which is equivalent to σ. Then V defines an irreducible representation $\widehat{\sigma}$ of L for which $\widehat{\sigma} \circ (\alpha_G|_N) = \sigma$ holds.

Let $\psi_N : \mathrm{Prof}(N) \to L$ be the homomorphism such that $\psi_N \circ \alpha_N = \alpha_G|_N$.

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(iii) *Third step.* We claim that

$$\mathrm{Ker}\psi_N = \bigcap_{\sigma \in \widehat{N}^{H-\mathrm{per}}_{\mathrm{finite}}} \mathrm{Ker}(\mathrm{Prof}(\sigma)).$$

Indeed, the proof is similar to the proof of the third step of Theorem A

(iv) *Fourth step.* We claim that the map $\psi_H : \mathrm{Prof}(H) \to \overline{\alpha_G(H)}$, defined by the relation $\varphi_H \circ \alpha_H = \alpha_G|_H$, is an isomorphism. Indeed, the proof is similar to the proof of the fourth step of Theorem A.

Every element $h \in H$ defines a continuous automorphism $\theta_p(h)$ of $\mathrm{Prof}(N)$. Let

$$\theta_p : H \to \mathrm{Aut}(\mathrm{Prof}(N))$$

be the corresponding homomorphism; as in Theorem A, we have an induced action $\overline{\theta_p}$ of H on $\mathrm{Prof}(N)/\mathrm{Ker}\psi_N$.

- *Fifth step.* We claim that the action $\overline{\theta_p}$ of H induces an action of $\mathrm{Prof}(H)$ by automorphisms on $\mathrm{Prof}(N)/\mathrm{Ker}\psi_N$ and that the map

$$(\mathrm{Prof}(N)/\mathrm{Ker}\psi_N) \rtimes \mathrm{Prof}(H) \to \mathrm{Prof}(G), (x\mathrm{Ker}\psi_N, y) \mapsto \psi_N(x)\psi_H(y)$$

is an isomorphism.

Indeed, the proof is similar to the proof of the fifth step of Theorem A.

4. *Proof of the Corollaries*

4·1. *Proof of Corollary* C

Assume that N is finitely generated. In view of Theorem B, we have to show that $\widehat{N}^{H-\mathrm{per}}_{\mathrm{finite}} = \widehat{N}_{\mathrm{finite}}$.

It is well known that, for every integer $n \geq 1$, there are only finitely many subgroups of index n in N. Indeed, since N is finitely generated, there are only finitely many actions of N on the set $\{1, \ldots, n\}$. Every subgroup M of index n defines an action of N on N/M and hence on $\{1, \ldots, n\}$ for which the stabiliser of, say, 1 is M. So, there are only finitely many such subgroups M.

Let $\sigma \in \widehat{N}_{\mathrm{finite}}$ and set $n := |\sigma(N)|$. Consider $N_\sigma = \cap_M M$, where M runs over the subgroups of N of index n. Then N_σ is a normal subgroup of N of finite index and, for every $h \in H$, the representation σ^h factorises to a representation of N/N_σ. Since N/N_σ is a finite group, it has only finitely many non equivalent irreducible representations and the claim is proved.

4·2. *Proof of Corollary* D

We assume that N is abelian. The dual group of $\mathrm{Bohr}(N)$ is $\widehat{N}$ and the dual of $\mathrm{Prof}(N)$ is $\widehat{N}_{\mathrm{finite}}$, viewed as discrete groups. With the notation as in Theorems A and B, the subgroups C and D are respectively the annihilators in $\mathrm{Bohr}(N)$ and in $\mathrm{Prof}(N)$ of the closed subgroups $\widehat{N}^{H-\mathrm{per}}$ and $\widehat{N}^{H-\mathrm{per}}_{\mathrm{finite}}$. Hence, $\mathrm{Bohr}(N)/C$ and $\mathrm{Prof}(N)/D$ are the dual groups of $\widehat{N}^{H-\mathrm{per}}$ and $\widehat{N}^{H-\mathrm{per}}_{\mathrm{finite}}$, viewed as discrete groups. So, the claim follows from Theorems A and B.

