Can. J. Math., Vol. XL, No. 3, 1988, pp. 742-749

CONJUGATES OF INFINITE MEASURE PRESERVING TRANSFORMATIONS

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1. Introduction. In this paper we consider a question concerning the conjugacy class of an arbitrary ergodic automorphism σ of a sigma finite Lebesgue space $(X, \mathscr{A}, \mu)$ (i.e., σ is a μ-preserving bimeasurable bijection of $(X, \mathscr{A}, \mu)$). Specifically we prove

THEOREM 1. *Let τ, σ be any pair of ergodic automorphisms of an infinite sigma finite Lebesgue space $(X, \mathscr{A}, \mu)$. Let F be any measurable set such that*

$$\mu(X - (F \cup \tau F)) = \infty.$$

Then there is some conjugate σ' of σ such that $\sigma'(x) = \tau(x)$ for μ-almost every x in F.

The requirement that $F \cup \tau F$ has a complement of infinite measure is, for example, satisfied when F has finite measure, and in that case, the theorem was proved by Choksi and Kakutani ([7], Theorem 6).

Conjugacy theorems of this nature have proved to be very useful in proving approximation results in ergodic theory. These conjugacy results all assert the denseness of the conjugacy class of an ergodic (or antiperiodic) automorphism in various topologies and subspaces. Furthermore, these results are closely linked to the basic Kakutani-Rohlin tower picture for an (antiperiodic) automorphism. We indicate for the reader some of the conjugacy theorems previously considered: Halmos's conjugacy lemma ([**9**], p. 77) and Alpern's various topological and measure theoretic extensions of Halmos's result ([**2**], [**3**]) are concerned with finite measure preserving systems; Chacon and Friedman (see [**8**], p. 111) obtain an analogue for nonsingular automorphisms of a Lebesgue probability space (see also [7] Theorem 2); [**11**], and ([7], Theorem 6) consider analogues of the Halmos lemma for infinite measure preserving systems. As mentioned above, the latter result is a special case of Theorem 1.

Theorem 1 can also be viewed as an extension problem for measure preserving transformations. Let $G(X, \mu)$ denote the group of all μ-preserving bijections of $(X, \mathscr{A}, \mu)$. Suppose F is a measurable set and $\bar{\tau}:F \rightarrow X$ is a μ-preserving injection. The "extension problem" asks

Received November 12, 1987. The last two authors acknowledge support from grants from NSERC. The second author was also supported by a grant from FCAR, Quebec.

what we can say, up to conjugacy in $G(X, \mu)$, about transformations τ in $G(X, \mu)$ which extend $\bar{\tau}$. Theorem 1 shows that when

$$\mu(X - (F \cup \bar{\tau}F)) = \infty$$

and there are no non-null $\bar{\tau}$-invariant subsets of F (so that $\bar{\tau}$ acts ergodically on F), then there exist extensions of $\bar{\tau}$ which are conjugate to any given ergodic transformation.

The conjugacy result of Theorem 1 is used in Section 3 to prove the existence of weak mixing homeomorphisms preserving a nonatomic locally positive sigma finite Borel measure μ on a sigma compact connected n-manifold M, $n \geqq 2$ (an infinite measure preserving system is weak mixing if its Cartesian square is ergodic). This answers the main question left open in **[6]** where the authors obtained a sufficient condition for the existence of weak mixing homeomorphisms; a condition on M which is not always satisfied. In fact, Theorem 2 shows that the sufficient condition is not necessary. Our methods show much more; namely that weak mixing can be replaced by any measure theoretic property that is "typical" (forms a dense G_δ in the coarse topology) for measure preserving automorphisms of the underlying measure space (M, μ).

2. The conjugacy theorem. Denote by $G(X, \mu)$ the group of all automorphisms (bimeasurable μ-preserving bijections) of the infinite sigma finite nonatomic Lebesgue space $(X, \mathscr{A}, \mu)$. Measure theoretically $(X, \mathscr{A}, \mu)$ is just the real line with the Lebesgue measurable sets and Lebesgue measure. An automorphism $\sigma \in G(X, \mu)$ is *ergodic* if there is no nontrivial σ-invariant set; i.e., $\sigma(A) = A$ implies either $\mu(A) = 0$ or $\mu(X - A) = 0$.