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4·3 *Proof of Corollary* E

In view of Theorems A and B, G is MAP, respectively RF, if and only if

$$\mathrm{Ker}(\varphi_N \circ \beta_N) = \{e\} \qquad \text{and} \qquad \mathrm{Ker}(\varphi_H \circ \beta_H) = \{e\},$$

respectively

$$\mathrm{Ker}(\psi_N \circ \alpha_N) = \{e\} \qquad \text{and} \qquad \mathrm{Ker}(\psi_H \circ \alpha_H) = \{e\}.$$

So, G is MAP, respectively RF, if and only if

$$\beta_N^{-1}(C) = \{e\} \qquad \text{and} \qquad \mathrm{Ker}(\beta_H) = \{e\},$$

respectively

$$\alpha_N^{-1}(D) = \{e\} \qquad \text{and} \qquad \mathrm{Ker}(\alpha_H) = \{e\}.$$

This exactly means that G is MAP, respectively RF, if and only if $\widehat{N}_{\mathrm{fd}}^{H-\mathrm{per}}$ separates the points of N and H is MAP, respectively $\widehat{N}_{\mathrm{finite}}^{H-\mathrm{per}}$ separates the points of N and H is RF.

4·4. *Proof of Corollary* F

We assume that $G = \Lambda \wr_X H$ is the wreath product of the groups Λ and H given by a transitive action $H \curvearrowright X$; set $N := \oplus_{x\in X}\Lambda$.

(a) Assume that X is finite. Then, of course, $\widehat{N}_{\mathrm{fd}}^{H-\mathrm{per}} = \widehat{N}_{\mathrm{fd}}$ and $\widehat{N}_{\mathrm{finite}}^{H-\mathrm{per}} = \widehat{N}_{\mathrm{finite}}$; so, the subgroups C and D from Theorems A and B are trivial. Since $\mathrm{Bohr}(N) = \oplus_{x\in X}\mathrm{Bohr}(\Lambda)$ and $\mathrm{Prof}(N) = \oplus_{x\in X}\mathrm{Prof}(\Lambda)$, we have

$$\mathrm{Bohr}(\Lambda \wr_X H) \cong (\oplus_{x\in X}\mathrm{Bohr}(\Lambda)) \rtimes \mathrm{Bohr}(H) \text{ and}$$
$$\mathrm{Prof}(\Lambda \wr_X H) \cong (\oplus_{x\in X}\mathrm{Prof}(\Lambda)) \rtimes \mathrm{Prof}(H).$$

(b) Assume that X is infinite.

(i) *First step*. We claim that, for every $\sigma \in \widehat{N}_{\mathrm{fd}}^{H-\mathrm{per}}$, we have $\dim\sigma = 1$, that is, $\sigma(N) \subset U(1) = \mathbf{S}^1$.

Indeed, assume by contradiction that $\dim\sigma > 1$. Let $\mathcal{F}$ be the family of finite subsets of X. For every $F \in \mathcal{F}$, let $N(F)$ be the normal subgroup of N given by

$$N(F) := \oplus_{x\in F}\Lambda$$

The restriction $\sigma|_{N(F)}$ of σ to $N(F)$ has a decomposition into isotypical components:

$$\sigma|_{N(F)} = \oplus_{\pi\in\Sigma_F} n_\pi \pi,$$

where Σ_F is a (finite) subset of $\widehat{N(F)}_{\mathrm{fd}}$ and the n_π's some positive integers. As is well known (see, e.g., [**Wei40**, section 17]), every representation in $\widehat{N(F)}_{\mathrm{fd}}$ is a tensor product $\otimes_{h\in F}\rho_h$ of irreducible representations ρ_h of Λ; so, we can view Σ_F as subset of $\prod_{x\in F}\widehat{\Lambda}_{\mathrm{fd}}$. If $F \subset F'$, then the obvious map $\prod_{x\in F'}\widehat{\Lambda}_{\mathrm{fd}} \to \prod_{x\in F}\widehat{\Lambda}_{\mathrm{fd}}$ restricts to a surjective map $\Sigma_{F'} \to \Sigma_F$.