LEMMA 1. *Let $\sigma \in G(X, \mu)$ be an ergodic automorphism and let m be any given natural number. For each measurable S with $\mu(S) < \infty$, and extended nonnegative real number ρ, $0 < \rho \leqq \infty$, there is a measurable subset $R \subseteq X$ such that $\mu(R) = \rho$ and $S, R, \sigma R, \dots, \sigma^{m-1}R$ are pairwise disjoint.*

Proof. We may assume that $\rho = \infty$, since a set of infinite measure contains subsets of arbitrary finite measure.

Define $S_0 = S$ and $S_i = \sigma S_{i-1} - S_0$ for $i = 1, 2, \dots$. The S_i are disjoint, and since σ is ergodic, $\mu(S_i)$ decrease to 0 and

$$X = \bigcup_{i=0}^{\infty} S_i;$$

i.e., we can view X as a skyscraper for σ built on $S_0 = S$ as base and having S_i as the $(i + 1)^{\text{st}}$ floor. For fixed integer m, $m > 0$ set

$$R = \bigcup_{i=1}^{\infty} \sigma^{-m+1} S_{i \cdot m}.$$

https://doi.org/10.4153/CJM-1988-032-1 Published online by Cambridge University Press

Note that R consists of those points in $X - S$ whose next $m - 1$ iterates do not enter S. From this it is clear that $S, R, \sigma R, \ldots, \sigma^{m-1}R$ are disjoint. Furthermore, since

$$\mu(X - (S \cup R \cup \sigma R \cup \ldots \cup \sigma^{m-1}R))$$
$$\leqq \mu(S \cup \sigma^{-1}S \cup \ldots \cup \sigma^{-m+1}S) \leqq m\mu(S) < \infty,$$

we obtain $\mu(R) = \infty$ as required.

We use this lemma repeatedly to prove

LEMMA 2. *Let $\sigma \in G(X, \mu)$ be an ergodic automorphism and let $(p_1, p_2, \ldots)$ be any denumerable distribution of extended nonnegative real numbers p_i $(0 \leqq p_i \leqq \infty)$ with $p_1 = \infty$. Then there is a partition of X,*

$$\{E_{i,j}: i = 1, 2, \ldots, j = 1, 2, \ldots, i\}$$

satisfying

1) $E_{i,j} = \sigma^{j-1}E_{i,1}$ *for* $i = 1, 2, \ldots$ *and* $j = 1, 2, \ldots, i$, *and*
2) $\mu(E_{i,j}) = p_i$ *for each* i.

Proof. First consider the case that $p_i < \infty$ for all $i \geqq 2$. Apply Lemma 1 to $S_2 = \emptyset$, $\rho_2 = p_2$ and $m_2 = 2$ to obtain a set R_2 with $\mu(R_2) = \rho_2$ such that $S_2, R_2, \sigma R_2$ are pairwise disjoint sets. Proceed inductively as follows. For $i \geqq 3$ let B_{i-1} be any set of measure 1 disjoint from the (finite measured) set

$$S^*_{i-1} = S_{i-1} \cup R_{i-1} \cup \sigma R_{i-1} \cup \ldots \cup \sigma^{i-2}R_{i-1}.$$

Apply Lemma 1 to the set

$$S_i = S^*_{i-1} \cup B_{i-1}, \quad \rho_i = p_i \text{ and } m_i = i,$$

to obtain a set R_i with $S_i, R_i, \sigma R_i, \ldots, \sigma^{i-1}R_i$ pairwise disjoint and $\mu(R_i) = p_i$. For $i \geqq 2$ and $1 \leqq j \leqq i$ define $E_{i,j} = \sigma^{j-1}R_i$, and set

$$E_{1,1} = X - \bigcup_{i=2}^{\infty} \bigcup_{j=1}^{i} E_{i,j}.$$

Since $E_{1,1} \supseteq \bigcup_{i=2}^{\infty} B_i$,

$$\mu(E_{1,1}) = p_1 = \infty.$$

Now we consider the general case where any p_i may be infinite. Let (m_k, ρ_k) be a sequence of pairs where m_k is an integer, $m_k \geqq 2$, and ρ_k is a finite nonnegative number such that for each $i \geqq 2$,