Since $\dim \sigma$ is finite, it follows that there exists $F_0 \in \mathcal{F}$ such that

$$\dim \pi = 1 \qquad \text{for all} \quad \pi \in \Sigma_F, F \in \mathcal{F} \quad \text{with} \quad F \cap F_0 = \emptyset$$

and

$$\dim \pi_0 > 1 \quad \text{for some} \quad \pi_0 \in \Sigma_{F_0}.$$

For $h \in H$ and $F \in \mathcal{F}$, observe that for the decomposition of $\sigma^h|_{N(h^{-1}F)}$ into isotypical components, we have

$$\sigma^h|_{N(h^{-1}F)} = \oplus_{\pi \in \Sigma_F} n_\pi \pi.$$

So, σ^h and σ are not equivalent if $h^{-1}F_0 \cap F_0 = \emptyset$.

Since X is infinite, we can choose inductively a sequence $(h_n)_{n \geq 0}$ of elements in H by $h_0 = e$ and

$$h_{n+1}^{-1} F_0 \cap \bigcup_{0 \leq m \leq n} h_m^{-1} F_0 = \emptyset \qquad \text{for all} \quad n \geq 0.$$

The σ^{h_n}'s are then pairwise not equivalent. This is a contradiction, since $\sigma \in \widehat{N}_{\rm fd}^{H-{\rm per}}$.

Let $p: \Lambda \wr_X H \to \Lambda^{\rm Ab} \wr_X H$ be the quotient map, which is given by

$$p((\lambda_x)_{x \in X}, h) = ((\lambda_x [\Lambda, \Lambda])_{x \in X}, h).$$

(ii) *Second step.* We claim that the induced maps

$$\mathrm{Bohr}(p) : \mathrm{Bohr}(\Lambda \wr_X H) \to \mathrm{Bohr}(\Lambda^{\rm Ab} \wr_X H)$$

and

$$\mathrm{Prof}(p) : \mathrm{Prof}(\Lambda \wr_X H) \to \mathrm{Prof}(\Lambda^{\rm Ab} \wr_X H)$$

are isomorphisms.

Indeed, by the first step, every $\sigma \in \widehat{N}_{\rm fd}^{H-{\rm per}}$ factorises through $N^{\rm Ab}$. Hence, by Theorems A and B, $[N, N]$ is contained in $C = \ker \varphi_N$ and $[N, N]$ is contained in $D = \ker \psi_N$. This means that $\beta_G(\ker p) = \{e\}$ and $\alpha_G(\ker p) = \{e\}$. The claim follows then from Proposition 7.

4·5. *Proof of Corollary* G

We assume that $G = \Lambda \wr_X H$ is the wreath product of the groups Λ and H given by an action $H \curvearrowright X$. We assume that Λ has at least two elements and, as before, we set $N = \oplus_{x \in X} \Lambda$.

(a) Assume that X is finite. Then G is MAP (respectively RF) if and only if Λ and H are MAP (respectively RF).

Indeed, $\widehat{N}_{\rm fd}^{H-{\rm per}} = \widehat{N}_{\rm fd}$ separates the points of N if and only if Λ is MAP and $\widehat{N}_{\rm finite}^{H-{\rm per}} = \widehat{N}_{\rm finite}$ separates the points of N if and only if Λ is RF. The claim follows then from Corollary E.

(b) Assume that X is infinite.

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Assume that G is MAP. Then, for every H-orbit Y in X, the wreath product $\Lambda \wr_Y H$, which embeds as subgroup of G, is MAP. Since some Y is infinite, Corollary F implies that Λ is abelian. So, we may and will from now assume that Λ (and hence N) is abelian.

(i) *First step.* We claim that, if $\widehat{N}^{H-\mathrm{per}}$ separates the points of N, then $H \curvearrowright X$ is RF.