$$\sum_{\{k: m_k = i\}} \rho_k = p_i.$$

As before, use Lemma 1 to construct sets R_k with $\mu(R_k) = \rho_k$ and

$R_k, \sigma R_k, \ldots, \sigma^{m_k - 1} R_k$ are pairwise disjoint and disjoint from S_k where

$$S_k = S^*_{k-1} \cup B_{k-1}$$

with S^*_{k-1} the union of the previous R's and S's and B_{k-1} a set of measure 1 disjoint from S^*_{k-1}. Now for $i \geqq 2$, set

$$E_{i,1} = \bigcup_{\{k:m_k = i\}} R_k \quad \text{and} \quad E_{i,j} = \sigma^{j-1} E_{i,1}$$

for $j = 1, 2, \ldots, i$. As before

$$E_{1,1} = X - \bigcup_{i=2}^{\infty} \bigcup_{j=1}^{i} E_{i,j}$$

and $\{E_{i,j}: i \geqq 1, 1 \leqq j \leqq i\}$ is the required partition of X.

Theorem 1 is now an easy consequence of this lemma.

THEOREM 1. *Let $\sigma, \tau \in G(X, \mu)$ be any two ergodic automorphisms of the infinite sigma finite nonatomic Lebesgue space $(X, \mathcal{A}, \mu)$. Let F be any measurable subset of X such that*

$$\mu(X - (F \cup \tau F)) = \infty.$$

Then there is an automorphism $\pi \in G(X, \mu)$ such that the conjugate $\sigma' = \pi^{-1} \sigma \pi$ of σ satisfies $\sigma'(x) = \tau(x)$ for μ-almost every x in F.

Proof. For each $i \geqq 2$, let

$$F_{i,1} = \{x \in F: \tau^{-1} x \notin F, x \in F, \tau x \in F, \ldots, \tau^{i-2} x \in F, \tau^{i-1} x \notin F\}.$$

For $j = 1, 2, \ldots, i - 1$, define

$$F_{i,j+1} = \tau F_{i,j}.$$

Then clearly these sets $\{F_{i,j}: i = 2, 3, \ldots, j = 1, 2, \ldots, i\}$ are disjoint and

$$F = \bigcup_{i=2}^{\infty} \bigcup_{j=1}^{i-1} F_{i,j}, \quad \tau F = \bigcup_{i=2}^{\infty} \bigcup_{j=2}^{i} F_{i,j}$$

(see [7] Theorem 6, Step 1). Define

$$F_{1,1} = X - (F \cup \tau F).$$

For each $i \geqq 1$, define $p_i = \mu(F_{i,j})$. Apply Lemma 2 to σ and the denumerable distribution $(p_1, p_2, \ldots)$. Define $\pi \in G(X, \mu)$ as follows: on $F_{i,1}$ for each $i \geqq 1$ set π to be any μ-preserving bijection from $F_{i,1}$ onto $E_{i,1}$; for each $i \geqq 2$ and $j > 1$, define $\pi: F_{i,j} \rightarrow E_{i,j}$ by

$$\pi(x) = \sigma^{j-1} \pi \tau^{-(j-1)}(x) \quad \text{for } x \in F_{i,j}.$$

By construction $\pi^{-1}\sigma\pi(x) = \tau(x)$ for μ-a.e. x in F.

3. Application to measure preserving homeomorphisms. Let $H = H(M, \mu)$ denote the group of all homeomorphisms of a σ-compact connected n-manifold M ($n > 1$) which preserve an infinite sigma finite, nonatomic locally positive and locally finite Borel measure μ. In this section we use the conjugacy theorem of the previous section to show that for any such manifold M, $H(M, \mu)$ always contains a homeomorphism h which is weak mixing (by which we mean $h \times h$ is ergodic). This answers the main open question in [**6**], where a sufficient condition for the existence of weak mixing homeomorphisms is established. That condition asserts that for the end compactification of M, some h in H induces an action on the ends (see below) which is topologically weak mixing on the ends of infinite measure. This condition, however is not always satisfied; e.g. for manifolds which have more than one but finitely many ends, no action on the ends can be topologically weak mixing.

In contrast to that condition we will show that every manifold M supports a weak mixing homeomorphism which is end preserving. Actually our methods apply equally well to any "typical property" $\mathscr{V}$, that is to any conjugate invariant property which constitutes a dense G_δ subset of $G(M, \mu)$, the group of all automorphisms endowed with the coarse topology, of the underlying measure space (M, μ). Weak mixing is such a property. We now proceed with the definition of these terms.

Each end of the manifold corresponds, roughly, to "a distinct way of going to infinity" on M. Formally an *end* of the manifold M is a map e which assigns to every compact subset K of M a connected unbounded component of $M - K$, in such a way that $K_1 \subset K_2$ implies $e(K_2) \subset e(K_1)$. The set of all ends is denoted by E. The manifold M is compactified by adjoining E, and defining suitable neighborhoods for the ends (see [**1**] or [**6**] for further details). Every homeomorphism $h \in H(M, \mu)$ induces a homeomorphism h^* of E defined by

$$(h^*e)(K) = h(e(h^{-1}(K)))$$

for each end $e \in E$ and compact subset K of M. A homeomorphism h is *end preserving* if h^* is the identity. An end preserving homeomorphism has *zero charge* if for each compact K,

$$\mu(e(K) - he(K)) - \mu(he(K) - e(K)) = 0 \quad \text{for all ends } e \in E.$$

Denote by H^0 the subset of $H(M, \mu)$ of zero charge end preserving homeomorphisms (M, μ) onto itself. With the compact open topology H^0 is a Baire space. Let d be a metric compatible with the topology on M.

The following proposition follows easily from Theorem 1.

PROPOSITION 1. *Let h be an ergodic end preserving homeomorphism in $H(M, \mu)$. Let K be any compact subset of M and B any connected component of $M - K$ of infinite measure. Let σ be any ergodic automorphism of the underlying measure space (M, μ) (i.e., $\sigma \in G(M, \mu)$). Then there is an automorphism $f \in G(M, \mu)$ conjugate to σ such that $f(x) = h(x)$ for* μ-a.e. *x in $M - B$. In particular $f = h$* μ-a.e. *on K and $f(A) = h(A)$ for every component A of $M - K$.*

Proof. Since h is end preserving and $\mu(B) = \infty$ it follows that

$$\mu(B \cap hB) = \infty$$

(see [**5**] Proposition 1 where it is shown that $\mu(A \cap hB) < \infty$ for all components A of $M - K$ with $A \neq B$). Apply Theorem 1 to $\tau = h$, $F = M - B$ and σ, and set $f = \sigma'$.

We need the following result from [**5**]:

THEOREM. *In the Baire space of zero charge end preserving homeomorphisms of (M, μ), the ergodic measure preserving homeomorphisms are dense and G_δ in H^0 with the compact open topology.*

To state our main result in this section we consider the embedding of $H(M, \mu)$ in the space $G(M, \mu)$ of all bimeasurable μ-preserving automorphisms of the infinite sigma finite Lebesgue space (M, μ). We endow $G(M, \mu)$ with the coarse topology, under which a sequence of automorphisms τ_n in $G(M, \mu)$ converges to a limit τ if and only if

$$\mu(\tau_n B \Delta \tau B) \to 0$$

for every finite measured subset B of M.

THEOREM 2. *For any conjugate invariant subset $\mathcal{V} \subset G(M, \mu)$ which is dense and G_δ in the coarse topology, $\mathcal{V} \cap H(M, \mu)$ is nonempty. More specifically $\mathcal{V} \cap H^0$ is dense and G_δ in H^0 (the set of zero charge end preserving homeomorphisms) with the compact open topology.*

Proof. Write

$$\mathcal{V} = \bigcap_{m=1}^{\infty} \mathcal{V}_m$$

where each $\mathcal{V}_m$ is open and dense in $G(M, \mu)$ and $\mathcal{V}$ is conjugate invariant. As it is well known (see for example [**7**] or [**11**]) that the ergodic automorphisms $\mathcal{E}$ in $G(M, \mu)$ constitute a coarse dense G_δ set it follows from Baire's Theorem that $\mathcal{E} \cap \mathcal{V}$ is also a coarse dense G_δ set and so nonempty. Therefore $\mathcal{V}$ contains an ergodic automorphism σ and hence its entire conjugacy class. We will show $\mathcal{V}_m \cap H^0$ is dense and open in H^0 with the compact-open topology. This set is clearly open in H^0 since the compact-open topology is finer than the coarse topology. To prove denseness we must show that $\mathscr{V}_m \cap H^0 \cap C(h, K, \epsilon)$ is nonempty where