Indeed, recall that the dual group $\widehat{\Lambda}$ of Λ, equipped with the topology of pointwise convergence, is a compact group. The dual group $\widehat{N}$ of N can be identified, as topological group, with the product group $\prod_{x\in X} \widehat{\Lambda}$, endowed with the product topology, by means of the duality

$$\left\langle \prod_{x\in X} \chi_x, \oplus_{x\in X}\lambda_x \right\rangle = \prod_{x\in X} \chi_x(\lambda_x) \quad \text{for all} \quad \prod_{x\in X} \chi_x \in \widehat{N}, \oplus_{x\in X}\lambda_x \in N.$$

(Observe that the product on the right hand side is well-defined since $\lambda_x = e$ for all but finitely many $x \in X$.) The dual action of H on $\widehat{N}$ is given by

$$\left(\prod_{x\in X} \chi_x\right)^h = \prod_{x\in X} \chi_{h^{-1}x} \qquad \text{for all} \quad h \in H.$$

For $\Phi := \prod_{x\in X} \chi_x \in \widehat{N}$, we have that $\Phi \in \widehat{N}^{H-\mathrm{per}}$ if and only if there exists a finite index subgroup H_Φ of H such that

$$\chi_{hx} = \chi_x \qquad \text{for all} \quad h \in H_\Phi, x \in X.$$

Let x_0, x_1 be two distinct points from X. By assumption, $\widehat{N}^{H-\mathrm{per}}$ separates the points of N; equivalently, $\widehat{N}^{H-\mathrm{per}}$ is dense in $\widehat{N}$. Since Λ has at least two elements, we can find $\chi^0 \in \widehat{\Lambda}$ and $\lambda_0 \in \Lambda$ with $\chi^0(\lambda_0) \neq 1$. Define $\Phi_0 = \prod_{x\in X} \chi_x \in \widehat{N}$ by $\chi_{x_0} = \chi^0$ and $\chi_x = 1_\Lambda$ for $x \neq x_0$. Set

$$\varepsilon := \frac{1}{2}\left|\chi^0(\lambda_0) - 1\right| > 0.$$

Since $\widehat{N}^{H-\mathrm{per}}$ is dense in $\widehat{N}$, we can find $\Phi' = \prod_{x\in X} \chi'_x \in \widehat{N}^{H-\mathrm{per}}$ such that

$$|\chi'_{x_0}(\lambda_0) - \chi_{x_0}(\lambda_0)| \leq \varepsilon/2 \quad \text{and} \quad |\chi'_{x_1}(\lambda_0) - \chi_{x_1}(\lambda_0)| \leq \varepsilon/2.$$

We claim that $H_{\Phi'}x_0 \neq H_{\Phi'}x_1$, where $H_{\Phi'}$ is the stabiliser of Φ'. Indeed, assume by contradiction that $x_0 \in H_{\Phi'}x_1$. Then $\chi'_{x_0} = \chi'_{x_1}$ and hence

$$\begin{aligned}
2\varepsilon &= |\chi^0(\lambda_0) - 1| \\
&\leq |\chi^0(\lambda_0) - \chi'_{x_0}(\lambda_0)| + |\chi'_{x_0}(\lambda_0) - 1| \\
&= |\chi_{x_0}(\lambda_0) - \chi'_{x_0}(\lambda_0)| + |\chi'_{x_1}(\lambda_0) - \chi_{x_1}(\lambda_0)| \\
&\leq \varepsilon
\end{aligned}$$

and this is a contradiction. Since $H_{\Phi'}$ has finite index, we have proved that $H \curvearrowright X$ is RF.

(ii) *Second step.* We claim that, if $H \curvearrowright X$ is RF, then $\widehat{N}^{H-\mathrm{per}}$ separates the points of N.

Indeed, let $\oplus_{x\in X}\lambda_x \in N \setminus \{e\}$. Then $F = \{x \in X : \lambda_x \neq e\}$ is a finite and non-empty subset of X. Let $(\chi^0_x)_{x\in F}$ be a sequence in $\widehat{\Lambda}$ such that $\prod_{x\in F} \chi^0_x(\lambda_x) \neq 1$ (this is possible, since

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abelian groups are MAP). Since $H \curvearrowright X$ is RF, we can find a subgroup of finite index L of H so that $Lx \neq Lx'$ for all $x, x' \in F$ with $x \neq x'$. Define $\Phi = \prod_{x' \in X} \chi_{x'} \in \widehat{N}$ by

$$\chi_{x'} = \begin{cases} \chi_x^0 & \text{if} \quad x' \in Lx \quad \text{for some} \quad x \in F, \\ 1_\Lambda & \text{if} \quad x' \notin \cup_{x \in F} Lh. \end{cases}$$

It is clear that $L \subset H_\Phi$ and hence that $\Phi \in \widehat{N}^{H-\mathrm{per}}$; moreover,

$$\Phi\left(\oplus_{x \in X} \lambda_x\right) = \prod_{x \in F} \chi_x^0(\lambda_x) \neq 1.$$

So, $\widehat{N}^{H-\mathrm{per}}$ separates the points of N.