$$C(h, K, \epsilon) = \{g \in H: \quad d(g(x), h(x)) < \epsilon \text{ for } \mu\text{-a.e. } x \text{ in } K\}$$

is a compact-open basic neighborhood of some ergodic homeomorphism h in H^0. Furthermore, we can choose our compact set K to be a special compact set; i.e., K is a compact connected n-manifold with boundary and $M - K$ has only unbounded components (see [**5**] for further details). Since $\mu(M) = \infty$, one of these components of $M - K$, call it B, must have infinite measure.

Apply Proposition 1 to h and $\sigma \in \mathscr{E} \cap \mathscr{V}$ and K as above to obtain f. Since $\mathscr{V}$ is conjugate invariant, f belongs to $\mathscr{V}_m$. Let $\delta = \omega(\epsilon)$ where ω is the uniform modulus of continuity of h on K. Thus the automorphism $g = h^{-1}f \in h^{-1}\mathscr{V}_m$, a coarse open set. The automorphism g satisfies the following three properties:

1) $g(K) = K$
2) $d(x, g(x)) = 0 < \delta$ for μ-a.e. x in K
3) $g(A) = A$ for all components A of $M - K$.

These are just the conditions needed to apply the Lusin Theorem stated in [**4**] to approximate the automorphism g by a homeomorphism h'. Namely under these conditions there is a compactly supported homeomorphism h' belonging to the coarse open neighborhood of g, $h^{-1}\mathscr{V}_m$, satisfying

$$h'(K) = K \quad \text{and} \quad d(h'(x), x) < \delta \quad \text{for all } x \text{ in } K.$$

It is easy to verify that hh' belongs to $\mathscr{V}_m \cap H^0 \cap C(h, K, \epsilon)$:

1) Since $h' \in h^{-1}\mathscr{V}_m$, $hh' \in \mathscr{V}_m$.

2) Because h' has compact support it therefore has zero charge and is end preserving. So hh' is also end preserving with zero charge, i.e., an element of H^0.

3) For each x in K, since $d(h'(x), x) < \delta = \omega(\epsilon)$ the uniform modulus of continuity of h on K,

$$d(hh'(x), h(x)) < \epsilon,$$

and so hh' is in $C(h, K, \epsilon)$.

This completes the proof of the theorem.

As applications of the above result we list two measure theoretic properties (weak mixing, and zero entropy) that are coarse dense G_δ in $G(M, \mu)$. Theorem 2 now implies the existence of μ-preserving homeomorphisms of the sigma compact, connected n-manifold M possessing the same properties.

(i) In [**11**, Theorem 2.4] Sachdeva proves that the set of weak mixing automorphisms,

$$\mathscr{W} = \{\sigma \in G(M, \mu):\sigma \times \sigma \text{ is ergodic on } (M \times M, \mu \times \mu)\}$$

is coarse dense G_δ in $G(M, \mu)$; n.b. $\mathscr{W}$ is called in **[11]** the set of automorphisms of ergodic index at least 2. What is referred to in **[11]** as "weak mixing" is just the set of automorphisms having no invariant sets of finite measure.

(ii) Krengel has shown [**10**, Theorem 8.2] that the ergodic automorphisms of zero entropy are coarse dense and G_δ in $G(M, \mu)$. The entropy $h(\sigma, \mu)$ of an infinite measure preserving ergodic automorphism σ is defined by setting for any set A of finite measure, $h(\sigma, \mu) = h(\sigma_A, \mu_A)$ the entropy of σ_A, the automorphism of (A, μ_A) induced on A by the first return map to A; i.e., for $x \in A$,

$$\sigma_A(x) = \sigma^{n(x)}(x)$$

where $n = n(x)$ is the smallest positive n such that $\sigma^n(x) \in A$ and μ_A is the restriction of μ to A.

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https://doi.org/10.4153/CJM-1988-032-1 Published online by Cambridge University Press