(iii) *Third step*. We claim that, if $H \curvearrowright X$ is RF and Λ is RF, then $\widehat{N}^{H-\mathrm{per}}_{\mathrm{finite}}$ separates the points of N.

The proof is the same as the proof of the second step, with only one difference: one has to choose a sequence $(\chi_x^0)_{x \in F}$ in $\widehat{\Lambda}_{\mathrm{finite}}$ such that $\prod_{x \in F} \chi_x^0(\lambda_x) \neq 1$; this is possible, since we are assuming that Λ is RF.

(iv) *Fourth step*. We claim that G is MAP if and only if H is RF and $H \curvearrowright X$ is RF. Indeed, this follows from Corollary E, combined with the first and second steps.

(v) *Fifth step*. We claim that G is RF if and only if Λ, H are RF and $H \curvearrowright X$ is RF. Indeed, this follows from Corollary E, combined with the first and third steps.

5. *Examples*

5·1. *Lamplighter group*

For $m \geq 1$, denote by C_m the finite cyclic group $\mathbf{Z}/m\mathbf{Z}$. Recall that

$$\mathrm{Bohr}(\mathbf{Z}) \cong \mathrm{Bohr}(\mathbf{Z})_0 \oplus \mathrm{Prof}(\mathbf{Z}).$$

and that

$$\mathrm{Prof}(\mathbf{Z}) = \varprojlim_m C_m \quad \text{and} \quad \mathrm{Bohr}(\mathbf{Z})_0 \cong \prod_{\omega \in \mathfrak{c}} \mathbf{A}/\mathbf{Q},$$

where $\mathbf{A}/\mathbf{Q}$ is the ring of adeles of $\mathbf{Q}$ and $\mathfrak{c} = 2^{\aleph_0}$ (see [**Bek23**, proposition 11]).

For an integer $n_0 \geq 2$, let $G = C_{n_0} \wr \mathbf{Z}$ be the lamplighter group. We claim that

$$\mathrm{Bohr}(G) \cong \mathrm{Bohr}(\mathbf{Z})_0 \times \mathrm{Prof}(G)$$

and

$$\mathrm{Prof}(G) = \varprojlim_m C_{n_0} \wr C_m.$$

Indeed, let $N := \oplus_{k \in \mathbf{Z}} C_{n_0}$. It will be convenient to describe N as the set of maps $f : \mathbf{Z} \to C_{n_0}$ such that $\mathrm{supp}(f) := \{k \in \mathbf{Z} : f(k) \neq 0\}$ is at most finite. The action of $m \in \mathbf{Z}$ on $f \in N$ is given by translation: $f^m(k) = f(k+m)$ for all $k \in \mathbf{Z}$.

We identify $\widehat{C_{n_0}}$ with the group μ_{n_0} of n_0-th roots of unity in $\mathbf{C}$ by means of the duality

$$\langle z, k\mathbf{Z} \rangle = z^k \qquad \text{for all} \quad z \in \mu_{n_0}, k \in \mathbf{Z}.$$

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Then $\widehat{N}$ can be identified with the set of maps $\Phi : \mathbf{Z} \to \mu_{n_0}$, with duality given by

$$\langle \Phi, f\rangle = \prod_{k\in\mathbf{Z}} \langle \Phi(k), f(k)\rangle \qquad \text{for all} \quad \Phi \in \widehat{N}, f \in N.$$

Observe that $\Phi(N) \subset \mu_{n_0}$ and so $\widehat{N} = \widehat{N}_{\text{finite}}$.

We have $\widehat{N}^{H-\text{per}} = \bigcup_{m\geq 1} \widehat{N}(m)$, where $\widehat{N}(m)$ is the subgroup

$$\widehat{N}(m) = \left\{\Phi : \mathbf{Z} \to \mu_{n_0} : \Phi(k+m) = \Phi(k) \quad \text{for all} \quad k \in \mathbf{Z}\right\}.$$

Observe that we have natural injections $i_{m_2}^{m_1} : \widehat{N}(m_2) \to \widehat{N}(m_1)$ if m_1 is a multiple of m_2. The dual group $A(m)$ of $\widehat{N}(m)$ can be identified with the set of maps $\overline{f} : C_m \to C_{n_0}$ by means of the duality

$$\langle \overline{f}, \Phi\rangle = \prod_{k+m\mathbf{Z}\in C_m} \Phi(k)^{\overline{f}(k+m\mathbf{Z})} \qquad \text{for all} \quad \Phi \in \widehat{N}(m), \overline{f} \in A(m).$$

If m_1 is a multiple of m_2, we have a projection $p_{m_1}^{m_2} : A(m_1) \to A(m_2)$ given by

$$\langle p_{m_1}^{m_2}(\overline{f}), \Phi\rangle = \langle \overline{f}, \Phi \circ i_{m_2}^{m_1}\rangle.$$

The dual group A of $\widehat{N}^{H-\text{per}} = \bigcup_{m\geq 1} \widehat{N}(m)$ can then be identified with the projective limit $\varprojlim_m A(m)$.

The action of $\mathbf{Z}$ by automorphisms of A is given, for $r \in \mathbf{Z}$ and $\overline{f} = (\overline{f}_m)_{m\geq 1} \in A$ by $(\overline{f})^r = (\overline{g}_m)_{m\geq 1}$, where

$$\overline{g}_m(k + m\mathbf{Z}) = \overline{f}_m(k + r + m\mathbf{Z}) \qquad \text{for all} \quad k \in \mathbf{Z}.$$

This action extends to an action of $\text{Proj}(\mathbf{Z}) = \varprojlim_m C_m$ by automorphisms on A in an obvious way. By Corollary D, the group $\text{Prof}(G)$ is isomorphic to the corresponding semi-direct product $A \rtimes \text{Prof}(\mathbf{Z})$ and hence

$$\text{Prof}(G) \cong \varprojlim_m C_{n_0} \wr C_m.$$

By Corollary D again, the action of $\mathbf{Z}$ on A extends to an action by automorphisms of $\text{Bohr}(\mathbf{Z})$. Since $\text{Bohr}(\mathbf{Z})_0$ is connected and A is totally disconnected, $\text{Bohr}(\mathbf{Z})_0$ acts as the identity on A. Since $\text{Bohr}(\mathbf{Z}) \cong \text{Bohr}(\mathbf{Z})_0 \times \text{Prof}(\mathbf{Z})$, it follows that

$$\text{Bohr}(G) \cong (A \rtimes \text{Proj}(\mathbf{Z})) \times \text{Bohr}(\mathbf{Z})_0 \cong \text{Prof}(G) \times \text{Bohr}(\mathbf{Z})_0.$$

For another description of $\text{Prof}(G)$, see [**GK14**, lemma 3·24].

5·2. *Heisenberg group*

Let R be a commutative unital ring. The Heisenberg group is the group

$$H(R) := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in R \right\}.$$

We can and will identify $H(R)$ with R^3, equipped with the group law

$$(a, b, c)(a', b', c') = (a + a', b + b', c + c' + ab').$$

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We will equip R with the discrete topology; in the sequel, $\mathrm{Bohr}(R)$, $\mathrm{Prof}(R)$, and $\widehat{R}$ will be the Bohr compactification, the profinite completion, and the dual group of $(R,+)$, the additive group of R.

Let $\mathcal{I}_{\text{finite}}$ be the family of *ideals* of the ring R with *finite* index (as subgroups of $(R,+)$). Every ideal I from $\mathcal{I}_{\text{finite}}$ defines two compact groups $H(\mathrm{Bohr}(R),I)$ and $H(\mathrm{Prof}(R),I)$ of Heisenberg type as follows:

$$H(\mathrm{Bohr}(R),I) := \mathrm{Bohr}(R)\times \mathrm{Bohr}(R)\times (R/I)$$

is equipped with the group law

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z'+p_I(x)p_I(y')),$$

where $p_I : \mathrm{Bohr}(R)\to R/I$ is the group homomorphism induced by the canonical map $R\to R/I$; the group $H(\mathrm{Prof}(R),I)$ is defined in a similar way.

Observe that, for two ideals I and J in $\mathcal{I}_{\text{finite}}$ with $J\subset I$, we have natural epimorphisms

$$H(\mathrm{Bohr}(R),J)\to H(\mathrm{Bohr}(R),I) \qquad \text{and} \qquad H(\mathrm{Prof}(R),J)\to H(\mathrm{Prof}(R),I).$$

We claim that the canonical maps $H(R)\to H(\mathrm{Bohr}(R),I)$ and $H(R)\to H(\mathrm{Prof}(R),I)$ induce isomorphisms

$$\mathrm{Bohr}(H(R))\cong \varprojlim_I H(\mathrm{Bohr}(R),I)$$

and

$$\mathrm{Prof}(H(R))\cong \varprojlim_I H(\mathrm{Prof}(R),I),$$

where I runs over $\mathcal{I}_{\text{finite}}$.

Indeed, $H(R)$ is a semi-direct product $N\rtimes H$ for

$$N=\{(0,b,c) : b,c\in R\}\cong R^2$$

and

$$H=\{(a,0,0) : a\in R\}\cong R.$$

Let $\chi\in\widehat{N}$. Then $\chi=\chi_{\beta,\psi}$ for a unique pair $(\beta,\psi)\in(\widehat{R})^2$, where $\chi_{\beta,\psi}$ is defined by

$$\chi_{\beta,\psi}(0,b,c) = \beta(b)\psi(c) \qquad \text{for } b,c\in R.$$

For $h=(a,0,0)\in H$, we have

$$\chi_{\beta,\psi}^h(0,b,c) = \beta(b)\psi(a^{-1}b)\psi(c) = \chi_{\beta\psi^a,\psi}(0,b,c) \qquad \text{for } b,c\in R,$$

where $\psi^a\in\widehat{R}$ is defined by $\psi^a(b)=\psi(a^{-1}b)$ for $b\in R$. It follows that the H-orbit of $\chi_{\beta,\psi}$ is

$$\{\chi_{\beta\psi^a,\psi} : a\in R\},$$

and that the stabiliser of $\chi_{\beta,\psi}$, which only depends on ψ, is

$$H_\psi = \{(a,0,0)\mid a\in I_\psi\},$$

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where I_ψ is the ideal of R defined by

$$I_\psi = \{a \in R \mid aR \subset \ker \psi\}.$$

Let $\widehat{R}_{\mathrm{per}}$ be the subgroup of all $\psi \in \widehat{R}$ which factorises through a quotient R/I for an ideal $I \in \mathcal{I}_{\mathrm{finite}}$. It follows that

$$\widehat{N}^{H-\mathrm{per}} = \{\chi_{\beta,\psi} : \beta \in \widehat{R}, \psi \in \widehat{R}_{\mathrm{per}}\} \cong \widehat{R} \times \widehat{R}_{\mathrm{per}}.$$

The dual group of $\widehat{R}_{\mathrm{per}}$ can be identified with $\varprojlim_I R/I$, where I runs over $\mathcal{I}_{\mathrm{finite}}$. So, the dual group A of $\widehat{N}^{H-\mathrm{per}}$ can be identified with $\varprojlim_I \mathrm{Bohr}(R) \times (R/I)$.

The action of $\mathrm{Bohr}(H) \cong \mathrm{Bohr}(R)$ on every $\mathrm{Bohr}(R) \times (R/I)$ is given by

$$x \cdot (y, z) = (y, z + p_I(x)p_I(y')) \qquad \text{for all} \quad x, y \in \mathrm{Bohr}(R), z \in R/I,$$

for the natural map $p_I : \mathrm{Bohr}(R) \to R/I$. This shows that

$$\mathrm{Bohr}(H(R)) \cong \varprojlim_I H(\mathrm{Bohr}(R), I).$$

Similarly, the dual group B of $\widehat{N}^{H-\mathrm{per}}_{\mathrm{finite}}$ can be identified with $\varprojlim_I \mathrm{Prof}(R) \times (R/I)$ and we have

$$\mathrm{Prof}(H(R)) \cong \varprojlim_I H(\mathrm{Prof}(R), I).$$

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